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مباحثی که به شکوه پدیده‌ای عمیقی شبکه‌های نوجه گرافی (Graph Attention Networks) می‌پردازند

آموزش استفاده از وب آسیس

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OD-characterization of $S_4(4)$ and its group of automorphisms

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Abstract. Let $G$ be a finite group and $\pi(G)$ be the set of all prime divisors of $|G|$. The prime graph of $G$ is a simple graph $\Gamma(G)$ with vertex set $\pi(G)$ and two distinct vertices $p$ and $q$ in $\pi(G)$ are adjacent by an edge if and only if $G$ has an element of order $pq$. In this case, we write $p \sim q$. Let $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$ are primes. For $p \in \pi(G)$, let $\deg(p) = |\{q \in \pi(G)| p \sim q\}|$ be the degree of $p$ in the graph $\Gamma(G)$, we define $D(G) = (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$ and call it the degree pattern of $G$. A group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $S$ such that $|G| = |S|$ and $D(G) = D(S)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group. Let $L = S_4(4)$ be the projective symplectic group in dimension 4 over a field with 4 elements. In this article, we classify groups with the same order and degree pattern as an almost simple group related to $L$. Since $\text{Aut}(L) \cong Z_4$ hence almost simple groups related to $L$ are $L$, $L : 2$ or $L : 4$. In fact, we prove that $L$, $L : 2$ and $L : 4$ are OD-characterizable.

Keywords: Finite simple group, OD-characterization, group of lie type

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1. Introduction

Let $G$ be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of $G$. The prime graph $\Gamma(G)$ of a finite group $G$ is a simple graph with vertex set $\pi(G)$ in which two distinct vertices $p$ and $q$ are joined by an edge if and only if $G$ has an element of order $pq$.

Definition 1.1 Let $G$ be a finite group and $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$. For $p \in \pi(G)$, let $\deg(p) = |\{q \in \pi(G)| p \sim q\}|$ be the degree of $p$ in the graph $\Gamma(G)$,
we define $D(G) = (\text{deg}(p_1), \text{deg}(p_2), \ldots, \text{deg}(p_k))$, which is called the degree pattern of $G$.

Given a finite group $G$, denote by $h_{OD}(G)$ the number of isomorphism classes of finite groups $S$ such that $|G| = |S|$ and $D(G) = D(S)$. In terms of the function $h_{OD}$, groups $G$ are classified as follows:

**Definition 1.2** A group $G$ is called $k$-fold OD-characterizable if $h_{OD}(G) = k$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable.

**Definition 1.3** A group $G$ is said to be an almost simple group if and only if $S \trianglelefteq G \trianglelefteq Aut(S)$ for some non-abelian simple group $S$.

### 2. Preliminaries

For any group $G$, let $\omega(G)$ be the set of orders of elements in $G$, where each possible order element occurs once in $\omega(G)$ regardless of how many elements of that order $G$ has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $\omega(G)$ is denoted by $\mu(G)$. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$. Let $\pi_i = \pi_i(G), 1 \leq i \leq t(G)$, be the $i$th connected component of $\Gamma(G)$. For a group of even order we let $2 \equiv (G)$.

The following lemmas are useful when dealing with a Frobenius group.

**Lemma 3.2** [3] Let $G$ be a 2-Frobenius group of even order which has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively. Then

- (a) $t(G) = 2$ and $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$.
- (b) $G/K$ and $K/H$ are cyclic groups, $|G/K| | |Aut(K/H)|$, and $\left(\frac{|G/K|}{|K/H|}\right) = 1$.
- (c) $H$ is a nilpotent group and $G$ is a solvable group.

The following lemmas are useful when dealing with a Frobenius group.

**Lemma 3.3** [5], [7] Let $G$ be a Frobenius group with complement $H$ and kernel $K$. Then the following assertions hold:

- (a) $K$ is a nilpotent group;
- (b) $|K| \equiv 1 (mod|H|)$;
- (c) Every subgroup of $H$ of order $pq$, with $p, q$ (not necessarily distinct)primes, is cyclic. In particular, every Sylow Subgroup of $H$ of odd order is cyclic and a
2-Sylow subgroup of $H$ is either cyclic or a generalized quaternion group. If $H$ is a non-solvable group, then $H$ has a subgroup of index at most 2 isomorphic to $Z \times SL(2,5)$, where $Z$ has cyclic Sylow $p$-subgroups and $\pi(Z) \cap \{2,3,5\} = \emptyset$. In particular, $15, 20 \notin \omega(H)$. If $H$ is solvable and $O(H) = 1$, then either $H$ is a 2-group or $H$ has a subgroup of index at most 2 isomorphic to $SL(2,3)$.

