FUNCTIONAL DECOMPOSITION OF STATE INDUCED $C^*$-MATRIX SPACES

TITARII WOOTIJIRATTIKAL$^1$ AND SING-CHEONG ONG$^2$

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ABSTRACT. A theorem of Dixmier states that each bounded linear functional $f$ on the algebra of bounded linear operators on a separable Hilbert space is a direct sum of a trace functional $g$ and a singular functional $h$, vanishing on the compact operators, such that $\|f\| = \|g\| + \|h\|$. We use elementary methods to construct, via the state space of a $C^*$-algebra, a Banach space of $C^*$ matrices that contains a closed subspace on which a version of Dixmier's theorem is proved. When the $C^*$-algebra is taken to be the complex numbers our approach gives elementary and transparent proofs of Dixmier's theorem and the trace formula $\text{tr}(AB) = \text{tr}(BA)$, without using the operator theoretical machineries used in the known proofs.

1. Introduction and notation

Let $f$ be a bounded linear functional on $\mathcal{B}(\ell^2)$ (the space of bounded linear operators on the Hilbert sequence space $\ell^2$). Then $f$ defines a bounded linear functional on $\mathcal{K}(\ell^2)$, the ideal of compact operators on $\ell^2$. Thus there is a trace class operator (or matrix) $A_f$ such that $f(B) = \text{tr}(A_f B)$, where $\text{tr}$ denotes the trace function, for all $B \in \mathcal{K}(\ell^2)$ [5, p. 46, Theorem 1]. Since the trace class operators form an ideal in $\mathcal{B}(\ell^2)$ [5, p. 42, Theorem 5], the function $g(B) = \text{tr}(A_f B)$ for $B \in \mathcal{B}(\ell^2)$ defines a bounded linear functional on $\mathcal{B}(\ell^2)$. The functional $h = f - g$ vanishes on $\mathcal{K}(\ell^2)$ is also known as a singular linear functional. Dixmier's theorem
Thus each scalar matrix into the second summand in the dual space $X^*$ of $X$, i.e., $X^* = J^\perp \oplus_1 E$ for some closed subspace $E$ of $X^*$. This theorem of Dixmier can now be restated as the compact operators form an $M$-ideal in $B(\ell^2)$ (later it is also known as the only nontrivial one [6, 7]). See also [3]. Most spaces with known $M$-ideal structures are Banach algebras, mainly bounded operators on certain Banach spaces.

Since a $C^*$-algebra resembles the complex field in many ways, here we will use a fixed $C^*$-algebra $A$ with identity $1$ and state space $s(A)$, together with the pair $K(\ell^2)$ and $B(\ell^2)$, to build a Banach space of matrices over $A$ with an $M$-ideal that corresponds to $K(\ell^2)$. The resulting space is not a Banach algebra. When the $C^*$-algebra is taken to be $C$, the space is exactly $B(\ell^2)$. Since there is no parallel machinery available for our setting, this approach also gives elementary alternate proofs of Dixmier’s theorem and the trace formula $\text{tr}(AB) = \text{tr}(BA)$, without using the theory of trace class operators and other machineries.

Let $A$ be a $C^*$-algebra with identity $1$ and state space $s(A)$ (consisting of all states, i.e., bounded positive linear functionals of norm $1$, on $A$) with the weak$^*$ topology (as a subspace of the dual space $A^*$ of $A$). For each matrix $B = [b_{jk}]$ with entries $b_{jk} \in A$, and each $\psi \in s(A)$, denote by $[\tilde{\psi}(b_{jk})]$ the complex matrix $[\psi(b_{jk})]$. Let $M$ be the space of all matrices $A = [a_{jk}]$ over $A$ such that (the scalar matrix)

$$\tilde{\varphi}(A) := [\varphi(a_{jk})] \in B(\ell^2) \quad \text{for all } \varphi \in s(A)$$

the map $\varphi \mapsto \tilde{\varphi}(A) = [\varphi(a_{jk})]$ is continuous from $s(A)$ with the weak$^*$ topology to $B(\ell^2)$ with the norm topology.

