A note On Topological Entropy and Transitive Tree Maps

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Abstract

In this note we investigate totally transitive maps on trees. We use some mild conditions and improve the lower bound of the topological entropy of transitive tree maps. Also we show that the totally transitive maps coincide with the mixing maps on trees.

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1 Introduction

The dynamics of continuous maps on interval, circle and graph have been studied by several authors (see for example [10], [3], [4]). Investigation the lower bound for the topological entropy of transitive maps is an interesting problem in dynamical systems. In [7] Blokh has shown that the entropy of interval transitive maps is greater than or equal to \( \frac{1}{2} \log 2 \). Also Alseda, Baldwin, Llibre and Misiurewicz [1] have shown that the entropy of tree maps is greater than or equal to \( \frac{1}{n} \log 2 \) (where \( n \) is the number of the end points of the tree). We show that if \( x \) is a fixed point of a transitive tree map \( f: T \to T \) and if \( T \) is an \( n \)-nonsymmetric tree, then \( h(f) \geq \frac{1}{\text{End}(T) - 1} \log 2 \).

A continuous map \( f \) on a metric space \( X \) is called transitive if for every nonempty open subsets \( U \) and \( V \) of \( X \), there exists a natural number \( n \) such that \( f^n(U) \cap V \neq \emptyset \). Also \( f \) is called totally transitive if \( f^n \) is transitive for all natural numbers \( n \). The map \( f \) is called topologically mixing or simply mixing if for every nonempty open subsets \( U \) and \( V \) of \( X \), there exists a natural number \( N \) such that for every \( h \geq N \), \( f^h(U) \cap V \neq \emptyset \). An element \( x \) of \( X \) is called a fixed point if \( f(x) = x \). The set of all fixed points of \( f \) is denoted by \( \text{Fix}(f) \). A subset \( A \) of \( X \) is called \( f \)-invariant if \( f(A) \subseteq A \). By an interval in \( X \) we mean any subspace of \( X \) homeomorphic with \([0,1]\). A connected space \( T \) which

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is a finite union of intervals and contains no subset homeomorphic with the unit circle is called a tree.

A finite union of disjoint trees is called a forest. A union of a forest and a finite set will be called a generalized forest. If $T$ is a tree and $x \in T$, the number of connected components of $T \setminus \{x\}$ is called the valence of $x$ and is denoted by $\text{Val}(x)$. An element of valence one is called an end point and an element of valence greater than 2 is called a branching point of $T$. The set (the number resp.) of all end points of $T$ is denoted by $E(T)$ (End$(T)$) resp.) and the set (the number resp.) of all branch points of $T$ is denoted by $b(T)$ (B$(T)$) resp.). An element $x$ of $X$ is called an interior point if it is not an end point.

A continuous map $f : T \to T$ on a tree $T$ is called a tree map. The topological entropy of a continuous map $f$ is denoted by $h(f)$ and it has been introduced in [8] (see also [11]). We shall use all definitions from [1] and [2].

2 Dynamic of The Transitive Tree Maps

Definition Let $T$ be a tree and let $x$ be a point of $T$. We say $T$ is $x$-symmetric provided that $x$ be an interior point of $T$ and the connected components of $T \setminus \{x\}$ have the same number of the end points, otherwise $T$ is $x$-nonsymmetric. Also the tree $T$ is called nonsymmetric if it is $x$-nonsymmetric for every $x \in T$.

Let $f : T \to T$ be a tree map and let $z$ be a fixed point of $T$. We say $z$ is a repelling point if there exists a neighborhood $U$ of $z$ such that $x \in (z, f(x))$ for every $x \in U \setminus \{z\}$.

We say a subtree (resp. subforest, generalized subforest) of a tree (resp. forest, generalized forest) $T$ is proper if it contains an interval and is not equal to $T$. The pair $(T, E)$ is called a marked tree if $T$ is a tree and $E$ is a subset of end points of $T$. Also $f : T \to T$ is called a marked map of $(T, E)$, if every element of $E$ is a repelling point and there is no $f$-invariant proper generalized subforest of $T$ disjoint from $E$.

Clearly every transitive map is a $(T, E)$ marked map.

