Heavy quarkonia with Cornell potential on noncommutative space

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Abstract

The effect of noncommutative space, to the second order in $\theta$, on heavy quarkonium spectra is studied within the framework of the nonrelativistic Schrödinger equation with the static Cornell potential. Due to this effect, a singular term $1/r^3$ appears in the potential, which is remedied using a regularization scheme by introducing a cut-off radius on the lower limit of the integrals needed for the perturbation computations. The results are applied to the 1S states of $c\bar{c}$, and it is found that the energy shifts are quadratic in the noncommutative parameter $\theta$. By comparing the results with those from experimental 1S-hyperfine splitting of $c\bar{c}$, we infer an upper limit on $\theta$.


Keywords: Noncommutative; Field theory; Quarkonia; Cornell potential; Heavy; QCD.

1 Introduction

Since their discoveries, investigation of heavy quarkonium systems ($c\bar{c}$, $b\bar{b}$, $c\bar{b}$, $b\bar{c}$) provides us with a great opportunity for quantitative tests of QCD. For detailed review of heavy quarkonium physics recent progress, see e.g. [1,2]. Because of the heavy masses of the constituent quarks ($m_q \gg \Lambda_{QCD}$, i.e. than about 200 MeV, where $\Lambda_{QCD}$ is the hadronization scale), a good description of many features of these systems can be obtained using nonrelativistic models, where one assumes that the motion of constituent quarks is nonrelativistic so that the quark-antiquark strong interaction is described by a static potential [1,2]. There are many potential models that are commonly used to study heavy quarkonium spectra; for instance, Martin, logarithmic, and Cornell potentials [1,2]. Any of these potential should take into account the two distinctive features of the strong interaction, namely, asymptotic freedom and confinement. In this paper, we will focus on Cornell potential, known from literature to phenomenologically describe the quark-antiquark strong interaction. The spin-averaged Cornell potential is defined as [2-5]:

$$V(r) = -\frac{4}{3} \frac{\alpha_s}{r} + kr,$$  \hfill (1)

where $r$ is the interquark distance, $\alpha_s$ is the quark-gluon coupling; it depends weakly on $r$, but this dependency will be neglected here, and $k$ is the confinement constant. The first part of this potential $-\frac{4}{3} \frac{\alpha_s}{r}$ corresponds to the potential induced by one-gluon exchange between the quark-antiquark system that dominated at short distances, while the second part $kr$ accounts for quark confinement at large distances. The $q\bar{q}$ wavefunction can be calculated for this system by applying the nonrelativistic Schrödinger equation, and then the parameters $\alpha_s$ and $k$ are determined by fitting the experimentally measured mass spectra of $c\bar{c}$ and $b\bar{b}$ systems. This technique works well after relativistic corrections, spin effects, and other effects on the mass spectra are taken into consideration [4,6].

In this work, we are interested in studying the effects of the non-commutative of space on heavy quarkonium spectra. Recently, physics on noncommutative space has been intensively studied; for a general review see [7,8] and references therein. For instance, energy levels of the hydrogen atom and as well the Lamb shift within the frame of noncommutative space have been investigated; for a classical and on the quantum levels. On both levels, the deviations depend on the non-commutative parameter. Also, the second-order non-commutative Stark effect was computed using the perturbation theory and Dalgarno-Lewis exact method in [10]. They found that the energy shift at the lowest order is quadratic in both the electric field and the non-commutative parameter. As a side result, they also obtain a sum rule for the mean oscillator strength, and suggest a way to infer an upper bound on the non-commutative parameter from the precision of Stark effect measurements.

This paper is organized as follows. In Section 2, we review the main formalism of non-commutative space and apply it on the Cornell potential. In Section 3, we study the effect of non-commutativity on the energy spectra of heavy quarkonia using Schrödinger perturbation theory, and we infer an upper limit on the

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noncommutative parameter from experimental hyperfine measurements. In the last section, summary and conclusions are presented.

