THE SOLVABILITY AND THE EXACT SOLUTION OF A SYSTEM OF REAL QUATERNION MATRIX EQUATIONS

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Abstract. In this paper, we establish necessary and sufficient conditions for the solvability of the system of real quaternion matrix equations

\[
\begin{align*}
A_1X &= C_1, \\
YB_1 &= D_1, \\
A_2Z &= C_2, ZB_2 = D_2, A_3ZB_3 = C_3, \\
A_4X + YB_4 + C_4ZD_4 &= E_1.
\end{align*}
\]

We also present an expression of the general solution to the system. The findings of this paper widely extend the known results in the literature.

1. Introduction and preliminaries

Throughout this paper we denote the set of all $m \times n$ matrices over the quaternion number field $\mathbb{H}$ by $\mathbb{H}^{m \times n}$. For a matrix $A$, $A^*$ and $\mathcal{R}(A)$ stand for the conjugate and the column space of $A$, respectively. $I_n$ denotes the $n \times n$ identity matrix. The Moore-Penrose inverse $A^\dagger$ of $A$ is defined to be the unique matrix $A^\dagger$, such that

\begin{enumerate}
\item[(i)] $AA^\dagger A = A,$
\item[(ii)] $A^\dagger AA^\dagger = A^\dagger,$
\item[(iii)] $(AA^\dagger)^* = AA^\dagger,$
\item[(iv)] $(A^\dagger A)^* = A^\dagger A.$
\end{enumerate}

Linear matrix functions and their special cases- linear matrix equations are fundamental subjects of study in matrix theory (e.g. \cite{3,8,21,27}). The
matrix function is a matrix-valve map between two linear spaces. The definition of matrix function and introduction of some matrix functions can be seen in [9]. In matrix theory and applications, many problems can be transformed in equivalent rank problems. In recent years this has been applied in seeking for the solvability for matrix equations (see, e.g. [16, 17, 24, 25]).

It is well known that in engineering and linear models, many problems can be expressed by some matrix functions. The limited conditions can be interpreted in limited matrix equations. With the developments of statistical and other science subjects, more parameters and variables are demanded for the matrix equations. Thus, investigations on some matrix functions with more parameters and variables are necessary for the matrix theory and the practical applications. For instance, Roth [13] developed the Sylvester’s matrix equation

\[
AX - XB = C,
\]

giving a necessary and sufficient condition for the consistency of

\[
AX - YB = C. \tag{1.1}
\]

In statistics, the growth curve model is consistent if and only if the more generalized matrix equation

\[
AY_3B + CY_4D = E \tag{1.2}
\]

is consistent [18]. A regression model related to equation (1.2) is \( M = AXB + CYD + \varepsilon \), where both \( X \) and \( Y \) are unknown parameter matrices and \( \varepsilon \) is a random error matrix. This matrix function is also called the nested growth curve model (see [14, 15]). In general, more limited equations means more complexity because more parameters and variables must be considered. Therefore, we first retrospect the development of some matrix equations and investigate the more complex ones.

There have been many papers discussing the classical system of matrix equations

\[
A_1XB_1 = C_1, \ A_2XB_2 = C_2. \tag{1.3}
\]

For instance, Mitra [10] first studied the system (1.3) over \( \mathbb{C} \). Vander Woude [21] investigated it over a field in 1987. Özküller and Akar [12] gave a condition for the solvability of the system over a principle domain in 1991. In 2004, Wang [26] gave some necessary and sufficient conditions for the existence of the solution to the system (1.3) and provided the expression of the general solution when it is solvable. Moreover, Wang, Chang and Ning [27] provided some necessary and sufficient conditions for the existence of an explicit expression for a common solution to the six classical linear quaternion matrix equations

\[
A_1X = C_1, \ XB_2 = C_2, \ A_2X = C_3, \ XB_2 = C_4, \ A_3XB_3 = C_5, \ A_4XB_4 = C_6. \tag{1.4}
\]
Observe that (1.1), (1.3) and (1.4) are special cases of the following system of real quaternion matrix equations

\[
\begin{align*}
A_1X &= C_1, \\
YB_1 &= D_1, \\
A_2Z &= C_2, ZB_2 &= D_2, A_3ZB_3 &= C_3, \\
A_4X + YB_4 + C_4ZD_4 &= E_1
\end{align*}
\]  

(1.5)

However, to our knowledge, so far there has been little information on the expression of the general solution to (1.5) with more variables and more parameters. This paper aims to give some solvability conditions and the expressions of the general solution to (1.5).

In order to get some necessary and sufficient conditions for the existence of the solution to the system (1.5), we need to derive the maximal and minimal ranks of the real quaternion matrix function with triple variables

\[
g(X, Y, Z) = E_1 - A_4X - YB_4 - C_4ZD_4, 
\]  

(1.6)

where \( X, Y \) and \( Z \) satisfy the following consistent matrix equations

\[
\begin{align*}
A_1X &= C_1, \\
YB_1 &= D_1, \\
A_2Z &= C_2, ZB_2 &= D_2, A_3ZB_3 &= C_3.
\end{align*}
\]  

(1.7)

The investigation on extremal ranks has been actively ongoing for more than 30 years. It is worthy to say that Professor Yongge Tian made great contributions in the literature. Minimal and maximal ranks and inertias are found to be useful in control theory (e.g. [1], [2]). In 2002, Tian [20] considered the maximal and minimal ranks of the matrix function

\[
p(X) = A_1 - B_1XC_1 
\]  

(1.8)

subject to

\[
B_2XC_2 = A_2.
\]  

(1.9)

In 2008, Wang, Yu and Lin [22] studied the extremal ranks of the quaternion matrix function

\[
f(X) = C_4 - A_4XB_4
\]  

(1.10)

subject to

\[
A_1X = C_1, XB_2 = C_2, A_3XB_3 = C_3.
\]  

(1.11)

Note that (1.8) and (1.10) are special cases of (1.6). The other goal of this paper is to consider the extremal ranks of (1.6) with more variables.

