Application of Homotopy Perturbation Method to Poisson Equation

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Abstract

This paper applies the homotopy perturbation method (HPM) to find the exact solution of Poisson equation with the Dirichlet and Neumann boundary conditions. To illustrate the simplicity and reliability of the method, two examples are provided. The results obtained reveal that the method is capable and easy to apply.

Keywords: Poisson Equation, Homotopy Perturbation Method.

1 Introduction

The homotopy perturbation method is a powerful and efficient technique to find the solutions of linear and nonlinear equations. The method was first introduced by He [1]. HPM is a combination of the perturbation and homotopy methods. This method can take the advantages of the conventional perturbation method while eliminating its restrictions. In general, this method has been successfully applied to solve many types of linear and nonlinear equations in science and engineering by many authors [2-9]. The aim of this paper is to employ HPM to obtain the exact solution of two Poisson equations, one with the Dirichlet boundary conditions and one with the Neumann boundary conditions.

The two dimensional Poisson equation has the following form [10]:

\[
\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = f(x, y),
\]

where \( f(x, y) \) is a known function.
2 Homotopy Perturbation Method (HPM)

To illustrate the basic idea of this method, we consider the following general nonlinear differential equation:

\[ A(u) - f(r) = 0, \quad r \in \Omega, \] (2)

with the following boundary conditions:

\[ B(u, \partial u / \partial n) = 0, \quad r \in \Gamma, \] (3)

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytical function and \( \Gamma \) is the boundary of the domain \( \Omega \). The operator \( A \) can be decomposed into a linear part and a nonlinear one, designated as \( L \) and \( N \) respectively. Hence Eq. (2) can be written as the following form:

\[ L(u) + N(u) - f(r) = 0. \]

Using homotopy technique, we construct a homotopy \( \nu(r, p) : \Omega \times [0,1] \rightarrow \mathbb{R} \) which satisfies:

\[ H(\nu, p) = (1 - p) \left[ L(\nu) - L(u_0) \right] + p \left[ A(\nu) - f(r) \right] = 0, \] (4)

where \( p \in [0,1] \) is an embedding parameter and \( u_0 \) is an initial approximation of Eq. (2) which satisfies the boundary conditions. Obviously, from Eq. (4) we have:

\[ H(\nu,0) = L(\nu) - L(u_0) = 0; \]
\[ H(\nu,1) = A(\nu) - f(r) = 0. \]

By changing the value of \( p \) from zero to unity, \( \nu(r, p) \) changes from \( u_0(r) \) to \( u(r) \), in topology this is called deformation and \( L(\nu) - L(u_0) \) and \( A(\nu) - f(r) \) are called homotopic. Due to the fact that \( p \in [0,1] \) can be considered as a small parameter, hence we consider the solution of Eq. (4) as a power series in \( p \) as the following:

\[ \nu = \nu_0 + p \nu_1 + p^2 \nu_2 + \cdots, \] (5)

setting \( p = 1 \) results in the approximate solution for Eq.(2)

\[ u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots. \]
3 Application

To illustrate the capability and reliability of the method, two examples are presented here.

Example 1. Consider the two dimensional Poisson equation in the form:

\[ u_{xx} + u_{yy} = xy, \quad 0 < x, y < \pi, \]  \hspace{1cm} (6)

subject to the following boundary conditions:

\[
\begin{aligned}
&u(0, y) = 0, \quad u(\pi, y) = \frac{1}{6} \pi y^3, \\
&u(x, 0) = 0, \quad u(x, \pi) = \frac{1}{6} x \pi^3 + \sin(x) \sin(\pi).
\end{aligned}
\]

HPM: Using HPM, we construct a homotopy in the following form:

\[ H(v, p) = (1 - p) \left[ \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} \right] + p \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - xy \right] = 0. \]  \hspace{1cm} (7)