Thus each $A \in M$ defines a continuous map, $\varphi \mapsto \tilde{\varphi}(A)$, from $s(A)$ to $B(\ell^2)$. Since $s(A)$, with the weak$^*$ topology, is a compact Hausdorff space [4, p. 257], it is well known that $C(s(A), B(\ell^2))$ is a Banach space with the norm

$$\|A\| = \sup_{\varphi \in s(A)} \|\tilde{\varphi}(A)\|_{B(\ell^2)}.$$  

Each $A \in M$ induces an element $\tilde{A}$ in $C(s(A), B(\ell^2))$:

$$\tilde{A}(\varphi) = \tilde{\varphi}(A), \quad \varphi \in s(A).$$

So $M$ can be considered as a subspace of the Banach space $C(s(A), B(\ell^2))$. The map $A \mapsto \tilde{A}$ does not map $M$ onto $C(s(A), B(\ell^2))$, even when $\ell^2$ is replaced by the one dimensional $C$ and in the very simple case of $A = C([0, 1])$ (the algebra of continuous complex-valued functions on the interval $[0, 1]$).

**Example 1.1.** With $A = C[0, 1]$ there is a continuous map $\Psi : s(A) \to \mathbb{C}$ such that there does not exist $a \in A$ that satisfies $\Psi(\varphi) = \varphi(a)$ for all $\varphi \in s(A)$.

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Proof. Each \( t \in [0,1] \) induces a state \( \varphi_t \) on \( \mathcal{A} \): the evaluation functional \( \varphi_t(a) = a(t) \) for all \( a \in \mathcal{A} \). Let \( a_1 \in \mathcal{A} \) be given by \( a_1(t) = t \) for all \( t \in [0,1] \). Let
\[
V = \left\{ \varphi \in s(\mathcal{A}) : \left| \varphi(a_1) - \varphi_{1/2}(a_1) \right| < \frac{1}{4} \right\},
\]
a weak* neighborhood of \( \varphi_{1/2} \). Since \( s(\mathcal{A}) \), with the relative weak* topology, being compact and Hausdorff [4, p. 257], is normal, there is a continuous map \( \Psi : s(\mathcal{A}) \to \mathbb{C} \) such that \( \Psi(\varphi_0) = 1 \) and \( \Psi(\varphi) = 0 \) for all \( \varphi \in s(\mathcal{A}) \setminus V \). In particular \( \Psi(\varphi_0) = 0 \). Suppose there is an \( a \in \mathcal{A} \) such that
\[
\Psi(\varphi) = \varphi(a) \quad \text{for all} \quad \varphi \in s(\mathcal{A}).
\]
Then \( 1 = \Psi(\varphi_{1/2}) = a(1/2) \) and \( 0 = \Psi(\varphi_0) = a(0) \). Let \( \hat{\varphi} := \frac{1}{5} \varphi_{1/2} + \frac{4}{5} \varphi_0 \). Then \( \hat{\varphi} \in s(\mathcal{A}) \) and
\[
\left| \hat{\varphi}(a_1) - \varphi_{1/2}(a_1) \right| = \left| \frac{1}{5} \varphi_{1/2}(a_1) + \frac{4}{5} \varphi_0(a_1) - \varphi_{1/2}(a_1) \right| = \frac{2}{5} > \frac{1}{4}.
\]
Thus \( \hat{\varphi} \in s(\mathcal{A}) \setminus V \), and hence,
\[
0 = \Psi(\hat{\varphi}) = \hat{\varphi}(a) = \frac{1}{5} a(1/2) + \frac{4}{5} a(0) = \frac{1}{5},
\]
which is a contradiction. \( \square \)

It will be shown in Proposition 2.1 that the image of \( \mathcal{M} \) under the map \( A \mapsto \tilde{A} \) is a closed subspace of \( C(s(\mathcal{A}), B(\ell^2)) \), and \( \mathcal{M} \) is a Banach space with the norm
\[
\|A\| = \sup_{\varphi \in s(\mathcal{A})} \| \varphi(A) \|_{B(\ell^2)}.
\]

