Special Case of Logarithmically Asymptotically Optimal Hypothesis Testing of Two Statistically Independent Random Variables

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Abstract

The problem of hypotheses testing for a model consisting of two independent objects is considered. It is assumed that five probability distributions are known and objects independently from each other follow one of them. In this paper using the theorem in [1] which Navaei has proved via the theory of large deviations the matrix of asymptotic inter-dependencies (reliability functions) of all possible pairs of the error probability exponents in optimal testing for this model is determined. The case with two independent objects and two given probability distributions was researched by Haroutunian and Ahlswede.

Keywords: Independent Objects, Reliability Matrix, Hypotheses Testing, Type.

1 Introduction

Many papers were devoted to the study of exponential decrease, as the sample size $N$ goes to infinity. In the book of Csiszar and Shields [5] for independent identically distributed observations different asymptotical aspects of the two hypotheses testing are considered via theory of large deviations. In [7] Navaei solved the problem proposed by R. Dobrushin for the case of one object and $M > 2$ hypotheses via the theory of large deviations. Recently Ahlswede and Haroutunian formulated an ensemble of new problems on multiple hypotheses testing for many objects and on identification of hypotheses. The problem of hypotheses testing for the model consisting of two independent objects with two possible hypothetical distributions was investigated in [1] without usage theory of large deviation.

In this paper the problem of hypotheses testing for the model consisting of two independent objects (random variables) is studied.
2 Problem Statements

We consider a generalization case of the problem of many hypotheses testing concerning one object [7], [6], to the similar problem for a model with two independent objects and also one from ensemble of problems considered and examined for the case of two known distributions in [1]. Let $X_i$ and $X_2$ be independent random variables taking values in the finite set $\chi$ and presenting characteristics of two independent objects. The cardinality of $\chi$ is $K$. Let $1, \ldots, K_i, 1, 2$ assume values in $\chi$, $i = 1, 2$. The random vector $(X_1, X_2)$ and also consider values $(x_{i_k_1}, x_{i_k_2}) \in \chi \times \chi$, $K_1, K_2 = 1, K$.

Let $P(\chi)$ be the space of all possible distributions on $\chi$. There are given five probability distributions $P = P_L$, $L = 1, 5$, from $P(\chi)$. We consider the model with two objects characteristics of which $X_i$, $i = 1, 2$, independently follow to one of the five distributions.

Let $\{X_{n_i_1}, X_{n_i_2}\}_{i=1}^{N}$, be a sequence of results of $N$ independent observations of the vector $(X_1, X_2)$. The statistician must define a pair of distributions corresponding to obtain data. The selection is made from five given distributions, i.e. We have five hypotheses: $H_l$ : $P = P_l$, $l = 1, 5$. We call the procedure of making decision on the base of $N$ observations of the test and denote it by $\phi_N$. The test $\phi_N$ of this model may be composed by the pair of the tests $\phi_N^1$ and $\phi_N^2$ for the respective object: $\phi_N = (\phi_N^1, \phi_N^2)$.

Let $\alpha_{m_i,m_j,l_i,l_j} (\phi)$, be the probability of the erroneous acceptance by the test $\phi_N$ of hypotheses pair $(H_l, H_l)$ provided that the pair $(H_{m_i}, H_{m_j})$ is true, where $(m_i, m_j) \neq (l_i, l_j)$, $m_i, l_i = 1, 5$, $i = 1, 2$.

The probability to reject a true pair of hypotheses $(H_{m_i}, H_{m_j})$, is defined by:

$$\alpha_{m_1, m_2 \neq m_1, m_2} (\phi_N) = \sum_{(m_1, m_2) \neq (l_1, l_2)} \alpha_{m_1, m_2 / l_1, l_2} (\phi_N)$$

(1)

We also define “reliabilities” of test $\phi$, in this way,

$$E_{m_i, m_j / l_i, l_j} (\phi) \overset{\Delta}{=} \lim_{N \to \infty} \frac{1}{N} \log E_{m_i, m_j / l_i, l_j} (\phi_N), m_i, l_i = 1, 5, i = 1, 2.$$  

(2)

It follows from (1) and (2) that:

$$E_{m_i, m_j / m_i, m_j} (\phi) = \min_{(l_1, l_2) \neq (m_i, m_j)} E_{m_i, m_j / l_i, l_j} (\phi).$$  

