Module-Amenability on Module Extension Banach Algebras

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Abstract. Let A be a Banach algebra and E be a Banach A-bimodule then $S = A \oplus E$, the $l^1$-direct sum of A and E becomes a module extension Banach algebra when equipped with the algebras product $(a, x)(a', x') = (aa', a.x' + x'.a')$. In this paper, we investigate $\triangle$-amenability for these Banach algebras and we show that for discrete inverse semigroup $S$ with the set of idempotents $E_S$, the module extension Banach algebra $S = l^1(E_S) \oplus l^1(S)$ is $\triangle$-amenable as a $l^1(E_S)$-module if and only if $l^1(E_S)$ is amenable as Banach algebra.

Keywords: Module-amenability, module extension, Banach algebras

1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in [8]. The main Theorem in [8] asserts that the group algebra $L^1(G)$ of a locally compact group $G$ is amenable if and only if $G$ is amenable. This is far from true for semigroups. If $S$ is a discrete inverse semigroup, $l^1(S)$ is amenable if and only if $E_S$ is finite and all the maximal subgroups of $S$ are amenable [6]. This failure is due to the fact that $l^1(S)$, for a discrete inverse semigroup $S$ with the set of idempotents $E_S$, is equipped with two algebraic structures. It is a Banach algebra and a Banach module over $l^1(E_S)$.

The concept of module amenability for Banach algebras was introduced by M.Amini in [1]. The main theorem in [1] asserts that for an inverse semigroup $S$, with the set of idempotents $E_S$, $l^1(S)$ is module amenable as a Banach module over $l^1(E_S)$ if and only if $S$ is amenable. Also the second named author study the concept of weak module amenability in [2] and showed that for a commutative inverse semigroup $S$, $l^1(S)$ is always weak module amenable as a Banach module over $l^1(E_S)$. There are many examples of Banach modules which do not have any natural algebra structure One example is $L^p(G)$ which is a left Banach $L^1(G)$-module, for a locally compact group $G$ [4]. The theory of amenability in [8] and module amenability developed in [1] does not cover these examples. There is one thing in common in these examples and that is the existence of a module homomorphism from the Banach module to the underlying Banach algebra. For instance if $G$ is a compact group and $f \in L^0(G)$, then on has the module homomorphism $\triangle_f : L^p(G) \to L^1(G)$ which sends $g$ to $f \ast g$. The concept of $\triangle$-amenability in [7] is defined for a Banach module $E$ over a Banach algebra $A$ with a given mod-

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ule homomorphism $\triangle : E \to A$. The authors in [7] gives the basic properties of $\triangle$-amenability and in particular establishes the equivalence of this concept with the existence of module virtual (approximate) diagonals in an appropriate sense. Also the main example in [7] asserts that for a discrete abelian group $G$, $L^p(G)$ is $\triangle$-amenable as an $L^1(G)$-module if and only if $G$ is amenable. In this paper we shall focus on an especial kind of Banach algebras which are constructed from a Banach algebra $A$ and a Banach $A$-bimodule $E$, called module extension Banach algebras and we verify the concept of $\triangle$-amenability for these Banach algebras.

2. Preliminaries

Let $A$ be a Banach algebra and $E$ be a Banach space with a left $A$-module structure such that, for some $M > 0$, \[ \|a.x\| \leq M \|x\| \quad (a \in A, x \in E). \] Then $E$ is called a left Banach $A$-module. Right and two-sided Banach $A$-modules are defined similarly. Throughout this section $E$ is a Banach $A$-bimodule and $\triangle : E \to A$ is a bounded Banach $A$-biomodule homomorphism.

**Definition 2.1** Let $X$ be a Banach $A$-Bimodule. A bounded linear map $D : A \to X$ is called a module derivation (or more specifically $\triangle$-derivation) if

\[
D(\triangle(a.x)) = a.D(\triangle(x)) + D(a).\triangle(x)
\]

\[
D(\triangle(x.a)) = D(\triangle(x)).a + \triangle(x).D(a)
\]

For each $a \in A$ and $x \in E$. Also $D$ is called inner (or $\triangle$-inner) if there is $f \in X$ such that

\[
D(\triangle(x)) = f.\triangle(x) - \triangle(x).f \quad (x \in E)
\]

**Definition 2.2** A bimodule $E$ is called module amenable (or more specifically $\triangle$-amenable as a $A$-bimodule) if for each Banach $A$-bimodule $X$, all $\triangle$-derivation from $A$ to $X^*$ are $\triangle$-inner.