The remaining of this paper is organized as follows. In Section 2, we consider the extremal ranks of the real quaternion matrix function (1.6) subject to (1.7). In Section 3, we give some necessary and sufficient conditions for the solvability to the system of real quaternion matrix equations (1.5) and present an expression of the general solution to system (1.5).

2. Extremal ranks of (1.6) subject to (1.7) with applications

In this section, we investigate the matrix function (1.6) subject to (1.7). The conclusion extends the known results in [20] and [22]. We begin with the following lemmas.
Lemma 2.1. [23] Let $A_1 \in \mathbb{H}^{m \times n_1}, B_1 \in \mathbb{H}^{m \times q}, C_3 \in \mathbb{H}^{m \times n_2}, D_3 \in \mathbb{H}^{p_2 \times q}, C_4 \in \mathbb{H}^{m \times n_3}, D_4 \in \mathbb{H}^{p_3 \times q}$, and $E_1 \in \mathbb{H}^{m \times q}$ be given. Set

$$A = R_A C_3, B = D_3 L_{B_1}, C = R_A C_4, D = D_4 L_{B_1},$$

$$E = R_A E_1 L_{B_1}, M = R_A C, N = D L_B, S = C L_M.$$  

Then the following statements are equivalent:

(1) Equation

$$A_1 X_1 + X_2 B_1 + C_3 X_3 D_3 + C_4 X_4 D_4 = E_1 \tag{2.1}$$

is consistent.

(2) $R_A E = M M^\dagger E,$ $E L_B = E N^\dagger N,$ $R_A E L_D = 0,$ $R_A E L_B = 0.$

(3)

$$r \begin{bmatrix} E_1 & C_4 & C_3 & A_1 \\ B_1 & 0 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} C_4 & C_3 & A_1 \end{bmatrix} + r(B_1),$$

$$r \begin{bmatrix} E_1 & C_4 & A_1 \\ B_1 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} C_4 & A_1 \end{bmatrix} + r \begin{bmatrix} D_3 & B_1 \end{bmatrix}.$$

In this case, the general solution of (2.1) can be expressed as

$$X_1 = A_1^\dagger (E_1 - C_3 X_3 D_3 - C_4 X_4 D_4) - A_1^\dagger W_2 B_1 + L_A W_1,$$

$$X_2 = R_A (E_1 - C_3 X_3 D_3 - C_4 X_4 D_4) B_1^\dagger + A_1 A_1^\dagger W_2 + W_3 R_{B_1},$$

$$X_3 = A_1^\dagger E B_1^\dagger - A_1^\dagger C M^\dagger E B_1^\dagger - A_1^\dagger S C^\dagger E N^\dagger D B_1^\dagger - A_1^\dagger S V_4 R_N D B_1^\dagger + L_A V_3 + V_4 R_B,$$

$$X_4 = M_1^\dagger E D^\dagger + S_1^\dagger S C^\dagger E N^\dagger + L_M L_S U_1 + L_M V_3 R_N + V_5 R_D,$$

where $V_1, V_2, V_3, V_4, V_5, W_1, W_2, W_3$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Lemma 2.2. [11] Let $A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{m \times k}, C \in \mathbb{H}^{l \times n}, D \in \mathbb{H}^{m \times p}, E \in \mathbb{H}^{q \times n}, Q \in \mathbb{H}^{m_1 \times k}$, and $P \in \mathbb{H}^{l \times n_1}$ be given. Then

(1) $r(A) + r(R_A B) = r(B) + r(R_B A) = r \begin{bmatrix} A & B \end{bmatrix}.$

(2) $r(A) + r(C L_A) = r(C) + r(AL_C) = r \begin{bmatrix} A \\ C \end{bmatrix}.$
\( (3) \ r(B) + r(C) + r(R_B AL_C) = r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \)

\( (4) \ r(P) + r(Q) + r \begin{bmatrix} A & BLQ \\ R_PC & 0 \end{bmatrix} = r \begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix}. \)

\( (5) \ r \begin{bmatrix} R_B AL_C & R_B D \\ EL_C & 0 \end{bmatrix} + r(B) + r(C) = r \begin{bmatrix} A & D & B \\ E & 0 & 0 \\ C & 0 & 0 \end{bmatrix}. \)

**Lemma 2.3.** [22] Let \( A_1 \) and \( C_1 \) be given. Then the equation \( A_1 X_1 = C_1 \) is consistent if and only if \( r \begin{bmatrix} A_1 & C_1 \end{bmatrix} = r(A_1) \). In this case, the general solution to \( A_1 X_1 = C_1 \) can be expressed as

\[ X_1 = A_1^+ C_1 + L_{A_1} U_1, \]

where \( U_1 \) is an arbitrary matrix over \( \mathbb{H} \) with appropriate size.

**Lemma 2.4.** [22] Let \( B_1 \) and \( D_1 \) be given. Then the equation \( X_2 B_1 = D_1 \) is consistent if and only if \( r \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} = r(B_1) \). In this case, the general solution to \( X_2 B_1 = D_1 \) can be expressed as

\[ X_2 = D_1 B_1^+ + U_2 R_{B_1}, \]

where \( U_2 \) is an arbitrary matrix over \( \mathbb{H} \) with appropriate size.