Let \( A \in \mathcal{M} \). For each \( n \in \mathbb{N} \), \( A_n \) denotes the \( n \)-th compression matrix of \( A \); that is, the \((j,k)\)-th entry of \( A_n \) is exactly the same as that of \( A \) for \( 1 \leq j, k \leq n \), and is zero otherwise. Denote by \( A_\uparrow \) [respectively, \( A_\downarrow \)] the matrix whose first \( n \) rows [respectively, columns] coincide with that of \( A \) and all other rows [respectively, columns] are zero. Dually, \( A_\downarrow \) [respectively, \( A_\uparrow \)] is the matrix whose first \( n \) rows [respectively, columns] are zero and all other rows [respectively, columns] coincide with that of \( A \). Denote by \( \mathcal{K} \) the space of all \( A \in \mathcal{M} \) with the property that
\[
\|A - A_\uparrow\| = \|A_\downarrow\| \to 0 \quad \text{as} \quad n \to \infty.
\]
Note that this is equivalent to the compactness of \( A \) (i.e., \( A \in \mathcal{K}(\ell^2) \)) when \( \mathcal{A} \) is the complex field \( \mathbb{C} \).

We will show that the annihilator \( \mathcal{K}^\perp \) of \( \mathcal{K} \) behaves in the dual space \( \mathcal{M}^* \) of \( \mathcal{M} \) just like \([\mathcal{K}(\ell^2)]^\perp \) in \([B(\ell^2)]^*\), as in Dixmier’s theorem. That is \( \mathcal{K} \) is an \( \mathcal{M} \)-ideal in \( \mathcal{M} \).
2. Preliminary results

We begin the section by showing that \( \mathcal{M} \) is a Banach space.

**Proposition 2.1.** \( \mathcal{M} \) is a Banach space with the norm
\[
\|A\| = \sup_{\varphi \in s(A)} \|\tilde{\varphi}(A)\|_{B(\ell^2)}.
\]

The state norm \( \|\cdot\|_s \) on \( A \) is defined by
\[
\|a\|_s = \sup_{\varphi \in s(A)} |\varphi(a)|, \quad a \in A.
\]

The state norm is a norm and \( \|a\|_s \leq \|a\| \leq 2\|a\|_s \) for all \( a \in A \).

Proof. It suffices to show that the image of \( \mathcal{M} \) under the map \( A \mapsto \tilde{A} \) is closed in \( C(s(A), B(\ell^2)) \). Let \( \{A_n\} \) be a sequence in \( \mathcal{M} \) such that \( \tilde{A}_n \to \Psi \) for some \( \Psi \in C(s(A), B(\ell^2)) \). Let \( A_n = [a^{(n)}_{jk}] \). For each \( j, k \in \mathbb{N} \),
\[
\|a^{(n)}_{jk} - a^{(m)}_{jk}\|_s = \sup_{\varphi \in s(A)} |\varphi(a^{(n)}_{jk}) - \varphi(a^{(m)}_{jk})| \\
\leq \sup_{\varphi \in s(A)} \|\tilde{\varphi}(A_n) - \tilde{\varphi}(A_m)\|_{B(\ell^2)} \to 0
\]
as \( n, m \to \infty \). By the equivalence of the state norm and the norm on \( A \), the sequence \( \{a^{(n)}_{jk}\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( A \). Thus there is an \( a_{jk} \in A \) such that \( \|a^{(n)}_{jk} - a_{jk}\| \to 0 \). We also have
\[
\|\tilde{A}_n(\varphi) - \Psi(\varphi)\|_{B(\ell^2)} \to 0 \quad \text{for all } \varphi \in s(A).
\]

For each \( \varphi \in s(A) \), let \( \Psi(\varphi) = [\psi_{jk}(\varphi)] \). It follows that
\[
\varphi(a^{(n)}_{jk}) \to \psi_{jk}(\varphi) \quad \text{for all } \varphi \in s(A).
\]

But we also have
\[
\varphi(a^{(n)}_{jk}) \to \varphi(a_{jk}) \quad \text{for all } \varphi \in s(A),
\]
and hence
\[
\varphi(a_{jk}) = \psi_{jk}(\varphi) \quad \text{for all } \varphi \in s(A).
\]

Let \( A = [a_{jk}] \). Then
\[
\Psi(\varphi) = [\psi_{jk}(\varphi)] = [\varphi(a_{jk})] = \tilde{\varphi}(A) = \tilde{A}(\varphi) \quad \text{for all } \varphi \in s(A).
\]

That is \( \tilde{A} = \Psi \in C(s(A), B(\ell^2)) \), and \( A \in \mathcal{M} \). \( \square \)

Now we prove some properties of \( \mathcal{K} \) that are parallel to well-known properties of compact operators.

