Homotopy Perturbation Method for the Generalized Fisher’s Equation

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Abstract


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1 Introduction

Wazwaz in [11], considered the Fisher’s equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u),$$  \hspace{1cm} (1)

but, in this paper, we consider the generalized Fisher’s equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u^\beta),$$  \hspace{1cm} (2)

where \( u_t = \frac{\partial u}{\partial t}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2} \).

Fisher proposed Eq. (1) as a model for the propagation of a mutant gene, with \( u \) denoting the density of an advantageous. This equation is encountered in chemical kinetics [7] and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models [1], flame propagation, neurophysiology, autocatalytic chemical reactions, and branching Brownian motion processes.

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In order to solve above equations, many researchers have used various methods. Authors studied variational iteration method for Fisher’s equation [8] and also, employed a modified of variational iteration method for generalized Fisher’s equation [9], and Wazwaz [10] studied Adomian decomposition method for Fisher’s equation. We know that Adomian decomposition method requires the use of Adomian polynomials for nonlinear terms, and this need more work.

In this paper, we solve the generalized Fisher’s equation (2) via homotopy perturbation method (in short HPM) to overcome the difficulty arising in calculating Adomian polynomials. HPM introduced by He [3, 4, 5, 6] has been used by many mathematicians and engineers to solve various functional equations. In this method the solution is considered as the sum of an infinite series which converges rapidly to the accurate solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter.

### 2 Homotopy perturbation methods

To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \quad (3)$$

with the boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \quad (4)$$

where $A$ is a general differential operator, $B$ a boundary operator, $f(r)$ a known analytical function and $\Gamma$ is the boundary of the domain $\Omega$. $A$ can be divided into two parts which are $L$ and $N$, where $L$ is linear and $N$ is nonlinear. Eq. (3) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \quad (5)$$

By the homotopy technique, we construct a homotopy $V(r, p): \Omega \times [0,1] \to R$, which satisfies:

$$H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \quad (6)$$

or

$$H(V, p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0, \quad (7)$$

where $p \in [0,1]$ is an embedding parameter, $u_0$ is an initial approximation of Eq. (3), which satisfies the boundary conditions. Obviously, from Eqs. (6) or (7) we will have

$$H(V, 0) = L(V) - L(u_0) = 0, \quad (8)$$
The changing process of \( p \) from zero to unity is just that of \( V(r, p) \) from \( u_0(r) \) to \( u(r) \). In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter \( p \) as a small parameter, and assume that the solution of Eqs. (6) or (7) can be written as a power series in \( p \):

\[
V(x,t)=V_0(x,t)+V_1(x,t)+V_2(x,t)+V_3(x,t)+\cdots,
\]

and the exact solution is obtained as follows:

\[
u = \lim_{p \to 1} V = \lim_{p \to 1} (V_0 + pV_1 + p^2 V_2 + p^3 V_3 + \cdots) = \sum_{j=0}^{\infty} V_j,
\]

The series (11) is convergent for most cases, and the rate of convergence depends on \( L(u) \) [2].

### 3 HPM for the generalized Fisher’s equation

In this section, we consider the generalized Fisher’s equation

\[
u_t = \nu_{xx} + \alpha \nu (1 - \nu^\beta),
\]

subject to the initial condition \( u(x,0) = f(x) \).

For solving Eq. (12), by homotopy perturbation method we construct a homotopy as follows:

\[
(1 - p) \left\{ \frac{\partial V}{\partial t} - \frac{\partial u_0}{\partial t} \right\} + p \left\{ \frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial x^2} - \alpha V + \alpha V^{\beta+1} \right\} = 0,
\]

or

\[
\frac{\partial V}{\partial t} - \frac{\partial u_0}{\partial t} + p \frac{\partial V}{\partial t} + p \left\{ - \frac{\partial^2 V}{\partial x^2} - \alpha V + \alpha V^{\beta+1} \right\} = 0.
\]

Suppose the solution of Eq. (14) has the form

\[
V(x,t)=V_0(x,t)+pV_1(x,t)+p^2V_2(x,t)+p^3V_3(x,t)+\sum_{j=0}^{\infty} p^jV_j(x,t),
\]

where \( V_j(x,t) \) are functions yet to be determined. Substituting (15) into (14), and equating the terms with identical powers of \( p \), we have

\[
p^0 : \quad \frac{\partial V_0}{\partial t} = \frac{\partial u_0}{\partial t}, \quad V_0(x,0) = f(x),
\]

\[
p^1 : \quad \frac{\partial V_1}{\partial t} = \frac{\partial^2 V_0}{\partial x^2} + \alpha V_0 - \alpha V_0^{\beta+1} - \frac{\partial u_0}{\partial t}, \quad V_1(x,0) = 0,
\]
\[ p^{k+1} : \frac{\partial V_{k+1}}{\partial t} = \frac{\partial^2 V_k}{\partial x^2} + \alpha V_k - \alpha \sum_{j_1=0}^{k} \sum_{j_2=0}^{l_1} \cdots \sum_{j_{p-1}=0}^{l_{p-1}} (V_{j_{p-1}} V_{j_{p-1}+1} \cdots V_{j_{k-l}}), \]  

where \( V_{k+1}(x,0) = 0, \ k \geq 1. \)