Let $f : T \to T$ be a tree map and let $P$ be an $f$-invariant finite set containing all branching points of $T$. We say that $f$ is $P$-monotone if for each connected component $K$ of $T \setminus P$ the map $f|_K$ is a homeomorphism onto its image.

The following graph (see [1]) is an example of a marked map $f : T \to T$ of the marked tree $(T, E)$ where $E = \{a, v\}$ and $f$ is a $P$-monotone map where $P = \{a, b, c, d, e, u, v\}$. 

![Diagram of a marked tree](image-url)
We state the following definition from [1].

Let $\{T_e, E\}$ be a marked tree and let $z$ be an interior point of $T$. The family $\{\{(T_e, E_i) : i = 1, \ldots, s - 1\}$ of marked trees is called a $z$-family if the following conditions are satisfied:

(i) Each $T_i$ is contained in the closure $S_i$ of some component of $T \backslash \{z\}$.
(ii) If $i \neq j$ then $S_i \neq S_j$.
(iii) Either $T_i = S_i \backslash \{z_i, z_i\}$ for some point $z_i$ in the interior point of the edge of $S_i$ containing $z$, or $T_i = S_i$ (and then we set $z_i = z$).
(iv) $E_i = (T_i \cap E) \cup \{z_i\}$

If $S$ is a subtree of $T$ then we denote by $r_S$ the natural retraction of $T$ into $S$, that is, the map such that $r_S(x)$ is the point of $S$ closest to $x$ (so, in particular, $r_S(x) = x$, if $x \in S$). Here “the closest” means the closest along the tree.

**Theorem A** Let $f: T \to T$ be a tree map and let $x$ be a point of $T$. If $f$ is totally transitive, then $\bigcup_{j=1}^{\infty} f^{-j}(x)$ is dense in $T$.

**Proof.** Suppose $\bigcup_{j=1}^{\infty} f^{-j}(x)$ is not dense in $T$. Let $U$ be a connected component of $T - \bigcup_{j=1}^{\infty} f^{-j}(x)$ with nonempty interior. Clearly $\bigcup_{j=1}^{\infty} f^{-j}(U)$ is an $f$-invariant set. Since $f$ is transitive, so $\bigcup_{j=1}^{\infty} f^{-j}(U)$ is dense in $T$.

For every subset $L$ of $T$, if $L$ and $\bigcup_{j=1}^{\infty} f^{-j}(x)$ are disjoint, then clearly $f(L) \cap (\bigcup_{j=1}^{\infty} f^{-j}(x)) = \emptyset$. So

$$\left(\bigcup_{j=1}^{\infty} f^{-j}(U)\right) \cap \left(\bigcup_{j=1}^{\infty} f^{-j}(x)\right) = \emptyset$$

(1)

On the other hand there exists a natural number $n$ such that:

$$f^n(U) \cap U = \emptyset$$

(2)

By definition of $U$ and relations (1) and (2) we have $f^n(U) \subseteq U$. So $U$ is $f^n$-invariant (Int($U) = \emptyset$). Hence $f^n$ is not transitive, which is a contradiction.

**Corollary 2.1** Let $T$ be a tree with $n$ end points and let $f: T \to T$ be totally transitive map. Then there are at most $n$ points in $T$ such that the set of their inverse images is not dense in $T$ and these points can only be the end points.

**Proof.** It can be conclude easily from Theorem A.

**Lemma 2.2** Let $f: T \to T$ be a transitive tree map and let $x \in T$. If $f^{-1}(x) = \{x\}$, then $h(f) \geq \frac{1}{\text{End}(T)} \log 3$.

**Proof.** See Proposition 4.2 of [1].

**Lemma 2.3** Let $I$ be a closed interval $[0,1]$ and let $f: I \to I$ be a transitive map. If $f$ has at least 2 fixed points, then $h(f) \geq \log 2$. 


Proof. See Proposition 4.3.9 of [10].

The following proposition gives a basis of marked maps which is proved in [1].

**Proposition 2.4** Let \( f \) be a marked map of a marked tree \((T, E)\). Let \( z \) be a fixed point of \( f \). Assume that \( z \) is an interior point of \( T \). Then there exists a \( z \)-family \( \{(T_i, E_i^i) : i = 1, \ldots, z - 1\} \) of marked trees such that for each \( i \) the map \( (f^i)_{T_i} = r_{T_i} \circ f^i \) is a marked map of \((T_i, E_i^i)\).