**Lemma 3.4** [3] Let $G$ be a Frobenius group of even order where $H$ and $K$ are Frobenius complement and Frobenius kernel of $G$, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$.

Let $G$ be a finite group with disconnected prime graph. The structure of $G$ is given in [8] which is stated as a lemma here.

**Lemma 3.5** Let $G$ be a finite group with disconnected prime graph. Then $G$ satisfies one of the following conditions:

a) $s(G) = 2$, $G = KC$ is a Frobenius group with kernel $K$ and complement $C$, and the two connected components of $G$ are $\Gamma(K)$ and $\Gamma(C)$. Moreover $K$ is nilpotent, and here $\Gamma(K)$ is a complete graph.

b) $s(G) = 2$ and $G$ is a 2-Frobenius group, i.e., $G = ABC$ where $A, AB \leq G$, $B \leq BC$, and $AB, BC$ are Frobenius groups.

c) There exists a non-abelian simple group $P$ such that $P \leq \bar{G} = \frac{G}{N} \leq Aut(P)$ for some nilpotent normal $\pi_1(G)$-subgroup $N$ of $G$ and $\frac{G}{P}$ is a $\pi_1(G)$-group. Moreover, $\Gamma(P)$ is disconnected and $s(P) \geq s(G)$.

If a group $G$ satisfies condition (c) of the above lemma we may write $P = B/N$, $B \leq G$, and $\frac{G}{P} = G/B = A$, hence in terms of group extensions $G = N \cdot P \cdot A$, where $N$ is a nilpotent normal $\pi_1(G)$-subgroup of $G$ and $A$ is a $\pi_1(G)$-group.

**Theorem 3.6** [6] The following assertions are equivalent:

(a) $G$ is a Frobenius group with kernel $K$ and complement $H$.

(b) $G = HK$ such that $K \triangleleft G$ and $H < G$ and $H$ act on $K$ without fixed point.

By [2] the outer automorphism group of $S_4(4)$ is isomorphic to $Z_4$; hence we have the following lemma:

**Lemma 3.7** If $G$ is an almost simple group related to $L = S_4(4)$, then $G$ is isomorphic to one of the following groups: $L$, $L : 2$ or $L : 4$.

4. **Main Results**

**Theorem 4.1** If $G$ is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where $M$ is an almost simple group related to $L = S_4(4)$, then the following assertions hold:

(a) If $M = L$, then $L$ is OD-characterizable.

(b) If $M = L : 2$, then $L : 2$ is OD-characterizable.

(c) If $M = L : 4$, then $L : 4$ is OD-characterizable.

**Proof.** We break the proof into a number of separate cases:

Case 1: If $M = L$, then $G \cong L$. This follows from [1].

Case 2: If $M = L : 2$, then $G \cong L : 2$.

If $M = L : 2$, by [2], we have $\mu(L : 2) = \{8, 10, 12, 15, 17\}$ from which we deduce that $D(L : 2) = (2, 2, 2, 0)$. The prime graph of $L : 2$ has the following form:
As $|G| = |L : 2| = 2^9 \cdot 3^2 \cdot 5^2 \cdot 17$ and $D(G) = D(L : 2) = (2, 2, 2, 0)$, then $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 3 \sim 5; 17\}$. Thus $G$ has a disconnected prime graph with $s(G) = 2$. Now, We show that $G$ is neither a Frobenius group nor 2-Frobenius group. If $G$ be a Frobenius group, then by Lemma 3.4(a), $G = KC$, with Frobenius kernel $K$ and Frobenius complement $C$ with connected components $\Gamma(K)$ and $\Gamma(C)$. $\Gamma(K)$ is a graph with vertex $\{17\}$ and $\Gamma(C)$ with vertices $\{2, 3, 5\}$. By Lemma 3.2(b), $|K| \mid (|C| - 1)$. Since $|K| = 17$ and $|C| = 2^9 \cdot 3^2 \cdot 5^2$ then $17 \mid (2^9 \cdot 3^2 \cdot 5^2 - 1)$ a contradiction. If $G$ be a 2-Frobenius group, then, there is a normal series $1 < H < G$ such that $K$ and $G/H$ are Frobenius groups with kernels $K$ and $H/K$, respectively. By Lemma 3.1(a), we have $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$. Therefore, $|K/H| = 17$. Also, by Lemma 3.1(b), we have $G/K \leq Aut(K/H) \cong Z_{16}$, hence $|G/K| \mid 2^4$, which implies that $\{3, 5, 17\} \subseteq \pi(K)$ from which we deduce that $5 \in \pi(H)$. Let $H_5 \in Syl_{15}(H)$ and $G_{17} \in Syl_{17}(G)$. Then $H_5 \text{char} H \subseteq G$. By nilpotency of $H$, we have $H_5 \triangleleft G$ and $H_5$ act on $G_{17}$ without fixed point, since $5 \approx 17$ in $\Gamma(G)$. Therefore, by Theorem 3.1, $H_5.G_{17}$ is a Frobenius group. So, $|G_{17}| \mid (|H_5| - 1)$, i.e., $17 \mid (5^i - 1)$, $i = 1$ or 2, a contradiction.