(3)

The matrix $E(\phi) = \{E_{m_i, m_j / l_i, l_j} (\phi)\}$ we call the reliability matrix of the sequence $\phi$ of tests.
Definition 1 The test sequence $\varphi^*$ is called logarithmically asymptotically optimal (LAO) for this model if for given values of the elements, $E_{1,1/1.5}, E_{1,1/5.1}, E_{2,2/5.2}, E_{2,2/2.5}, E_{3,3/3.5}, E_{3,3/5.3}, E_{4,4/4.5},$ and $E_{4,4/4.5},$ it provides maximal values for all other elements of $E(\varphi^*)$.

Our aim is to define conditions on $E_{1,1/1.5}, E_{1,1/5.1}, E_{2,2/5.2}, E_{2,2/2.5}, E_{3,3/3.5}, E_{3,3/5.3}, E_{4,4/4.5},$ and $E_{4,4/4.5},$ under which there exists LAO sequence of tests $\varphi^*$ and to show how other elements of $E(\varphi^*)$ can be found from them.

In paper [1] the case of one object and $M$ hypothetical distributions was considered via the theory of large deviations. Let us recall main definitions for the case $M = 5$. The random variable $X$ taking values in the finite set $\chi$ and follow one of the five distributions $P_l$, $l = 1, 5$. Let $X = (x_1, ..., x_N)$ is a sequence of results $N$ observations of the object. The procedure of decision making is a non–randomized test $\varphi_N$, it can be defined by decision of the sample space $\chi^N$ on five disjoint subsets $A_l^N = \{X: \varphi^N(X) = l\}$, $l = 1, 5$. The set $A_l^N$ consist of all vectors $X$ for which the hypothesis $H_l$ is adopted. The probabilities of the erroneous acceptance of hypothesis $H_l$ provided that $H_m$ is true $\alpha_{m/l}(\varphi_N) = P_m(\{A_l^N\})$

The probability to reject $H_m$, when it is true, is $\alpha_{m/m}(\varphi_N) = \sum_{l\neq m} \alpha_{m/l}(\varphi_N)$.

Corresponding error probability exponents, call “reliability “are defined as $E_{m/l}(\varphi) = \lim_{N\to\infty} -\frac{1}{N} \log \alpha_{m/l}(\varphi_N)$, $m, l = 1, 5$.

It follows (see (3)) that $E_{m/m}(\varphi) = \min_{l\neq m} E_{m/l}(\varphi)$, $m = 1, 5$.

(4)

The matrix $E(\varphi) = \{E_{m/l}(\varphi)^{\Delta}\}$ is called reliability matrix of the sequence $\varphi$ of the tests.

In this paper we use the type of a vector $X = (x_1, x_2, ..., x_N) \in \chi^N$ that is the empirical distribution given by $Q(x) = N^{-1} N(x \mid X)$ for all $x \in \chi$, where $N(x \mid X)$ denotes the number of occurrences of $x$ in $X$.

Definition 2 We call the sequence of tests logarithmically asymptotically optimal (LAO) if for given positive values of 4 diagonal elements of the matrix $E$ the procedure provides maximal values for other elements of it.

We also define the reigns of tests that is:

$\mathcal{R}_l = \{Q : D(Q \parallel P_l) < E_{l/l}\}$, $l = 1, 4$

$\mathcal{R}_l = \{Q : D(Q \parallel P_l) > E_{l/l}\}$, $l = 1, 4$.

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Now in Theorem (1) we reformulate the result of [1] for the case \( M = 5 \). For given positive elements \( E_{1/1}, E_{2/2}, E_{3/3} \) and \( E_{4/4} \) let us denote :

\[
E_{1/1}^*, \ E_{2/2}^*, \ E_{3/3}^* = E_{3/3}, \ E_{4/4}^* = E_{4/4}^*.
\] (5.a)

\[
E_{m/l}^* = \inf_{Q : D(Q \| P_l) \leq l} D(Q \| P_m), \quad m = 1, 5, \quad l = 1, 4, \quad m \neq l.
\] (5.b)

\[
E_{5/5}^* = \min_{l = 1, 4} E_{5/l}^*.
\] (5.c)