We select \( u_0(x, y) = \frac{1}{6} xy^3 + y \sin(x) \) as an initial approximation that satisfies the three boundary conditions. Substituting Eq. (5) into Eq. (7) and equating the terms with identical powers of \( p \), we drive:

\[
\begin{aligned}
p^0: & \quad \frac{\partial^2 v_0}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = 0, \quad v_0(0, y) = 0, \\
& \quad v_0(\pi, y) = \frac{1}{6} \pi y^3, \quad v_0(x, 0) = 0, \\
& \quad v_0(x, \pi) = \frac{1}{6} x \pi^3 + \sin(x) \sin(\pi).
\end{aligned}
\]

\[
\begin{aligned}
p^1: & \quad \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x^2} - xy = 0, \quad v_1(0, y) = 0, \\
& \quad v_1(\pi, y) = 0, \quad v_1(x, 0) = 0, \\
& \quad v_1(x, \pi) = \frac{1}{6} x \pi^3 + \sin(x) \sin(\pi).
\end{aligned}
\]

\[
\begin{aligned}
p^2: & \quad \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_1}{\partial x^2} = 0, \quad v_2(0, y) = 0, \\
& \quad v_2(\pi, y) = 0, \quad v_2(x, 0) = 0, \\
& \quad v_2(x, \pi) = \frac{1}{6} x \pi^3 + \sin(x) \sin(\pi).
\end{aligned}
\]

We set \( v_0(x, y) = u_0(x, y) = \frac{1}{6} xy^3 + y \sin(x) \). From (8) and (9), we have:
\[ v_1(x, y) = \frac{1}{6} y^3 \sin(x), \]

\[ v_2(x, y) = \frac{1}{120} y^5 \sin(x), \]

Therefore, the solution of Eq. (6) when \( p \to 1 \) will be as:

\[ u(x, y) = \frac{1}{6} xy^3 + \sin(x) \left( y + \frac{1}{6} y^3 + \frac{1}{120} y^5 + \cdots \right), \]

or

\[ u(x, y) = \frac{1}{6} xy^3 + \sin(x) \sinh(y). \]

**Example 2** Consider the two dimensional Poisson equation in the form below:

\[ u_{xx} + u_{yy} = xy, \quad 0 < x, y < \pi, \quad (10) \]

subject to the following Neumann boundary conditions:

\[
\begin{align*}
&\begin{cases}
  u_x(0, y) = \frac{1}{6} y^3, & u_x(\pi, y) = \frac{1}{6} y^3,
  
  u_y(x, 0) = \cos(x), & u_y(x, \pi) = \frac{1}{2} x \pi^2 + \cos(x) \cosh(\pi).
\end{cases}
\end{align*}
\]

**HPM:** Using HPM, a homotopy can be constructed as follows:

\[ H(v, p) = (1 - p) \left[ \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} \right] + p \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - xy \right] = 0. \quad (11) \]

We select \( u_0(x, y) = \frac{1}{6} xy^3 + y \cos(x) \) as an initial approximation for the solution that satisfies the three boundary conditions. Substituting Eq. (5) into Eq. (11) and equating the terms with identical powers of \( p \), leads to:

\[
\begin{align*}
&\begin{cases}
  p^0: \quad \frac{\partial^2 v_0}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = 0, & v_{0x}(0, y) = \cos(x), \\
  & v_{0y}(0, y) = \frac{1}{6} y^3, & v_{0x}(\pi, y) = \frac{1}{6} y^3.
\end{cases}
\end{align*}
\]
\[ p^1 : \begin{aligned}
\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x^2} - xy &= 0, \\
v_{1x}(0, y) &= 0, \\
v_{1x}(\pi, y) &= 0,
\end{aligned} \tag{12}
\]

\[ p^2 : \begin{aligned}
\frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_1}{\partial x^2} &= 0, \\
v_{2y}(x, 0) &= 0, \\
v_{2x}(\pi, y) &= 0,
\end{aligned} \tag{13}
\]

We always set \( v_0(x, y) = u_0(x, y) = \frac{1}{6} xy^3 + y \cos(x) \). From (12) and (13), we will have:

\[ v_1(x, y) = \frac{1}{6} y^3 \cos(x), \]

\[ v_2(x, y) = \frac{1}{120} y^5 \cos(x), \]

\[ \vdots \]

Therefore, the solution of Eq. (10) when \( p \to 1 \) will be as follows:

\[ u(x, y) = \frac{1}{6} xy^3 + \cos(x) \left( y + \frac{1}{6} y^3 + \frac{1}{120} y^5 + \cdots \right). \]

Because the boundary conditions are the Neumann boundary conditions, an arbitrary constant must be added. Therefore, the exact solution in the closed form will be as follows:

\[ u(x, y) = \frac{1}{6} xy^3 + \cos(x) \sinh(y) + C. \]

4 Conclusion

In this paper, the homotopy perturbation method has been successfully applied to obtain the exact solution of Poisson equation with the Dirichlet and Neumann boundary conditions. It is clearly seen that the used method is straightforward, powerful and efficient. The computations associated with examples provided here were performed using Maple 10.
References