**Lemma 2.5.** [22] Let \( A_2, B_2, C_2, D_2, A_3, B_3 \) and \( C_3 \) be given. Set

\[ A_5 = A_3 L_{A_2}, \quad B_5 = R_{B_2} B_3, \quad C_5 = C_3 - A_3 (A_2^+ C_2 + L_{A_2} D_2 B_2^+) B_3. \]

Then the following statements are equivalent:

1. System of real quaternion matrix equations

\[ A_2 Z = C_2, \quad Z B_2 = D_2, \quad A_3 Z B_3 = C_3 \tag{2.2} \]

is consistent.

2. \( R_{A_2} C_2 = 0, \quad D_2 L_{B_2} = 0, \quad R_{A_3} C_5 = 0, \quad C_5 L_{B_3} = 0, \quad A_2 D_2 = C_2 B_2. \)

3. \( r \begin{bmatrix} A_2 & C_2 \\ B_2 \end{bmatrix} = r(A_2), \quad \begin{bmatrix} D_2 \\ B_2 \end{bmatrix} = r(B_2), \quad A_2 D_2 = C_2 B_2, \)

\[ r \begin{bmatrix} A_3 & C_3 \\ A_2 & C_2 B_3 \end{bmatrix} = r \begin{bmatrix} A_3 \\ B_3 \\ B_2 \end{bmatrix}, \quad r \begin{bmatrix} C_3 & A_3 D_2 \\ B_3 & B_2 \end{bmatrix} = r \begin{bmatrix} B_3 & B_2 \end{bmatrix}. \]

In this case, the general solution to (2.2) can be expressed as

\[ Z = A_1^+ C_2 + L_{A_2} D_2 B_2^+ + L_{A_2} A_3^+ C_2 B_3^+ R_{B_2} + L_{A_2} L_{A_3} U_3 R_{B_2} + L_{A_2} U_4 R_{B_3} R_{B_2}, \]

where \( U_3 \) and \( U_4 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

The next Lemma is due to Tian.

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Lemma 2.6. [19] Let
\[ p(X_1, X_2, X_3, X_4) = A - B_1X_1 - X_2C_2 - B_3X_3C_3 - B_4X_4C_4 \]
be a matrix expression over \( \mathbb{H} \), where \( A \in \mathbb{H}^{m \times n} \). Then the extremal ranks of \( p(X_1, X_2, X_3, X_4) \) are the following
\[
\max_{\{X_i\}} r[p(X_1, X_2, X_3, X_4)] = \min \left\{ m, n, r \begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix}, r \begin{bmatrix} A & B_1 & B_3 & B_4 \\ 0 & C_2 & 0 & 0 \\ 0 & C_3 & 0 & 0 \end{bmatrix} \right\},
\]
and
\[
\min_{\{X_i\}} r[p(X_1, X_2, X_3, X_4)] = r \begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix} + r \begin{bmatrix} A & B_1 & B_3 & B_4 \\ 0 & C_2 & 0 & 0 \\ 0 & C_3 & 0 & 0 \end{bmatrix} - (r(B_1) - r(C_2)) + \max \left\{ r \begin{bmatrix} A & B_1 & B_3 \\ 0 & C_2 & 0 \\ 0 & C_3 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_4 \\ 0 & C_2 & 0 \\ 0 & C_3 & 0 \end{bmatrix}, r \begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_3 & B_4 \\ 0 & C_2 & 0 & 0 \\ 0 & C_3 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_4 \\ 0 & C_2 & 0 \\ 0 & C_3 & 0 \end{bmatrix} \right\}.
\]

For convenience, we adopt the following notations:
\[
J_1 = \left\{ X | A_1X = C_1 \right\}, J_2 = \left\{ Y | YB_1 = D_1 \right\},
\]
\[
J_3 = \left\{ Z | A_2Z = C_2, ZB_2 = D_2, A_3ZB_3 = C_3 \right\}.
\]

Theorem 2.7. Let \( A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, A_4, B_4, C_4, D_4 \) and \( E_1 \in \mathbb{H}^{m \times n} \) be given. Assume that \( J_1 - J_3 \) are not empty sets. Denote that
\[
N_1 = \begin{bmatrix}
E_1 & A_4 & D_1 & C_4 D_2 \\
B_4 & 0 & B_1 & 0 \\
D_4 & 0 & 0 & B_2 \\
C_1 & A_1 & 0 & 0
\end{bmatrix},
N_2 = \begin{bmatrix}
E_1 & A_4 & C_4 & D_1 \\
B_4 & 0 & 0 & B_1 \\
C_1 & A_1 & 0 & 0 \\
C_2 D_4 & 0 & A_2 & 0
\end{bmatrix},
\]
Then we have the following:

(a) The maximal rank of (1.6) subject to (1.7) is

\[
\max_{X \in J_1, Y \in J_2, Z \in J_3} r[g(X, Y, Z)] = \min \left\{ m, n, r(N_1) - r(A_1) - r(B_1) - r(B_2),
\right. \\
\left. r(N_2) - r(A_1) - r(B_1) - r(A_2),
\right. \\
\left. r(N_3) - r(A_1) - r(B_1) - r(A_2) + r \left[ \begin{bmatrix} A_2 \\ A_3 \\ B_2 \end{bmatrix} \right] - r \left[ \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} \right] \right\}.
\]

(b) The minimal rank of (1.6) subject to (1.7) is

\[
\min_{X \in J_1, Y \in J_2, Z \in J_3} r[g(X, Y, Z)] = r(N_1) - r(N_2) - r \left[ \begin{bmatrix} A_1 \\ B_1 \\ B_4 \end{bmatrix} \right]
\]

\[
+ \max \{ r(N_3) - r(N_5) - r(N_6), -r(N_4) \}.
\]