Let \( z \) be an interior point of the tree \( T \). A sequence \( \{x_{z1}, \ldots, x_{zn}\} \) is called \( z \)-independent if there are no \( i, j \in \{1, \ldots, n\} \) such that \( i \neq j \) and \( x_i \in [z, x_j] \). Also when \( f \) is a tree map, the sequence \( \{x_{z0}, x_{z1}, \ldots, x_{zn}\} \) of elements of \( T \) is said an inverse sequence if for every \( i \) \((1 \leq i \leq n - 1)\), \( f(x_i) = x_{i+1} \).

A collection \( \{I_{z1}, \ldots, I_{zk}\} \) of pairwise disjoint (except perhaps end points) subintervals of \( T \) is called a \( k \)-horseshoe if \( I_{zj} \subset f(I_{zj}) \) for every \( i \in \{1, \ldots, k\} \). It is well known (see for example [10]) that if \( f \) has a \( k \)-horseshoe then \( h(f) \geq k \log 2 \).

**Lemma 2.5** Let \( T \) be a tree and let \( z \) be an interior point of \( T \). Then the length of each \( z \)-independent sequence is less than or equal to \( \text{End}(T) \). Also if \( z \) is an end point, then the length of any \( z \)-independent sequence is less than or equal to \( \text{End}(T) - 1 \).

Proof. Let \( E(T) = (z_1, \ldots, z_n) \) with \( \text{End}(T) = n \). Then \( T = \cup_{i=1}^{n} [z_i, z_{i+1}] \) and the lemma follows.

**Lemma 2.6** (Lemma 2.2 of [12]). Let \( f : T \to T \) be a tree map and let \( z \) be a fixed point. If there exists an inverse sequence \( \{x_{z0}, x_{z1}, \ldots, x_{zn}\} \) with \( x_{z1} \neq x_0 \) and \( x_{zn} \in (x_0, x_z) \) for some \( 0 \leq i \leq n \), then \( f^k \) has a horseshoe for some \( k \leq n \).

The following Proposition is the Theorem 2.4 of [12]. Since we need some parts of the proof in Remark 2.8, so the proof is included.

**Proposition 2.7** Let \( f \) be a transitive map of a tree \( T \). Then there is some \( 0 \leq k \leq \text{End}(T) \) such that \( f^k \) has a horseshoe, and consequently \( k(f) \geq \frac{1}{\text{End}(T) - 1} \log 2 \).

Furthermore, if some end point of \( T \) is fixed by \( f \). Then there is some \( 0 \leq k \leq \text{End}(T) - 1 \) such that \( f^k \) has a horseshoe, and consequently \( k(f) \geq \frac{1}{\text{End}(T) - 2} \log 2 \).

Proof. Let \( x_0 \) be a fixed point of \( f \). If \( f^{-1}(x_0) = \{x_0\} \), then the Proposition follows from Lemma 2.2, and we are done. Assume \( f^{-1}(x_0) \neq \{x_0\} \) so there exists \( x_1 \neq x_0 \) such that \( f(x_1) = x_0 \). First assume \( x_u \in \text{int}(T) \). If for every \( 1 \leq k \leq \text{End}(T) \), \( f^k \) has no horseshoe,

we set \( A_1 = \{x_1\} \) and for every \( 2 \leq i \leq \text{End}(T) \), define:

\[ A_i = \{ y \in T : f^{i-1}(y) = x_0, (f^{i-1}(y), \ldots, f(y), y) \text{ is an } x_0 \text{ - independent sequence} \} \]
Let \( j_0 = \max \{ k : A_i = \emptyset \} \) and for each \( 1 \leq j \leq j_0 \), let \( T_j \) be the closure of the connected component of \( T \setminus \bigcup_{i=1}^{j} A_i \) containing \( x_0 \) (so \( T_1 = T \setminus \{ x_1 \} \)). Note that:

1. For every \( 1 \leq j \leq j_0 \), \( x_0 \in cl(\bigcup_{i=1}^{j} A_i) \).
2. If we set \( T = T_0 \), then \( T_j \) is a subtree of \( T_{j-1} \) and \( E(T_j) \subset E(T_{j-1}) \cup cl(\bigcup_{i=1}^{j} A_i) \) for each \( 1 \leq j \leq j_0 \).