2. Heavy quarkonia in non-commutative space

Let \( x_i, p_i \) be the position and momentum operators. Then, in standard quantum mechanics, these operators obey the familiar commutators:

\[
[x_i, x_j] = 0,
\]
\[
[x_i, p_j] = i\hbar \delta_{ij},
\]
\[
[p_i, x_j] = 0.
\]

However, at very short scales, the space coordinates may not commute but obey the following commutators [7-10]:

\[
\left[ \hat{x}_i, \hat{x}_j \right] = i\hbar \delta_{ij},
\]
\[
\left[ \hat{x}_i, \hat{p}_j \right] = i\hbar \delta_{ij},
\]
\[
\left[ \hat{p}_i, \hat{p}_j \right] = 0,
\]

where \( \theta_{ij} \) is the non-commutative parameter and has dimensions of \((\text{length})^2\). The hat symbol indicates that the coordinate and momentum operators are in the non-commutative space. Let \( H(x, p) \) be the Hamiltonian operator of the system in the ordinary quantum mechanics, then the static the nonrelativistic Schrödinger equation in NC space is usually written as:

\[
H(x, p) \ast \psi(\vec{r}) = -\frac{\hbar^2}{2\mu} \nabla^2 \psi(\vec{r}) + V(\vec{r}) \ast \psi(\vec{r}) = E \psi(\vec{r}),
\]

(4)

where the Moyal-Weyl or the star product \( \ast \) between two arbitrary functions \( f(x) \) and \( g(x) \) is defined as [9,11]:

\[
(f \ast g)(x) \equiv e^{i\theta_{\mu\nu} \partial_{\mu} f(y) \partial_{\nu} g(z)}|_{y=x}.
\]

(5)

This formula that appeared in the field theory has an older version known as Bopp’s shift, namely, we replace the star product in the Schrodinger equation by the usual product, using [11]:

\[
x_i = \hat{x}_i + \frac{1}{2\hbar} \theta_{ij} p_j,
\]
\[
p_i = \hat{p}_i,
\]

(6)

which is, for computational purposes, and is more convenient to use to recast in terms of the new coordinates \( x_i \) and \( p_i \), which satisfy the usual canonical commutation relations of Eq. (2). Therefore, the only modification in Schrodinger equation is to replace \( V(\vec{r}) \ast \psi(\vec{r}) \) with \( V\left(\vec{r} - \frac{1}{2\hbar} \theta_{ij} p_j\right) \ast \psi(\vec{r}) \). Thus, Schrodinger equation in NC space becomes:

\[
V(\vec{r}) = -\frac{4}{3} a_s \alpha_s + k \theta.
\]

(8)

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\[
H(\hat{x}, \hat{p}) \ast \psi(\vec{r}) = H\left(x_i + \frac{1}{2\hbar} \theta_{ij} p_i, \hat{p}_i\right) \ast \psi(\vec{r}) = E \psi(\vec{r}).
\]

(7)

We assume next that Cornell potential in non-commutative space has the same expression as in Eq. 1 but with \( x_i \) is replaced by \( \hat{x}_i \), i.e.,

\[
V(\vec{r}) = -\frac{4}{3} a_s \alpha_s + k \theta.
\]

Defining the vector \( \hat{\theta} \) whose components are written in terms of the NC parameters \( \theta_{ij} \), as \( \theta_1 = \epsilon_{ijk} \theta_{jk} \), and using the transformation relations (4), it is straightforward to show that:

\[
\hat{\theta} = \sqrt{\epsilon_{\alpha\beta}} \hat{\theta}_\alpha = \hat{\theta}_1 \left(1 - \frac{1}{4\hbar^2} \left(\frac{1}{\alpha s} \hat{L} \cdot \hat{\theta} - \frac{1}{\alpha s} \theta_{ij} \theta_{ik} \hat{p}_j \hat{p}_k + O(\theta^3)\right)\right)
\]

(9)

and

\[
\frac{1}{r} = \frac{1}{\sqrt{2}} \left(1 + \frac{1}{2\hbar^2} \left(\frac{1}{\alpha s} \hat{L} \cdot \hat{\theta} - \frac{1}{\alpha s} \theta_{ij} \theta_{ik} \hat{p}_j \hat{p}_k + O(\theta^3)\right)\right).
\]