**Theorem 1** Let different distributions \( P_l, \quad l = 1, 5, \quad D(P_l \| P_m) < \infty, \quad l \neq \infty \) and positive numbers \( E_{1/1}, E_{2/2}, E_{3/3} \) and \( E_{4/4} \) are given and the following inequalities take place:

\[
E_{1/1} \leq \min_{l = 1, 4} D(P_l \| P_l),
\] (6.a)

\[
E_{2/2} \leq \min_{l = 1, 5} (E_{2/l}^*, \min_{l = 1, 5} D(P_l \| P_2)),
\] (6.b)

\[
E_{3/3} \leq \min_{l = 1, 3} (E_{3/l}^*, \min_{l = 1, 3} (D(P_l \| P_3), D(P_3 \| P_3))),
\] (6.c)

\[
E_{4/4} \leq \min_{l = 1, 3} (E_{4/l}^*, D(P_l \| P_4)),
\] (6.d)

Then:

a) There exists a LAO sequence of tests \( \varphi^* \), as elements of the reliability matrix \( E(\varphi^*) \) are defined in (5).

b) If one of the inequalities (6) is violated, then at least one element of the matrix \( E(\varphi^*) \) is equal to 0. [1]

**Remark 1** From the definition of \( E_{1/5}^*, E_{2/5}^*, E_{3/5}^*, \) and \( E_{4/5}^* \) in (5) it follows,

\[
E_{4/5}^* \geq E_{m/l}^*, \quad m = 1, 4.\]

Taking into account the latter and (4) we obtain that

\[
E_{1/1}^* = E_{1/5}^*, \ E_{2/2}^* = E_{2/5}^*, \ E_{3/3}^* = E_{3/5}^*, \ E_{4/4}^* = E_{4/5}^*.
\] (7)

**3 Results and Proofs**

According to the case of two objects and the sequence of tests \( \varphi^i \), let the reliability matrix \( E(\varphi^i) \), of the i-th object, \( i = 1, 2 \), be denoted:

\[
E(\varphi^i) = \begin{bmatrix}
E_{1/1}(\varphi^i) & E_{1/2}(\varphi^i) & \cdots & E_{1/5}(\varphi^i) \\
E_{2/1}(\varphi^i) & E_{2/2}(\varphi^i) & \cdots & E_{2/5}(\varphi^i) \\
\vdots & \vdots & \ddots & \vdots \\
E_{5/1}(\varphi^i) & E_{5/2}(\varphi^i) & \cdots & E_{5/5}(\varphi^i)
\end{bmatrix}.
\]
We begin by proving the following lemma.

**Lemma 1.** Let positive elements \( E_{1/1}(\varphi^1), E_{2/2}(\varphi^2), E_{3/3}(\varphi^3) \) and \( E_{4/4}(\varphi^4) \) satisfy the condition \( (6) \), then the following equalities hold true for the test \( \varphi = (\varphi^1, \varphi^2) \) for two objects:

\[
E_{m_1,m_2/l_1,l_2}(\varphi) = E_{m_1/l_1}(\varphi^1) + E_{m_2/l_2}(\varphi^2) \quad \text{if} \quad m_1 \neq l_1, \ m_2 \neq l_2, \quad (8.a)
\]
\[
E_{m_1,m_2/l_1,l_2}(\varphi) = E_{m_1/l_1}(\varphi^1) \quad \text{if} \quad m_{3-i} = l_{3-i}, \ m_i \neq l_i, \ i = 1, 2. \quad (8.b)
\]

**Proof.** According the independence of the objects it follows that:

\[
\alpha_{m_1,m_2/l_1,l_2}(\varphi_N) = \alpha_{m_1/l_1}(\varphi^1_N) \alpha_{m_2/l_2}(\varphi^2_N) \quad \text{if} \quad m_1 \neq l_1, \ m_2 \neq l_2, \quad (9.a)
\]
\[
\alpha_{m_1,m_2/l_1,l_2}(\varphi_N) = \alpha_{m_1/l_1}(\varphi^1_N) \left[ 1 - \alpha_{m_{3-i}/l_{3-i}}(\varphi^2_{N-i}) \right] \quad \text{if} \quad m_{3-i} = l_{3-i}, \ m_i \neq l_i. \quad (9.b)
\]

According to (2), (9.a), (9.b) and Theorem 1, we obtain (8).