To see (2) let \( x \in \text{Int}(T_j) \). Then \( [x_0, x] \subset T_j \). This implies that \( [x_0, x] \cap \bigcup_{i=1}^{j} A_i = \emptyset \). Hence \( [x_0, x] \cap \bigcup_{i=1}^{j-1} A_i = \emptyset \). So \( x \in \text{Int}(T_{j-1}) \). Thus (2) holds.

(3) \( \bigcup_{i=2}^{j} A_i \subset T_1 \) and hence \( j_0 \leq \text{End}(T_1) \) (see Lemma 2.5).

We claim that for every \( 1 \leq j \leq j_0 \), \( cl(\bigcup_{i=1}^{j} A_i) \cap T_j = E(T_j) \cap \bigcup_{i=1}^{j} A_i \).

For \( j = 1 \) it is obvious. Suppose the claim is true for \( 1, \ldots, j \) with \( j \leq j_0 - 1 \). We show that \( cl(\bigcup_{i=1}^{j+1} A_i) \cap T_{j+1} = E(T_{j+1}) \cap \bigcup_{i=1}^{j+1} A_i \).

We first show that \( f(T_{j+1}) \subset T_j \). By induction we have \( E(T_j) = E(T) \cap \bigcup_{i=1}^{j} A_i \).

Assume there exists \( y \in T_{j+1} \) such that \( \{ y \} \not\subset T_j \). Hence there is \( z \in T_j \) (hence \( z \in E(T_j) \)) such that \( f(z) = z \). As \( z \in E(T_j) \), \( f(z) \in E(T) \cap \bigcup_{i=1}^{j} A_i \). Assume that \( f(z) \in A_i \) with \( i \leq j \). Then \( (x_1, \ldots, f(z)) \) is \( x_0 \)-independent. By the construction of \( T_{j+1} \), \( f^k(z) \in E(T_{j+1}) \) for \( k = 1, \ldots, j+1 - 1 \). At the same time by Lemma 2.6 and (*) \( z \in (x_0, f^k(z)) \) for \( k = 1, \ldots, j+1 - 1 \). Thus \( (x_1, \ldots, f(z), x) \) is \( x_0 \)-independent, i.e. \( z \in \bigcup_{i=1}^{j+1} A_i \). Since \( f(T_{j+1}) \subset T_j \), this is a contradiction. This proves that \( f(T_{j+1}) \subset T_j \).

It is clear that \( T_{j+1} \cap cl(\bigcup_{i=1}^{j+1} A_i) = E(T_{j+1}) \cap \bigcup_{i=1}^{j+1} A_i \). So we only have to prove the other inclusion. Set \( n = \text{End}(T_1) \) and let \( x \in T_{j+1} \cap cl(\bigcup_{i=1}^{j+1} A_i) \). Since \( x \in cl(\bigcup_{i=1}^{j+1} A_i) \), so

\[
\{ f(x) \} \subset cl(\bigcup_{i=1}^{j+1} A_i) \subset cl(\bigcup_{i=1}^{j+1} A_i) \] for each \( 1 \leq i \leq j+1 - 1 \), where \( j \) is defined so that \( f^{j-1}(x) = x_2 \). Thus \( (x_1, \ldots, f(x), x) \) is \( x_0 \)-independent, i.e. \( x \in \bigcup_{i=1}^{j+1} A_i \). This completes the proof of the claim.

Now we show that \( n = j_0 \). Assume that \( n > j_0 \). If \( T = T_{j_0} \), then \( A_{i_0} \subset E(T_{i_0}) \) and there is \( x \in T \) with \( f(x) \in A_{i_0} \) by the surjectivity of \( f \), and if \( T \neq T_{i_0} \), then there is \( x \in \text{Int}(T_{i_0}) \) with \( f(x) \in E(T_{i_0}) \) (as \( T_{i_0} \) is not invariant under \( f \) and \( x_0 \in T_{i_0} \)). In either case there exists \( T_{i_0} \) such that \( f(x) \in E(T_{i_0}) \cap \bigcup_{i=1}^{j_0} A_i \). By (*) and Lemma 2.6, \( (x_1, \ldots, f(x), x) \) is \( x_0 \)-independent. Moreover, by the construction of
Hence \( f(x) \in A_n \) and \( x \in A_{n+1} \), which is a contradiction. Thus \( n = j_0 \).