Proof. It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that the general solutions of

\[ A_1X = C_1, \quad YB_1 = D_1, \quad A_2Z = C_2, \quad ZB_2 = D_2, \quad A_3ZB_3 = C_3 \]

can be expressed as

\[ X = X_0 + L_{A_1}U_1, \quad Y = Y_0 + U_2R_{B_1}, \quad Z = Z_0 + L_{A_2}L_{A_3}U_3R_{B_2} + L_{A_2}U_4R_{B_3}R_{B_2}, \]

where \( X_0, Y_0, Z_0 \) are special solutions of the corresponding matrix equations, \( U_1 - U_3 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes. Substituting (2.5) into (1.6) yields

\[ g(X, Y, Z) = A - A_1L_{A_1}U_1 - U_2R_{B_1}B_4 - C_4L_{A_2}L_{A_3}U_3R_{B_2}D_4 - C_4L_{A_2}U_4R_{B_3}R_{B_2}D_4, \]

where

\[ A = E_1 - A_4X_0 - Y_0B_4 - C_4Z_0D_4.\]
Applying Lemma 2.6 to (2.6) gives
\[
\max_{X \in J_1, Y \in J_2, Z \in J_3} r \{ g(X, Y, Z) \} = \min \{ m, n, l_1, l_2, l_3 \}. \tag{2.7}
\]

\[
\min_{X \in J_1, Y \in J_2, Z \in J_3} r \{ g(X, Y, Z) \} = l_1 + l_2 - r(A_4L_{A_1}) - r(R_{B_1}B_4) + \max \{ t_3 - t_5 - t_6, -t_4 \}, \tag{2.8}
\]

where
\[
l_1 = r \begin{bmatrix} A & A_4L_{A_1} \\ R_{B_1}B_4 & 0 \\ R_{B_1}B_4 & 0 \\ R_{B_1}B_4 & 0 \end{bmatrix}, \quad l_2 = r \begin{bmatrix} A & A_4L_{A_1} & C_4L_{A_2} \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \end{bmatrix},
\]
\[
l_3 = r \begin{bmatrix} A & A_4L_{A_1} & C_4L_{A_2}L_{A_5} \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \end{bmatrix}, \quad l_4 = r \begin{bmatrix} A & A_4L_{A_1} & C_4L_{A_2} \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \end{bmatrix},
\]
\[
l_5 = r \begin{bmatrix} A & A_4L_{A_1} & C_4L_{A_2} \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \end{bmatrix}, \quad l_6 = r \begin{bmatrix} A & A_4L_{A_1} & C_4L_{A_2}L_{A_5} \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_1}B_4 & 0 & 0 \end{bmatrix}.
\]

By Lemma 2.2 and

\[
A_1X_0 = C_1, \quad Y_0B_1 = D_1, \quad A_2Z_0 = C_2, \quad Z_0B_2 = D_2, \quad A_3Z_0B_3 = C_3,
\]

we obtain that
\[
l_1 = r(N_1) - r(A_1) - r(B_1) - r(B_2), \tag{2.9}
\]
\[
l_2 = r(N_2) - r(A_1) - r(B_1) - r(A_2), \tag{2.10}
\]
\[
l_3 = r(N_3) - r(A_1) - r(B_1) - r \begin{bmatrix} A_2 \\ A_3 \end{bmatrix} - r \begin{bmatrix} B_2 & B_3 \end{bmatrix}, \tag{2.11}
\]
\[
l_4 = r(N_4) - r(A_1) - r(B_1) - r(A_2) - r(B_2), \tag{2.12}
\]
\[
l_5 = r(N_5) - r(A_1) - r(B_1) - r(A_2) - r \begin{bmatrix} B_2 & B_3 \end{bmatrix}, \tag{2.13}
\]
\[
l_6 = r(N_6) - r(A_1) - r(B_1) - r \begin{bmatrix} A_2 \\ A_3 \end{bmatrix} - r(B_2). \tag{2.14}
\]

Substituting (2.9)-(2.14) into (2.7) and (2.8) yields (2.3) and (2.4).

In Theorem 2.7, let \( A_1, B_1, C_1, D_1, A_4 \) and \( B_4 \) vanish. Then we can obtain the extremal ranks of (1.10) subject to (1.11).

**Corollary 2.8.** The extremal ranks of the quaternion matrix expression \( f(X) = C_4 - A_4XB_4 \) subject to the consistent system (1.11) are the following:

\[
\max_{A_1X = C_1, \quad XB_2 = C_2, \quad A_3XB_3 = C_3} r(f(X)) = \min \{ a, b, c \},
\]

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where
\[ a = r \begin{bmatrix} C_1 B_4 & A_1 \\ C_4 & A_4 \end{bmatrix} - r(A_1), \]
\[ b = r \begin{bmatrix} B_2 & B_1 \\ A_4 C_2 & C_4 \end{bmatrix} - r(B_2), \]
\[ c = r \begin{bmatrix} A_1 & 0 & 0 & C_1 B_4 \\ A_3 & -A_3 C_2 & -C_3 & 0 \\ A_4 & 0 & 0 & C_4 \\ 0 & B_2 & B_3 & B_4 \end{bmatrix} - r \begin{bmatrix} A_1 \\ A_3 \\ A_4 \\ 0 \end{bmatrix} - r \begin{bmatrix} B_2 & B_3 \end{bmatrix}. \]

\[ \min_{A_1 X_1 = C_1, \ X B_2 = C_2, \ A_3 X B_3 = C_3} r(f(X)) = r \begin{bmatrix} C_1 B_4 & A_1 \\ C_4 & A_4 \end{bmatrix} + r \begin{bmatrix} B_2 \\ A_4 C_2 \\ C_4 \end{bmatrix} \]
\[ + r \begin{bmatrix} A_1 & 0 & 0 & C_1 B_4 \\ A_3 & -A_3 C_2 & -C_3 & 0 \\ A_4 & 0 & 0 & C_4 \\ 0 & B_2 & B_3 & B_4 \end{bmatrix} - r \begin{bmatrix} A_1 \\ A_3 \\ A_4 \\ 0 \end{bmatrix} - r \begin{bmatrix} B_2 & B_3 \end{bmatrix}. \]

**Remark 2.9.** Corollary 2.8 is Theorem 2.5 in [22].

In Theorem 2.7, let \( A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_4 \) and \( B_4 \) vanish. Then we can obtain the extremal ranks of (1.8) subject to (1.9).