(10)

where \( \hat{L} = \hat{r} \times \hat{p} \) is the orbital angular momentum. Therefore, Cornell potential in non-commutative space is:

\[
V = -\frac{4}{3} a_s \alpha_s + k r - \left(\frac{\alpha_s}{4\hbar^2} + \frac{k}{4\hbar^2}\right) \hat{L} \cdot \hat{\theta} + \left(\frac{\alpha_s}{6\hbar^2 r^3} + \frac{k}{32\hbar^2 r^3}\right) \theta_{ij} \theta_{ik} \hat{p}_j \hat{p}_k.
\]

(11)

The effect of non-commutativity is then:

\[
V^{\text{NC}} = -\left(\frac{\alpha_s}{4\hbar^2} + \frac{k}{4\hbar^2}\right) \hat{L} \cdot \hat{\theta} + \left(\frac{\alpha_s}{6\hbar^2 r^3} + \frac{k}{32\hbar^2 r^3}\right) \theta_{ij} \theta_{ik} \hat{p}_j \hat{p}_k.
\]

(12)

This result can be rewritten in a more accessible form by setting \( \theta = \theta_z \) by assuming that \( \theta_x = \theta_y = 0 \), which can be done by a rotation of coordinates, and thus \( \hat{L} \cdot \hat{\theta} = L_{\theta} \theta \), and by noting that \( \hat{\theta} \cdot j \hat{p} = -2\theta_i j \hat{p}_j \) and thus \( \theta_i j \theta_{ik} \hat{p}_j \hat{p}_k = \frac{1}{4} (p_x^2 + p_z^2) \theta^2 \). Then, we obtain:

\[
V^{\text{NC}} = -\left(\frac{\alpha_s}{3\hbar^2 r^2} + \frac{k}{4\hbar^2 r}\right) L \theta + \left(\frac{\alpha_s}{24\hbar^2 r^2} + \frac{k}{32\hbar^2 r^2}\right) (p_x^2 + p_z^2) \theta^2,
\]

(13)

which is to the second order in \( \theta \).

3. Application, results and discussion

Because of the smallness of the non-commutative parameter \( \theta \), the non-commutative of space effects on
heavy quarkonium spectra may be studied using the nonrelativistic Schrodinger perturbation theory. As an application, we will focus on the 1S states of $c\bar{c}$ for which $l = 0$. The first-order perturbation correction to the 1S energy levels due to non-commutative of space is:

$$E_{1}^{NC} = \langle \psi \vert [V^{NC}] \vert \psi \rangle$$

$$= \frac{\alpha_{e}^{2}}{24\hbar^{2}} \left( \frac{p_{x}^{2} + p_{y}^{2}}{r^{3}} \right) + \frac{k\theta^{2}}{32\hbar^{2}} \left( \frac{p_{x}^{2} + p_{y}^{2}}{r} \right),$$

(14)

where we have used the facts that $\langle p_{x}^{2} \rangle = \langle p_{y}^{2} \rangle = 1/3 \langle p_{x}^{2} \rangle$ due to symmetries. To find the exact expectation values, it is necessary to solve the nonrelativistic Schrodinger equation with linear plus Coulomb potential. Unfortunately these analytical solutions are not well known yet. However, for 1S-states, $l = 0$ and $m = 0$, and their wavefunctions are purely radial. In this case, the wavefunction can be constructed using variational method. According to [4,5], one may choose a trial wave function where there is only one variational parameter to study the 1S states of $c\bar{c}$. The general form of such a trial wave function is:

$$\Psi_{\text{trial}}(r) = N e^{-a r^{b}},$$

(15)

where

$$N = \left[ \frac{b(2a)^{3/2}}{4\pi \Gamma(3/2)} \right]^{1/2},$$

(16)

is the normalization constant, $a$ denotes the variational parameter, which is fixed by minimizing the expectation value of Hamiltonian and $b = 4/3$ is the model parameter, which determines the type of the trial wave function. These calculations were carried out in [4], and the value of $a$ was found to be $a = 0.522357$. The following numerical values of the various needed quantities were used: $\alpha_{e} = 0.39, k = \left( \frac{1}{2.34} \right)^{2}$ GeV$^{-2}, m_{c} = 1.84$ GeV.