It is clear that $A$ is $A$-module amenable (which $\triangle =$id) if and only if it is amenable as a Banach algebra. A right bounded approximate identity of $E$ is a bounded net $a_{\alpha}$ in $A$ such that for each $x \in E$, $(\triangle(x).a_{\alpha} - \triangle(x)) \to 0$ as $\alpha \to 0$. The left and two sided approximate identities are defined similarly.

**Proposition 2.3** If $E$ is module amenable, then $E$ has a bounded approximate identity.

**Proposition 2.4** If $I$ is a closed ideal of $A$ which contains a bounded approximate identity, $E$ is a Banach $A$-bimodule with module homomorphism $\triangle : E \to A$, and $X$ is an essential Banach $I$-module, then $X$ is a Banach $A$-module and each $\triangle_I$-derivation $D : I \to X$ uniquely extends to a $\triangle$-derivation $D : A \to X$ which is continuous with respect to the strict topology of $A$ (induced by $I$) and $W$-topology of $X^*$.

**Proposition 2.5** If $\triangle : E \to A$ has a dense range, then $\triangle$-amenability of $E$ is equivalent to amenability of $A$.

**Definition 2.6** Let $\triangle : A \hat{\otimes} A \to A$ be the continuous lift of the multiplication map of $A$ to the projective tensor product $A \hat{\otimes} A$. A module approximate diagonal of $E$ is
a bounded net \( e_\alpha \) in \( A \hat{\otimes} A \) such that
\[
\|e_\alpha \triangle(x) - \triangle(x).e_\alpha\| \to 0 \\
\|\pi(e_\alpha \triangle(x) - \triangle(x))\| \to 0, \quad (x \in E)
\]
As \( \alpha \to \infty \). A module virtual diagonal of \( E \) is an element \( M \) in \( (A \hat{\otimes} A)^{**} \) such that
\[
M.\triangle(x) - \triangle(x).M = 0 \\
\pi^{**}(M) . \triangle(x) - \triangle(x) = 0, \quad (x \in E)
\]
It is clear that if \( E \) has a module virtual diagonal, then \( A \) contains a bounded approximate identity.

**Theorem 2.7** Consider the following assertions

i) \( E \) is module amenable,

ii) \( E \) has a module virtual diagonal,

iii) \( E \) has a module approximate diagonal.

We have (i) \( \to \) (ii) \( \to \) (iii). If moreover \( \triangle \) has a dense range, all the assertions are equivalent.

**Example 2.8** let \( 1 < P < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) then \( l^1 \) is a Banach algebra and \( l^p \) is a Banach \( l^1 \) module, both with respect to pointwise multiplication. Also each \( f \in l^q \) defines a module homomorphism \( \triangle_f : l^p \to l^1 \) by \( \triangle_f(g) = g^* f \). If \( f = \sum_{k=-\infty}^{\infty} \delta_k \), then \( \triangle_f \) has dense range and \( l^p \) is \( \triangle_f \)-amenable.

### 3. \( \triangle \)-amenability of Module extension Banach algebras

The module extension Banach algebra corresponding to \( A \) and \( E \) is \( S = A \oplus E \), the \( l^1 \)-direct sum of \( A \) and \( E \), with the algebra product defined as follows:
\[
(a, x). (a', x') = (aa', ax' + x.a') \quad (a, a' \in A, x, x' \in E).
\]
Some aspects of algebras of this form have been discussed in [3] and [5] also the amenability and \( n \)-weak amenability of module extension Banach algebras investigated by zhang in [? ?]. In this section we show that the amenability of Banach algebra \( A \) is equivalent to \( \triangle \)-amenability \( A \oplus E \) as a Banach \( A \)-module.

By the following module actions the module extension Banach algebra \( A \oplus E \) is a Banach \( A \)-module
\[
a.(b, x) = (ab, x), \quad (b, x).a = (ba, x) \quad (a, b \in A, x \in E).
\]
Also \( \triangle : A \oplus E \to A \) by \( (a, x) \to a(a \in A, x \in E) \) is a surjective \( A \)-module homomorphism, so we have:

**Proposition 3.1** The Banach algebra \( A \) is amenable if and only if the module extension Banach algebra \( A \oplus E \) is \( \triangle \)-amenable as a \( A \)-module.

**Example 3.2** Let \( S \) is a discrete inverse semigroup with the set of idempotents \( E_S \) and \( E = l^1(S) \), \( A = l^1(E_S) \) and \( l^1(E_S) \) act on \( l^1(S) \) by multiplication in this case; the module extension Banach algebra \( S = l^1(E_S) \oplus l^1(S) \) is \( \triangle \)-amenable as a \( l^1(E_S) \)-module if and only if \( l^1(E_S) \) is amenable.
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