**Corollary 2.10.** Suppose that the matrix equation \( B_2 X C_2 = A_2 \) is consistent. Then
\[ (a) \] The maximal rank of \( p(X) = A_1 - B_1 X C_1 \) subject to \( B_2 X C_2 = A_2 \) is

\[ \max_{B_2 X C_2 = A_2} r(p(X)) = \min \left\{ r \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} - r(B_2) - r(C_2), r \begin{bmatrix} A_1 \\ C_1 \end{bmatrix}, r \begin{bmatrix} A_1 & B_1 & 0 \\ C_1 & 0 & C_2 \end{bmatrix} \right\}. \]

\[ (b) \] The minimal rank of \( p(X) = A_1 - B_1 X C_1 \) subject to \( B_2 X C_2 = A_2 \) is

\[ \min_{B_2 X C_2 = A_2} r(p(X)) = r \begin{bmatrix} A_1 \\ C_1 \end{bmatrix} + r \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix} - r \begin{bmatrix} A_1 & B_1 & 0 \\ C_1 & 0 & C_2 \end{bmatrix} \]
\[ - r \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix} + r \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix}. \]

**Remark 2.11.** Corollary 2.10 is Theorem 3.2 in [20].
3. The solvable conditions and the expression of the general solution to (1.5)

Our goal in this section is to give some solvable conditions for (1.5) and to provide an expression of this general solution when the solvability conditions are met.

**Theorem 3.1.** Let $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, A_4, B_4, C_4, D_4$ and $E_1$ be as in Theorem 2.7. Set

$$A_5 = A_3L_{A_2}, \quad B_5 = R_{B_2}B_3, \quad C_5 = C_3 - A_3(A_2^tC_2 + L_{A_2}D_2B_2^t)B_3, \quad A_6 = A_4L_{A_1},$$

$$B_6 = R_{B_1}B_4, \quad C_6 = C_4L_{A_2}L_{A_5}, \quad D_6 = R_{B_2}D_4, \quad C_7 = C_4L_{A_2}, \quad D_7 = R_{B_2}R_{B_2}D_4,$$

$$E_2 = E_1 - A_4A_1^tC_1 - D_1B_1^tB_4 - C_4(A_2^tC_2 + L_{A_2}D_2B_2^t + L_{A_2}A_3^tC_5B_3^tR_{B_2})D_4,$$

$$A = R_{A_6}C_6, \quad B = D_6L_{B_6}, \quad C = R_{A_6}C_7, \quad D = D_7L_{B_6},$$

$$E = R_{A_6}E_2L_{B_6}, \quad M = R_{A}C, \quad N = DL_{B}, \quad S = C_{LM}.$$

Then the following statements are equivalent:

(a) System (1.5) is consistent.

(b) $R_{A_6}C_i = 0, \quad D_iL_{B_i} = 0, \quad i = 1, 2, \quad R_{A_6}C_5 = 0, \quad C_5L_{B_5} = 0, \quad A_2D_2 = C_2B_2,$

$$R_{A_6}E = MM^tE, \quad EL_B = EN^tN, \quad R_{A_6}EL_D = 0, \quad R_{C_6}EL_B = 0.$$

(c)

$$r\left[A_i \quad C_i\right] = r(A_i); \quad \left[D_i \quad B_i\right] = r(B_i), \quad i = 1, 2, \quad A_2D_2 = C_2B_2,$$

$$r\left[A_3 \quad C_3 \quad A_2 \quad C_2B_3\right] = r\left[A_3 \quad A_2\right], \quad r\left[C_3 \quad A_3D_2 \quad B_3 \quad B_2\right] = r\left[B_3 \quad B_2\right],$$

$$r(N_1) = r\left[A_1 \quad C_4 \quad A_1 \quad B_1 \quad 0 \quad B_2\right] r(N_2) = r\left[A_4 \quad C_4 \quad A_1 \quad 0 \quad B_2\right] + r\left[B_4 \quad B_1\right],$$

$$r(N_3) = r\left[A_4 \quad C_4 \quad A_1 \quad 0 \quad A_2 \quad 0 \quad A_3\right] r(N_4) = r\left[B_4 \quad B_1 \quad 0 \quad B_2\right] r(N_5) = r\left[B_4 \quad B_1 \quad 0 \quad B_2\right] + r\left[A_4 \quad C_4 \quad A_1 \quad 0 \quad A_3\right].$$

In this case, the general solution of (1.5) can be expressed as

$$X = A_1^tC_1 + L_{A_1}U_1,$$

$$Y = D_1B_1^t + U_2R_{B_1},$$

$$Z = A_2^tC_2 + L_{A_2}D_2B_2^t + L_{A_2}A_3^tC_5B_3^tR_{B_2} + L_{A_2}L_{A_5}U_3R_{B_2} + L_{A_2}U_4R_{B_2},$$
Lemma 2.5 that