For given \( E_{1,1/5}, E_{1,1/5,1}, E_{2,2/5,2}, E_{2,2/5,5}, E_{3,3/5,3}, E_{3,3/5,5}, E_{4,4/5,4} \) and \( E_{4,4/5,5} \).

Let us consider the following sets:

\[
R^1_{l} = \{ Q : D(Q || P_l) \leq E_{l,l/5,l} \}, \quad l = 1, 4, \quad R^1_{5} = \{ Q : D(Q || P_l) > E_{l,l/5,l} \}, \quad l = 1, 4, \quad (10.a)
\]
\[
R^2_{l} = \{ Q : D(Q || P_l) \leq E_{l,l/5,l} \}, \quad m = 1, 4, \quad R^2_{5} = \{ Q : D(Q || P_l) > E_{l,l/5,l} \}, \quad l = 1, 4. \quad (10.b)
\]

The optimal values of the reliabilities of the LAO test sequence will be the following:

\[
E_{1,1/5}^* = E_{1,1/5}, \ E_{1,1/5,1}^* = E_{1,1/5,1}, \ E_{2,2/5,2}^* = E_{2,2/2/5}, \ E_{2,2/2/5}^* = E_{2,2/2/5}, \ E_{3,3/5,3}^* = E_{3,3/5,3}, \ E_{3,3/5,5}^* = E_{3,3/5,5}, \ E_{4,4/5,4}^* = E_{4,4/5,4}, \ E_{4,4/5,5}^* = E_{4,4/5,5}, \quad (10.a)
\]
\[
E_{m_{1,m_2/l_1,l_2}}^* = \inf_{Q \in R^1_{l}} D(Q || P_{m_i}), \ m_i \neq l_i, \ m_{3-i} = l_{3-i}, \ i = 1, 2, \quad (10.b)
\]
\[
E_{m_{1,m_2/l_1,l_2}}^* = E_{m_{1,m_2/l_1,l_2}}^* + E_{m_1,m_2/l_1,m_2}^* \quad \text{if} \quad m_i \neq l_i, \ i = 1, 2, \quad (10.c)
\]
\[
E_{m_{1,m_2}/m_1,m_2}^* = \min \ E_{m_{1,m_2}/l_1,l_2}^* \quad (10.d)
\]

According to the lemma and theorem (1) we can write the main result of the present paper is:
Theorem 2
Let all distributions $P_l$, $l = 1,5$ be different, and absolutely continuous each relative to others $0 < D(P_l || P_m) < \infty, l \neq m$. If positive elements $E_{1,1/5}$, $E_{1,1/5}, E_{2,2/5}, E_{2,2/5}$, $E_{3,3/5}, E_{3,3/5}, E_{4,4/5}$ and $E_{4,4/5}$ are given and the following inequalities hold: A

$$E_{1,1/5} < \min_{l=1,5} \left[ D \left( P_l \| P_1 \right) \right], \quad \text{(11.a)}$$

$$E_{1,1/5} < \min_{l=1,5} \left[ D \left( P_l \| P_1 \right) \right], \quad \text{(11.b)}$$

$$E_{2,2/5} < \min \left[ E_{2,2/5} \min_{l=1,5} \left( D \left( P_l \| P_2 \right) \right) \right], \quad \text{(11.c)}$$

$$E_{2,2/5} < \min \left[ E_{2,2/5} \min_{l=1,5} \left( D \left( P_l \| P_2 \right) \right) \right], \quad \text{(11.d)}$$

$$E_{3,3/5} < \min \left[ \min(E_{3,3/5}, E_{3,3/5}), \min(D \left( P_3 \| P_3 \right) , D(P_3 \| P_3) \right), \quad \text{(11.e)}$$

$$E_{3,3/5} < \min \left[ \min(E_{3,3/5}, E_{3,3/5}), \min(D \left( P_3 \| P_3 \right) , D(P_3 \| P_3) \right), \quad \text{(11.f)}$$

$$E_{4,4/5} < \min \left[ \min \left( E_{4,4/5}, D(P_5 \| P_5) \right) \right], \quad \text{(11.g)}$$

$$E_{4,4/5} < \min \left[ \min \left( E_{4,4/5}, D(P_5 \| P_5) \right) \right], \quad \text{(11.h)}$$

Then:
a) There exists a LAO test sequence $\phi^*$, the reliability matrix of which $E(\phi^*) = \{E_{m,nl/l',l'2}(\phi^*)\}$ is defined in (10).
b) If one of the inequalities (11) is violated, then there exists at least one element equal to 0 in the matrix $E(\phi^*)$.