Next, we need to calculate the expectation values $\langle p_{x}^{2} \rangle$ and $\langle p_{y}^{2} \rangle$, using the above trial wavefunction (15). Using $p^{2} = -\hbar^{2} \left( \frac{2}{dr} \frac{d}{dr} + \frac{a^{2}}{dr^{2}} \right)$, it is straightforward to compute the second integral to be $\langle p^{2} \rangle = 1.082111\hbar^{2}$. However, the first integral diverges because it has a singularity at $r = 0$ due to the $1/r^{3}$ term, and should be treated carefully. To compute this integral we use the regularization scheme, described in [4], to get rid of the singularity at $r = 0$, by introducing a "cut-off" radius $r_{0}$ on the lower limit of the integral. In [4], they adopted the value $r_{0} = 0.068$ GeV$^{-1}(\approx 0.014$ fm), after comparing their results with other methods, to regulate such a divergent integral associated with the $c\bar{c}$ bound state energy levels of Cornell potential. We also tested this cut-off with the second integral, and with the energy expectation value, and the deviations from the exact values were found to be very small. Using this cut-off, the integral $\langle p^{2} \rangle$ is then:

$$\langle p^{2} \rangle = 4\pi \int_{r_{0}}^{\infty} \psi^{*}(r) \frac{p^{2}}{r^{2}} \psi(r) r^{2} dr = 13.1504\hbar^{2}.$$  

(17)

Gathering the above numerical results, we obtain the first-order perturbation correction to the 1S energy levels due to non-commutative of space as:

$$E_{1}^{NC} = 0.142466\theta^{2} + 0.00412\theta^{2} = 0.146580\theta^{2}.$$  

(18)

Therefore, the effect of non-commutativity to the second order in $\theta(\theta^{2})$ results in a positive non-trivial correction to the 1S states. Thus, the energy of the 1S level is $E_{1S} = E_{0} + E_{1S}^{NC} = 0.257761 + 0.146580\theta^{2}$ (in GeV), where $E$ is the exact energy of 1S state and was computed by finding the expectation value of the Hamiltonian, $H = \frac{p^{2}}{2\mu} - \frac{4\alpha}{3r} + kr$, of the system with reduced mass $\mu = \frac{m_{c}m_{b}}{m_{c}+m_{b}} = \frac{m_{c}}{2}$, using the trial wavefunction (15).

In addition, one may use the best experimental hyperfine measurements of the $c\bar{c}$ spectrum for the 1S state [12]:

$$\Delta M_{hf}(1S)_{c\bar{c}} = 116.6 \pm 1.0$ MeV,$$

(19)

to infer some upper bound on the non-commutative parameter $\theta$. Since non-commutative of space has not been detected yet, one can set $0.146580\theta^{2} \leq 117 \text{ MeV}$, from which one gets $\theta \leq (1.69 \text{ GeV})^{-2}$. It is a little smaller than the limits obtained in [13], where they reported that $\theta \leq (0.16 \text{ GeV})^{-2}$, and in [14] where they obtained $\theta \leq (0.6 \text{ GeV})^{-2}$.

4 Conclusions

In this paper, we have studied heavy quarkonium bound states using the nonrelativistic Schrodinger equation with the phenomenological Cornell potential in non-commutative space. We have obtained correction to the 1S energy level for charmonium in terms of the non-commutative parameter using perturbation theory. The computed shift is $0.146580\theta^{2}$. Also, an upper limit on the non-commutative parameter $\theta \leq (1.69 \text{ GeV})^{-2}$, which corresponds to a distance of $\sqrt{\theta} \approx 0.12 \text{ fm}$, was imposed by comparing with the experimental measurements of the hyperfine splitting $\Delta M_{hf}(1S)_{c\bar{c}} = 117 \text{ MeV}$.
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