Proof. (b) \iff (c) : It follows from Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 that

\[ R_{A_i}C_i = 0 \iff r \begin{bmatrix} A_i & C_i \end{bmatrix} = r(A_i), \quad D_iL_{B_i} = 0 \iff r \begin{bmatrix} B_i \end{bmatrix} = r(B_i), \quad i = 1, 2, \]

\[ R_{A_5}C_5 = 0 \iff r \begin{bmatrix} A_3 & C_3 \\ A_2 & C_2B_4 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_2 \end{bmatrix}, \]

\[ C_5L_{B_5} = 0 \iff r \begin{bmatrix} C_3 & A_5D_2 \\ B_3 & B_2 \end{bmatrix} = r \begin{bmatrix} B_3 & B_2 \end{bmatrix}, \]

\[ R_{A}E = MM^{\dagger}E \iff r \begin{bmatrix} E_2 & C_7 \\ B_6 & C_6 \end{bmatrix} = r \begin{bmatrix} C_7 & C_6 & A_6 \end{bmatrix} + r(B_6) \]

\[ \iff r \begin{bmatrix} E_2 & A_4L_{A_1} \\ R_{B_1}B_4 & C_4L_{A_2} \end{bmatrix} = r \begin{bmatrix} E_2 & A_4L_{A_1} & C_4L_{A_2} \end{bmatrix} + r(R_{B_1}B_4) \]

\[ \iff r(N_2) = r \begin{bmatrix} A_4 & C_4 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} + r \begin{bmatrix} B_4 & B_1 \end{bmatrix}, \]

\[ EL_{B} = EN^{\dagger}N \iff r \begin{bmatrix} E_2 & A_6 \\ D_6 & 0 \\ D_7 & 0 \end{bmatrix} = r \begin{bmatrix} D_6 \\ D_7 \\ B_6 \end{bmatrix} + r(A_6) \]

\[ \iff r \begin{bmatrix} E_2 & A_4L_{A_1} \\ R_{B_1}B_4 & 0 \\ R_{B_2}D_4 & 0 \end{bmatrix} = r \begin{bmatrix} R_{B_1}B_4 \\ R_{B_2}D_4 \end{bmatrix} + r(A_4L_{A_1}) \]

\[ \iff r(N_1) = r \begin{bmatrix} A_4 \\ A_1 \end{bmatrix} + r \begin{bmatrix} B_4 & B_1 & 0 \\ D_4 & 0 & B_2 \end{bmatrix}, \]

\[ R_{A}EL_{D} = 0 \iff r \begin{bmatrix} E_2 & C_6 & A_6 \\ B_6 & 0 & 0 \\ D_7 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} C_6 & A_6 \\ D_7 & B_6 \end{bmatrix} \]
Lemma 2.3, Lemma 2.4 and Lemma 2.5 that applying and elementary matrix operations, we obtain

\[ \begin{bmatrix} E_2 & A_4 L_{A_1} & C_4 L_{A_2} L_{A_5} \\ R_{B_1} B_4 & 0 & 0 \\ R_{B_2} R_{B_2} D_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{B_1} B_1 & C_4 L_{A_2} L_{A_5} + \end{bmatrix} \begin{bmatrix} R_{B_2} D_4 \end{bmatrix} \]

\[ \iff r(N_3) = r \begin{bmatrix} A_4 & C_4 \\ A_1 & 0 \\ -I \\ 0 & A_2 \\ 0 & A_3 \end{bmatrix} + r \begin{bmatrix} B_4 & B_1 & 0 & 0 \\ D_4 & 0 & B_3 & B_2 \end{bmatrix} \]

\[ R_{C} E L_B = 0 \iff r \begin{bmatrix} E_2 & C_7 & A_6 \\ B_6 & 0 & 0 \\ D_6 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} C_7 & A_6 \end{bmatrix} + r \begin{bmatrix} D_6 \end{bmatrix} \]

\[ \iff r(M_4) = r \begin{bmatrix} A_4 & C_5 \\ A_1 & 0 \\ 0 & A_3 \end{bmatrix} + r \begin{bmatrix} B_4 & B_1 & 0 & 0 \\ D_4 & 0 & B_2 \end{bmatrix} \]

\((a) \implies (c)\) : Suppose that \((X_0, Y_0, Z_0)\) is a solution of (1.5). It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that

\[ r \begin{bmatrix} A_i & C_i \end{bmatrix} = r(A_i), \quad \begin{bmatrix} D_i \\ B_i \end{bmatrix} = r(B_i), \quad i = 1, 2, \quad A_2 D_2 = C_2 B_2, \]

\[ r \begin{bmatrix} A_3 & C_3 \\ A_2 & C_2 B_2 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_2 \end{bmatrix}, \quad r \begin{bmatrix} C_3 & A_3 D_2 \\ B_3 & B_2 \end{bmatrix} = r \begin{bmatrix} B_3 & B_2 \end{bmatrix}. \]

Applying

\[ A_1 X_0 = C_1, \quad Y_0 B_1 = D_1, \quad A_2 Z_0 = C_2, \quad Z_0 B_2 = D_2, \quad A_3 Z_0 B_3 = C_3 \]

and elementary matrix operations, we obtain

\[ \begin{bmatrix} I & \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \end{bmatrix} N_1 \begin{bmatrix} I & 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_4 & 0 \\ B_4 & 0 & B_1 \\ D_4 & 0 & B_2 \\ 0 & A_1 & 0 \end{bmatrix}, \]

\[ \begin{bmatrix} I & \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \end{bmatrix} N_2 \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_4 & C_4 & 0 \\ B_4 & 0 & 0 & B_1 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \end{bmatrix}, \]
Hence, the system (1.5) has a solution.

(c) \implies (a): Suppose that the equalities in (c) hold. It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that the equations in
\[ A_1X = C_1, \ YB_1 = D_1, \ A_2Z = C_2, \ ZB_2 = D_2, \ A_3ZB_3 = C_3, \]
are consistent, respectively. On the other hand, by Theorem 2.7, we obtain that
\[ (3.7) \]
are consistent, respectively, if and only if
\[ \min_{X \in J_1, Y \in J_2, Z \in J_3} r(E_1 - A_4X - YB_1 - C_4ZD_4) = 0. \]
Hence, the system (1.5) has a solution.