**Proof.** Conditions (11) imply that inequalities (6) hold simultaneously for the both objects. Using (6), we can write inequalities (6) for both objects as follows:

$$E_{1/5}^1 < \min_{l=1,5} \left[ D \left( P_l \| P_1 \right) \right], \quad \text{(12.a)}$$

$$E_{1/5}^2 < \min_{l=1,5} \left[ D \left( P_l \| P_1 \right) \right], \quad \text{(12.b)}$$

$$E_{2/5}^2 < \min \left[ E_{2,2/5} \min_{l=1,5} \left( D \left( P_l \| P_2 \right) \right) \right], \quad \text{(12.c)}$$

$$E_{2/5}^1 < \min \left[ E_{2,2/5} \min_{l=1,5} \left( D \left( P_l \| P_2 \right) \right) \right], \quad \text{(12.d)}$$

$$E_{3/5}^2 < \min \left[ \min(E_{3,3/5}, E_{3,3/5}), \min(D \left( P_3 \| P_3 \right) , D(P_3 \| P_3) \right), \quad \text{(12.e)}$$

$$E_{3/5}^1 < \min \left[ \min(E_{3,3/5}, E_{3,3/5}), \min(D \left( P_3 \| P_3 \right) , D(P_3 \| P_3) \right), \quad \text{(12.f)}$$

$$E_{4/5}^2 < \min \left[ \min \left( E_{4,4/5}, D(P_5 \| P_5) \right) \right], \quad \text{(12.g)}$$

$$E_{4/5}^1 < \min \left[ \min \left( E_{4,4/5}, D(P_5 \| P_5) \right) \right], \quad \text{(12.h)}$$
We shall prove, for example, (12.c); which is the consequence of the inequality (11.c). Let us consider a test \( \varphi = (\varphi^1, \varphi^2) \), such that \( E_{2,2/2,5}(\varphi) = E_{2,2/2,5} \) and \( E_{2,2/2,1}(\varphi) = E_{2,2/2,1}^* \). The corresponding error probabilities \( \alpha_{2,2/5}(\varphi_N) \) and \( \alpha_{2,2/2,1}(\varphi_N) \) are given as products defined by (9.b). According to (2) and (9) we obtain that

\[
E_{2,2/2,1}(\varphi) = E_{2/1}^{2,*} + \lim_{N \to \infty} - \frac{1}{N} \log(1 - \alpha_{2/2}(\varphi_N)),
\]

(13.a)

\[
E_{2,2/2,5}(\varphi) = E_{2/5}^2 + \lim_{N \to \infty} - \frac{1}{N} \log(1 - \alpha_{2/2}(\varphi_N)),
\]

(13.b)

where \( E_{2/1}^2 = E_{2/1}(\varphi^2) \) and \( E_{2/5}^2 = E_{2/5}(\varphi^2) \).

When \( \min_{i=3,5} \left( E_{2,2/i,1}(\min(D(P_i^1P_2^2)) = E_{2,2/2,1} \right) \), from (13) and (11.c) we conclude that \( E_{2/1}^2 < E_{2/5}^{2,*} \). We shall show that also \( E_{2/1}^{2,*} < \min_{i=3,5}(D(P_i^1P_2^2) \). That again follows from (13) and \( E_{2,2/2,1}^* < \min_{i=3,5}(D(P_i^1P_2^2) \). Hence in this case the inequality (12.c) holds.

Let us assume that \( \min_{i=3,5} \left( E_{2,2/i,1}(\min(D(P_i^1P_2^2)) = \min_{i=3,5}(D(P_i^1P_2^2) \). First we shall prove that when the reliability \( E_{2,2/2,5} \) satisfies the condition (11.c), the \( E_{2/5}^2 < \min_{i=3,5}(D(P_i^1P_2^2) \).