Considering \( u_0(x,t) = u(x,0) = f(x) \), we have

\[ V_0(x,t) = u_0(x,t) = f(x), \]  

\[ V_k(1+1) \int_0^t \left\{ \frac{\partial^2 V_k}{\partial x^2} + \alpha V_k - \alpha \sum_{j_1=0}^{k} \sum_{j_2=0}^{l_1} \cdots \sum_{j_{p-1}=0}^{l_{p-1}} (V_{j_{p-1}} V_{j_{p-1}+1} \cdots V_{j_{k-l}}) \right\} ds, \]  

where \( k \geq 0 \). Therefore, the exact solution of (12) can be obtained by setting \( p = 1 \), i.e.

\[ u(x,t) = \lim_{p \to 1} V(x,t) = \sum_{k=0}^\infty V_k(x,t). \]

In Eq. (12), following [11], we set \( \alpha = 1, \beta = 6 \) and \( u(x,0) = f(x) = \frac{1}{(1 + e^{2x})^\frac{1}{2}}. \)

Therefore, by Eqs. (20) and (21) we have

\[ V_0(x,t) = \frac{1}{(1 + e^{2x})^\frac{1}{2}}, \]

\[ V_k(x,t) = \int_0^t \left\{ \frac{\partial^2 V_k}{\partial x^2} + V_k - \sum_{j_1=0}^{k} \sum_{j_2=0}^{l_1} \cdots \sum_{j_{p-1}=0}^{l_{p-1}} (V_{j_{p-1}} V_{j_{p-1}+1} \cdots V_{j_{k-l}}) \right\} ds, \]

where \( k \geq 0 \). Consequently, we obtain

\[ V_0(x,t) = \frac{1}{(1 + e^{2x})^\frac{1}{2}}, \]

\[ V_1(x,t) = \frac{5e^{\frac{3}{2}}}{4(1 + e^{2x})^\frac{3}{2}}, \]

\[ V_2(x,t) = \frac{25e^{\frac{3}{2}} (e^{2x} - 3) t^2}{16 (1 + e^{2x})^\frac{3}{2} 2}, \]

\[ V_3(x,t) = \frac{125e^{\frac{3}{2}} (18e^{\frac{3}{2}} - e^{3x} - 9) t^3}{64 (1 + e^{2x})^\frac{3}{2} 3}. \]
\[ V_d(x,t) = \frac{625 e^{\frac{3}{2}x} (e^{6x} - 80 e^{\frac{9}{2}x} + 90 e^{3x} + 144 e^{\frac{3}{2}x} - 27) t^4}{6 (1 + e^{\frac{3}{2}x})^{\frac{16}{3}}} \]

\[ V_s(x,t) = \frac{78125 e^{\frac{3}{2}x} (e^{9x} - 334 e^{\frac{15}{2}x} + 1255 e^{6x} + 220 e^{\frac{9}{2}x} - 585 e^{3x} - 1134 e^{\frac{3}{2}x} + 81) t^5}{24576 (1 + e^{\frac{3}{2}x})^{\frac{22}{3}}} \]

and so on. The solution in a closed form is given by

\[ u(x,t) = \sum_{k=0}^{\infty} V_k \]

\[ = \frac{1}{(1 + e^{\frac{3}{2}x})^{\frac{1}{3}}} + \frac{5 e^{\frac{3}{2}x} t}{4(1 + e^{\frac{3}{2}x})^{\frac{1}{3}}} + \frac{25 e^{\frac{3}{2}x} (e^{\frac{3}{2}x} - 3) t^2}{16 (1 + e^{\frac{3}{2}x})^{\frac{5}{3}}} + \frac{125 e^{\frac{3}{2}x} (18 e^{\frac{3}{2}x} - e^{3x} - 9) t^3}{64 (1 + e^{\frac{3}{2}x})^{\frac{8}{3}}} + \cdots \]

\[ = \left\{ \frac{1}{2} \tanh \left( -\frac{3}{4} (x - \frac{5}{2} t) \right) + \frac{1}{2} \right\}^{\frac{1}{3}}, \]

which is exactly the same as obtained by Adomain decomposition method [11]. Therefore, the HPM avoids the need for calculating the Adomian polynomials which can be difficult.

4 Conclusions

In this paper, the homotopy perturbation method has been successfully used to study generalized Fisher’s equation. An important结论 can be made here. The homotopy perturbation method avoids the need for calculating the Adomian polynomials which can be difficult in some cases. The results show that the homotopy perturbation method is a powerful mathematical tool for finding the exact and approximate solutions of nonlinear equations. In our work we use the MATLAB to calculate the series obtained from the homotopy perturbation method.

References

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