Repeating the argument in the above paragraph with \( T_{j_0} \) replaced by \( T_n \) and noticing that \( A_n \subset A_{n+1} \) and by considering \( T = T_n \) or \( T = T_1 \) we may find \( x \in \text{Int}(T_n) \) such that \( f(x) \in E(T_n) \cap (\cup_{k=1}^n U_k \cup A_n) \). Again by the construction of \( T_n \) we have \( f(x) \in A_n \). By Lemma 2.5 we have \( x \in (x_0 f^k(x)) \), for some \( 1 \leq l \leq n \). According to Lemma 2.6 there is \( 1 \leq k \leq n \leq \text{End}(T) \) such that \( f^k \) has a horseshoe; which is a contradiction. Hence in the case when \( x_0 \in \text{Int}(T) \) there is \( 1 \leq k \leq n \) such that \( f^k \) has a horseshoe.

We now consider the case when \( x_0 \) is an end point of \( T \). Using the arguments analogous to the case \( x_0 \in \text{Int}(T) \) (replacing assumption (*) on page 6, by “there is no \( 1 \leq k \leq \text{End}(T) - 1 \) such that \( f^k \) has a horseshoe”) and noting that any \( x_0 \)-independent sequence has length less than or equal to \( n - 1 \), we conclude that \( f^k \) has a horseshoe for some \( 1 \leq k \leq n - 1 \). This completes the proof of the Proposition.

Remark 2.8
(i) If \( f: T \rightarrow T \) is a transitive tree map and \( x_0 \) is a fixed point such that \( f^{-1}(x_0) \neq \{x_0\} \)
then \( h(f) \geq \frac{1}{\text{End}(T)} \log 2 \) (where \( T_1 \) is the tree which was built in the proof of Proposition 2.7).
(ii) The transitivity of \( f \) in the proof of Proposition 2.7 under the above assumption was used in a weak setting. Namely, \( f \) is surjective and there is no \( f \)-invariant subset containing an interval.

Proposition 2.9
Let \( f: T \rightarrow T \) be a transitive map and let \( x \) be a point of \( T \), then:
(i) \( h(f) \geq \frac{1}{\text{End}(T)} \log 2 \)
(ii) If \( f^{-1}(x) = \{x\} \), then \( h(f) \geq \frac{1}{\text{End}(T)} \log 3 \).

Proof. Part (i) is the Theorem B of [1] and part (ii) is the Proposition 4.2 of [1].

Theorem B Let \( f: T \rightarrow T \) be a transitive map, then:
(i) If \( f \) has a fixed point \( x \) and if \( T \) is an \( x \)-nonsymmetric tree, then \( h(f) \geq \frac{1}{\text{End}(T)-1} \log 2 \)
(ii) If \( |f(x)| \neq 1 \), then \( h(f) \geq \frac{1}{\text{End}(T)-1} \log 2 \).

Proof. Part (i). Let \( x \) be an end point, then by Proposition 2.7 the result follows. Now assume \( x \in \text{Int}(T) \) and \( h(f) < \frac{1}{\text{End}(T)-1} \). Let \( V \) be the closure of connected components of \( T \setminus \{x\} \). By Proposition 2.4 there exists an \( x \)-family with \( s \leq m \) element \( T_1, \ldots, T_s \). Set \( i = \text{End}(A_i) - 1 \) \((1 \leq i \leq m) \) and \( l = \min(l_i, 1 \leq i \leq m) \). Obviously \( \sum_{i=1}^s l_i = \text{End}(T) \) and there exists a \( k \) \((1 \leq k \leq s) \) such that \( l_k = l \). By Remark 2.8 we have \( h(f^k) \geq \frac{1}{l} \log 2 \). Now since for every natural number \( n \) we have \( h(f^n) = n.h(f) \), so

\[
\frac{1}{\text{End}(T)-1} \log 2 \geq h(f) = \frac{1}{s} h(f^s) \geq \frac{1}{s} \log 2
\]