Assume that the opposite statement is true, i.e., \( E_{2/1}^2 \geq \min_{i=3,5}(D(P_i^1P_2^2) \). In this case using (13.b) and (11.c) we can derive

\[
\min_{i=3,5}(D(P_i^1P_2^2) + \lim_{N \to \infty} - \frac{1}{N} \log(1 - \alpha_{2/2}(\varphi_N)) \leq E_{2/5}^2 + \lim_{N \to \infty} - \frac{1}{N} \log(1 - \alpha_{2/2}(\varphi_N)) < \min_{i=3,5}(D(P_i^1P_2^2)
\]

\[
\lim_{N \to \infty} - \frac{1}{N} \log(1 - \alpha_{2/2}(\varphi_N)) < 0.
\]

The last inequality states an index \( N_0 \) exists, as for subsequence of \( N > N_0 \). We will have \( 1 - \alpha_{2/2}(\varphi_N^i) > 1 \), i.e the assumption was wrong.

Now we will prove that also \( E_{2/5}^2 < E_{2/1}^{2,*} \). From (11.c) \( E_{2,2/2,5} < E_{2,2/2,4} \) comes. Using (13) we can derive \( E_{2/5}^2 < E_{2/1}^{2,*} \). Hence in this case (12.c) also takes place.

It follows from (7) and (12) that conditions (6) of Theorem 1 hold for both objects. According to Theorem 1 there exist LAO sequences of tests \( \varphi^{1,*} \) and \( \varphi^{2,*} \), for the first and the second objects, such the elements of the matrices \( E(\varphi^{1,*}) \) and \( E(\varphi^{2,*}) \) are determined in (5). We will take \( \varphi^* = (\varphi^{1,*}, \varphi^{2,*}) \) as a test for the model and show that it is LAO and that other elements of the matrix \( E(\varphi^*) \) are determined in (10).

It follows from (12) and (7) that the requirements of Lemma are accomplished. Applying Lemma we can deduce that the reliability matrix \( E(\varphi^*) \) can be obtained from matrices \( E(\varphi^{1,*}) \) and \( E(\varphi^{2,*}) \) as in (8).

\[
E_{1,1/5,1} = E_{1/5}^1, E_{1,1/1,5} = E_{1/5}^2, E_{2,2/2,5} = E_{2/5}^2, E_{2,2/2,2} = E_{2/2}^2, E_{3,3/3,5} = E_{3/3}^3, E_{3,3/3,3} = E_{3/3}^1
\]

(14)
When (11) takes place according to (8.b), (5), (7) and (14) we obtain, that the elements $E_{m_i,m_j,i,j} \left( \varphi' \right)$, $m_i \neq l_i$, $m_{3-i} = l_{3-i}$, $i=1,2$, of the matrix $E(\varphi^*)$ are determined by relations (10.b). From (8.a) and (10.b) we obtain (10.c). The equality in (10.d) is the particular case of (3). When one of the inequalities (11) is violated, then from (9) and (10.b) we see, that some elements in the matrix $E(\varphi^*)$ must be equal to 0.

Now let us show that the compound test for two objects is LAO, i.e. It is optimal. Suppose that for given $E_{1,1/1.5}$, $E_{1,1/5.1}$, $E_{2,2/5.2}$, $E_{2,2/5.5}$, $E_{3,3/3.5}$, $E_{3,3/5.3}$, $E_{4,4/5.4}$ and $E_{4,4/4.5}$ there exists a test $\varphi' = \left( \varphi'_1, \varphi'_2 \right)$ with matrix $E(\varphi')$, such as it has at least one element exceeding the respective element of the matrix $E(\varphi^*)$. It is contrary to fact, that LAO tests $\varphi'^*_{1}$ and $\varphi'^*_{2}$ have been used for the objects $X_1$ and $X_2$.

**Remark 2**

The similar result may be received if we take alternatively: $E_{2,1/2.5}$ or $E_{3,1/3.5}$ or $E_{4,1/4.5}$ or $E_{5,1/5.5}$ instead of $E_{1,1/1.4}$ $E_{1,2/5.2}$ or $E_{1,3/5.3}$ or $E_{2,4/5.4}$ or $E_{5,1/5.5}$ instead of $E_{1,1/1.4}$ and etc.

4 Conclusion

According the result of this paper (Theorem 1, Theorem 2 and Remark 1), this is not difficult that, determination of logarithmically asymptotically optimal hypotheses testing for the general case of the model of two independent objects and $M > 2$, hypotheses testing.

References