Now by Lemma 3.4(c), there exists a non-abelian simple group $P$ such that $P \leq \overline{G} = G/N \leq Aut(P)$ for some nilpotent normal $\{2, 3, 5\}$-subgroup $N$ of $G$ and $\overline{G}/P$ is a $\{2, 3, 5\}$-group.

$17 \in \pi(P)$. Since $\overline{G}/P$ is a $\{2, 3, 5\}$-group and $17 \mid |G|$, therefore, we have $17 \mid |P|$, i.e., $P \in \mathfrak{S}_{17}$, which implies that $\pi(P) \subseteq \{2, 3, 5, 17\}$. Using [9], we list the possibilities for $P$ in the following table.

| Table 1: Simple groups in $\mathfrak{S}_p$, $p \leq 17, p \neq 7, 11, 13$. |
|---|---|---|
| $P$ | $|P|$ | $|\text{out}(P)|$ |
| $L_2(17)$ | $2^3 \cdot 3^2 \cdot 5^2 \cdot 17$ | 2 |
| $L_2(16)$ | $2^3 \cdot 3 \cdot 5 \cdot 17$ | 4 |
| $S_4(4)$ | $2^3 \cdot 3 \cdot 5 \cdot 17$ | 4 |

If $P \cong L_2(17)$ we get $L_2(17) \leq G/N \leq Aut(L_2(17))$. It follows that $|N| = 2^5 \cdot 5^2 \leq |\text{out}(P)|$ or $|N| = 2^4 \cdot 5^2 \leq |\text{out}(P)|$. Let $N_5 \in Syl_{15}(N)$ and $G_{17} \in Syl_{17}(G)$. Then $N_5 \text{char} N \leq G$. By the nilpotency of $N$, which implies that $N_5 \leq G$ and $N_5$ act on $G_{17}$ without fixed point, since $5 \approx 17$ in $\Gamma(G)$. Therefore, by Theorem 3.1, $N_5.G_{17}$ is a Frobenius group. So, $|G_{17}| \mid (|N_5| - 1)$, i.e., $17 \mid (5^i - 1)$, $i = 1$ or 2, a contradiction.

If $P \cong L_2(16)$ we get $L_2(16) \leq G/N \leq Aut(L_2(16))$. It follows that $|N| = 2^5 \cdot 3 \cdot 5$ or $|N| = 2^4 \cdot 3 \cdot 5$. Let $N_5 \in Syl_{15}(N)$ and $G_{17} \in Syl_{17}(G)$. Then $N_5 \text{char} N \leq G$. By the nilpotency of $N$, which implies that $N_5 \leq G$ and $N_5$ act on $G_{17}$ without fixed point, since $5 \approx 17$ in $\Gamma(G)$. Therefore, by Theorem 3.1, $N_5.G_{17}$ is a Frobenius group. So, $|G_{17}| \mid (|N_5| - 1)$, i.e., $17 \mid (5^i - 1)$ a contradiction.

Therefore, $P \cong S_4(4)$. We have $S_4(4) \leq G/N \leq Aut(S_4(4))$. It follows that $|N| = 2$ or $|N| = 1$. 

Figure 1: The prime graph of $S_4(4) : 2$
If \(|N| = 1\), then \(G \cong S_4(4) : 2\).
If \(|N| = 2\), then \(G/C_G(N) \leq Aut(N) = 1\), therefore \(G/C_G(N) = 1\), hence \(G = C_G(N)\) and \(N \leq Z(G)\). Let \(G_{17} \in Syl_{17}(G)\). Then \(N.G_{17}\) is a subgroup of \(G\), therefore, \(N.G_{17}\) has an element of order 2,17, which implies that 2 \(\sim\) 17 in \(\Gamma(G)\), a contradiction.