(a) \iff (b): We separate the equations in system (1.5) into two groups
\[ A_1X = C_1, \ YB_1 = D_1, \ A_2Z = C_2, \ ZB_2 = D_2, \ A_3ZB_3 = C_3, \] \tag{3.7}
\[ A_4X + YB_4 + C_4ZD_4 = E_1. \] \tag{3.8}
It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that matrix equations in (3.7) are consistent, respectively, if and only if
\[ R_{A_i}C_i = 0, \ D_iL_{Bi} = 0, \ i = 1, 2, \ R_{A_3}C_5 = 0, \ C_5L_{B_5} = 0, \ A_2D_2 = C_2B_2. \]
And the general solutions to these matrix equations in (3.7) can be expressed as
\[ X = A_1^tC_1 + L_{A_1}U_1, \] \tag{3.9}
\[ Y = D_1B_1^t + U_2R_{B_1}, \] \tag{3.10}
\[ Z = A_2^tC_2 + L_{A_2}D_2B_2^t + L_{A_2}A_3^tC_5B_3^tR_{B_2} + L_{A_2}L_{A_5}U_3R_{B_2} + L_{A_2}U_4R_{B_5}R_{B_2}. \] \tag{3.11}
Substituting (3.9)-(3.11) into (3.8) gives
\[ A_6U_1 + U_2B_6 + C_6U_3D_6 + C_7U_4D_7 = E_2. \] \tag{3.12}
Hence, the system (1.5) is consistent if and only if the matrix equations in (3.7) and (3.12) are consistent, respectively. By Lemma 2.1, we know that the matrix equation (3.12) is consistent if and only if
\[ R_AE = MM^tE, \ EL_B = EN^tN, \ R_AEL_D = 0, \ R_CE = 0. \]
We know by Lemma 2.1 that the general solutions of equation (3.12) can be expressed as (3.3)-(3.6). □
In Theorem 3.1, let $A_2, B_2, C_2$ and $D_2$ vanish. We can obtain the general solution to the following system

\[
\begin{cases}
A_1X = C_1, YB_1 = D_1, \\
A_2ZB_2 = C_2 \\
A_3X + YB_3 + C_3ZD_3 = E_1.
\end{cases}
\] (3.13)

**Corollary 3.2.** Let $A_1, B_1, C_1, D_1, A_2, B_2, C_2, A_3, B_3, C_3, D_3, E_1$ and $N_1 - N_0$ be given. Set

\[
A_4 = A_3L_{A_1}, B_4 = R_{B_1}B_3, C_4 = C_3L_{A_2}, D_4 = R_{B_2}D_3,
\]

\[
E_4 = E_1 - A_3A_1^iC_1 - D_1B_1^iB_3 - C_3A_2^iC_2B_2^iD_3,
\]

\[
A = R_{A_4}C_4, B = D_3L_{B_4}, C = R_{A_4}C_3, D = D_4L_{B_4},
\]

\[
E = R_{A_4}E_4L_{B_4}, M = RA_C, N = DL_B, S = CL_M.
\]

Then the following statements are equivalent:

(a) System (3.13) is consistent.
(b) $R_{A_4}C_2 = 0, D_2L_{B_2} = 0, R_{A_4}C_5 = 0, C_5L_{B_3} = 0, A_2D_2 = C_2B_2,$

\[
R_{A_4}E = MM^\perp E, EL_B = EN^\perp N, R_{A_4}EL_D = 0, R_{C_4}EL_B = 0.
\]

(c)

\[
r \begin{bmatrix} A_i & C_i \end{bmatrix} = r(A_i),
\]

\[
r \begin{bmatrix} B_i \end{bmatrix} = r(B_i), \quad i = 1, 2,
\]

\[
r \begin{bmatrix} E_1 & A_3 & D_1 \\ B_3 & 0 & B_1 \\ D_3 & 0 & 0 \\ C_1 & A_1 & 0 \end{bmatrix} = r \begin{bmatrix} A_1 \\ A_3 \\ B_1 \\ 0 \end{bmatrix} + r \begin{bmatrix} B_3 \\ B_1 \end{bmatrix},
\]

\[
r \begin{bmatrix} E_1 & A_3 & C_3 & D_1 \\ B_3 & 0 & 0 & B_1 \\ D_3 & 0 & 0 & B_2 \\ C_1 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 -C_2 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} B_3 \\ B_1 \\ 0 \\ D_3 \end{bmatrix},
\]

\[
r \begin{bmatrix} E_1 & A_3 & C_3 & D_1 \\ B_3 & 0 & 0 & B_1 \\ D_3 & 0 & 0 & 0 \\ C_1 & A_1 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_1 & 0 \\ A_3 & C_3 \\ 0 & A_2 \\ D_3 \end{bmatrix} + r \begin{bmatrix} B_3 & B_1 \\ 0 & D_3 & 0 & B_2 \end{bmatrix},
\]