Case 3: If \(M = L : 4\), then \(G \cong L : 4\).
If \(M = L : 4\), by \([2]\), we have \(\mu(M) = \{12, 15, 16, 17, 20\}\) from which we deduce that \(D(L : 4) = (2, 2, 2, 0)\). The prime graph of \(L : 4\) has the following form:

![Prime graph of S4(4) : 4](image)

As \(|G| = |L : 4| = 2^{10}.3.5.17\) and \(D(G) = D(L : 4) = (2, 2, 2, 0)\), then \(\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 3 \sim 5; 17\}\). Thus \(G\) has a disconnected prime graph with \(s(G) = 2\).

Now, we show that \(G\) is neither a Frobenius group nor 2-Frobenius group. If \(G\) is a Frobenius group, then by Lemma 3.4(a), \(G = KK\), with Frobenius kernel \(K\) and Frobenius complement \(C\) with connected components \(\Gamma(K)\) and \(\Gamma(C)\). \(\Gamma(K)\) is a graph with vertex \(\{17\}\) and \(\Gamma(C)\) with vertices \(\{2, 3, 5\}\). By Lemma 3.2(b), \(|K| \mid (|C| - 1)\).

Since \(|K| = 17\) and \(|C| = 2^{10}.3.5^2\) then \(17 \mid (2^{10}.3.5^2 - 1)\) a contradiction. If \(G\) is a 2-Frobenius group, then, there is a normal series \(1 < H < K < G\) such that \(K\) and \(G/H\) are Frobenius groups with kernels \(H\) and \(K/H\), respectively. By Lemma 3.1(a), we have \(T(G) = \{p_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}\). Therefore, \(|K/H| = 17\). Also, by Lemma 3.1(b), we have \(G/K \leq Aut(K/H) \cong Z_{16}\), hence \(|G/K| \mid 2^4\), which implies that \(\{3, 5, 17\} \subseteq \pi(K)\) from which we deduce that 5 \(\in\) \(\pi(H)\).

Let \(H_5 \in Syl_{15}(H)\) and \(G_{17} \in Syl_{17}(G)\). Then \(H_5charH \leq G\). By nilpotency of \(H\), we have \(H_5 \leq G\) and \(H_5\) act on \(G_{17}\) without fixed point, since 5 \(\sim\) 17 in \(\Gamma(G)\). Therefore, by Theorem 3.1, \(H_5.G_{17}\) is a Frobenius group. So, \(|G_{17}| \mid |(H_5 - 1)|\), i.e., 17 \(\mid (5^i - 1), i = 1\) or 2, a contradiction.

Now by Lemma 3.4(c), there exists a non-abelian simple group \(P\) such that \(P \leq G = G/N \leq Aut(P)\) for some nilpotent normal \(\{2, 3, 5\}\)-subgroup \(N\) of \(G\) and \(G/P\) is a \(\{2, 3, 5\}\)-group.

Similarly to case 2, we deduce that \(P \cong S_4(4)\). We have \(S_4(4) \leq G/N \leq Aut(S_4(4))\).

It follows that \(|N| = 4, 2\) or 1.

If \(|N| = 1\), then \(G \cong S_4(4) : 4\).
If \(|N| = 2\), then \(G/C_G(N) \leq Aut(N) = 1\), therefore \(G/C_G(N) = 1\), hence \(G = C_G(N)\) and \(N \leq Z(G)\). Let \(G_{17} \in Syl_{17}(G)\). Then \(N.G_{17}\) is a subgroup of \(G\), therefore, \(N.G_{17}\) has an element of order 2,17, which implies that 2 \(\sim\) 17 in \(\Gamma(G)\), a contradiction.

If \(|N| = 4\), then \(G/C_G(N) \leq Aut(N) \cong Z_2\). Thus, \(|G/C_G(N)| = 1\) or 2. If \(|G/C_G(N)| = 1\), then, we have \(N \leq Z(G)\). Let \(G_{17} \in Syl_{17}(G)\). Then \(N.G_{17}\) is a subgroup of \(G\), therefore, \(N.G_{17}\) has an element of order 2,17, which implies that 2 \(\sim\) 17 in \(\Gamma(G)\), a contradiction. If \(|G/C_G(N)| = 2\), then \(N < C_G(N)\) and \(1 \neq C_G(N)/N \leq G/N \cong L\).

Therefore, from simplicity \(L\) we deduce that \(G = C_G(K)\), a contradiction.

References


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مباحث پیشرفته یادگیری عمقی؛ شبکه‌های توجه گرافی (Graph Attention Networks)

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