\[
r \begin{bmatrix} E_1 & A_3 & C_3 & D_1 \\ B_3 & 0 & 0 & B_1 \\ D_3 & 0 & 0 & 0 \\ C_1 & A_1 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_1 & 0 \\ A_3 & C_3 \\ 0 & A_2 \\ D_3 \end{bmatrix} + r \begin{bmatrix} B_3 & B_1 \\ 0 & D_3 \end{bmatrix}.
\]
In this case, the general solution of (3.13) can be expressed as
\[ X = A_1^tC_1 + L_{A_1}U_1, \quad Y = D_1B_1^t + U_2R_{B_1}, \quad Z = A_2^tC_2B_2^t + L_{A_2}U_3 + U_4R_{B_2}, \]
\[ U_1 = A_4^t(E_4 - C_4U_3D_3 - C_3U_4D_4) - A_1^tW_2B_4 + L_{A_4}W_1, \]
\[ U_2 = R_{A_4}(E_4 - C_4U_3D_3 - C_3U_4D_4)B_4^t + A_4A_1^tW_2 + W_3R_{B_1}, \]
\[ U_3 = A_4^tEB^t - A_3CM^tEB^t = A_3SC^tEN^tDB^t - A_1^tSV_4R_NDB^t + L_AV_1 + V_2R_B, \]
\[ U_4 = M^tED^t + S^tSC^tEN^t + L_ML_SV_3 + L_MV_4R_N + V_5R_D, \]
where \( V_1, V_2, V_3, V_4, V_5, W_1, W_2, W_3 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

In Theorem 3.1, let \( A_1, B_1, C_1, D_1, A_4 \) and \( B_4 \) vanish. We can derive the general solution to the following system
\[ A_2Z = C_2, \quad ZB_2 = D_2, \quad A_3ZB_3 = C_3, \quad C_3ZD_4 = E. \] 

**Corollary 3.3.** Let \( A_2, B_2, C_2, D_2, A_3, B_3, C_3, C_4, D_4 \) and \( E_1 \) be given. Set
\[ A_5 = A_3L_{A_2}, \quad B_5 = R_{B_2}B_3, \quad C_5 = C_3 - A_3(A_2^tC_2 + L_{A_2}D_2B_2^t)B_3, \]
\[ A = C_4L_{A_2}L_{A_5}, \quad B = R_{B_2}D_4, \quad C = C_4L_{A_2}, \quad D = R_{B_2}R_{B_2}D_4, \]
\[ E = E_1 - C_4(A_2^tC_2 + L_{A_2}D_2B_2^t + L_{A_2}A_3^tC_5B_5^tR_{B_2})D_4, \]
\[ M = R_{A_4}C_5N = DL_B, \quad S = CL_M. \]

Then the following statements are equivalent:
(a) System (3.14) is consistent.
(b) \( R_{A_i}C_i = 0, \; D_iL_{B_i} = 0, \; i = 1, 2, \quad R_{A_5}C_5 = 0, \; C_5L_{B_5} = 0, \quad A_2D_2 = C_2B_2, \)
\[ R_{A_4}E = MM^tE, \quad EL_B = EN^tN, \quad R_{A_4}EL_D = 0, \quad R_CE_L_B = 0. \]
(c) \[ r \begin{bmatrix} A_2 & C_2 \end{bmatrix} = r(A_2), \quad \begin{bmatrix} D_2 \\ B_2 \end{bmatrix} = r(B_2), \quad A_2D_2 = C_2B_2, \]
\[ r \begin{bmatrix} A_3 & C_3 \\ A_2 & C_2B_3 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_2 \end{bmatrix}, \quad r \begin{bmatrix} C_3 & A_3D_2 \\ B_3 & B_2 \end{bmatrix} = r \begin{bmatrix} B_3 & B_2 \end{bmatrix}, \]
\[ r \begin{bmatrix} E_1 & C_4 \\ C_2D_4 & A_2 \end{bmatrix} = r \begin{bmatrix} C_4 \\ A_2 \end{bmatrix}, \quad r \begin{bmatrix} E_1 & C_4D_2 \\ D_4 & B_2 \end{bmatrix} = r \begin{bmatrix} B_2 & D_4 \end{bmatrix}, \]
\[ r \begin{bmatrix} E_1 & C_4 & 0 & 0 \\ D_4 & 0 & B_3 & B_2 \\ 0 & A_3 & 0 & 0 \\ 0 & A_2 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} C_4 \\ A_3 \\ A_2 \end{bmatrix} + r \begin{bmatrix} B_3 & D_4 & B_2 \end{bmatrix}. \]
\[
\begin{bmatrix}
E_1 & C_4 & 0 \\
D_4 & 0 & B_3 \\
0 & A_2 & 0
\end{bmatrix}
= \begin{bmatrix}
C_4 \\
A_2
\end{bmatrix} + \begin{bmatrix}
D_4 & B_2
\end{bmatrix}.
\]

In this case, the general solution to (3.14) can be expressed as

\[
Z = A_2^t C_2 + L_{A_2} D_2 B_2^t + L_{A_2} A_2^t C_5 B_5^t R_{B_2} + L_{A_2} L_{A_5} U_3 R_{B_2} + L_{A_2} U_4 R_{B_5} R_{B_2},
\]

\[
U_3 = A^t E B^t - A^t C M^t E B^t - A^t S C^t E N^t D B^t - A^t S V_4 R_N D B^t + L_A V_1 + V_2 R_B,
\]

\[
U_4 = M^t E D^t + S^t S C^t E N^t + L_M L_S V_3 + L_M V_4 R_N + V_5 R_D,
\]

where \( V_1, V_2, V_3, V_4, V_5 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

**Remark 3.4.** Our expression of the general solution to system (3.14) is different from the expression in [27].

4. Conclusions

In this paper we have given the extremal ranks of the matrix function (1.6) subject to (1.7), which extend the known results in [20] and [22]. We have derived some solvable conditions for the existence of the general solution to system (1.5), and proved that (3.1)-(3.6) are solutions of system (1.5) when the solvability conditions are met. Using the results on (1.5), we have established some necessary and sufficient conditions for the existence of the general solution to (3.13) and (3.14), respectively. The expressions of such solutions to (3.13) and (3.14) have also been given, respectively. There is no doubt that most of the results in this paper can be extended to the corresponding system for linear operators on a Hilbert space or elements in a ring with involution.

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