**Proposition 2.2.** \( \mathcal{K} \) is a closed proper subspace of \( \mathcal{M} \).
Proof. Let \( \{A_k\}_{k=1}^\infty \) be a sequence in \( \mathcal{K} \) such that \( \|A_k - A\| \to 0 \) for some \( A \in \mathcal{M} \). Let \( \epsilon > 0 \). There exists an \( N \in \mathbb{N} \) such that
\[
\|A_k - A\| < \frac{\epsilon}{4} \quad \text{for all } k \geq N.
\]
Since \( A_N \in \mathcal{K} \), there is an \( n_0 \in \mathbb{N} \) such that
\[
\left\| (A_N)_{n} - A_N \right\| < \frac{\epsilon}{4} \quad \text{for all } n \geq n_0.
\]
Let \( n \geq n_0 \).
\[
\|A_n - A\| \leq \|A_n - (A_N)_{n}\| + \|(A_N)_{n} - A_N\| + \|A_N - A\|
\leq \left\| (A_N)_{n} - A_N \right\| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \|A_N - A\| + \frac{\epsilon}{2} < \epsilon
\]
That is \( \|A - A_n\| \to 0 \) as \( n \to \infty \), and hence \( A \in \mathcal{K} \).

By definition, we have \( \mathcal{K} \subseteq \mathcal{M} \). To see that the inclusion is proper, we note that the matrix \( A \) with 1 (the identity of \( \mathcal{A} \)) on the diagonal and 0 elsewhere (i.e., \( A(j, k) = \delta_{j,k} \)) is in \( \mathcal{M} \) but not in \( \mathcal{K} \). Weak* norm continuity of the map \( \varphi \mapsto \tilde{\varphi}(A) \) follows immediately from the fact that \( \tilde{\varphi}(A) \) is the identity matrix in \( B(\ell^2) \) for each \( \varphi \in s(A) \). Thus \( A \in \mathcal{M} \). But \( \|\tilde{\varphi}(A - A_n)\| = 1 \) for all \( \varphi \in s(A) \) and all \( n \in \mathbb{N} \), which implies that \( A \notin \mathcal{K} \).

Proposition 2.3. Let \( A \in \mathcal{M} \) satisfy \( A = A_N \) (respectively, \( A = A_{\infty} \)) for some fixed \( N \in \mathbb{N} \). Then \( A \in \mathcal{K} \), and \( \|A - A_{\nu}\| \to 0 \) as \( \nu \to \infty \).

Proof. Suppose \( A = A_{\infty} \in \mathcal{M} \). For \( n \geq N \), we have \( A_n = A_{\infty} = A \). Thus \( \|A - A_n\| = 0 \) for all \( n \geq N \), and hence \( A \in \mathcal{K} \).

If \( A = A_N \in \mathcal{M} \), then the transpose of \( A \),
\[
B = A^T \quad \left( B_{jk} = (A^T)_{jk} = A_{kj} \quad \forall \ j, k \in \mathbb{N} \right),
\]
satisfies
\[
B = A^T = [A_N]^T = B_{\infty},
\]
and hence,
\[
\|B - B_n\| = 0 \quad \text{for all } n \geq N.
\]
For each \( n \geq N \) we have
\[
\|A - A_n\| = \sup_{\varphi \in s(A)} \left\| \tilde{\varphi}(A - A_n) \right\|_{B(\ell^2)} = \sup_{\varphi \in s(A)} \left\| (\tilde{\varphi}(A - A_n))^T \right\|_{B(\ell^2)}
\leq \sup_{\varphi \in s(A)} \left\| \tilde{\varphi}(A^T - (A_n)^T) \right\|_{B(\ell^2)} = \sup_{\varphi \in s(A)} \left\| \tilde{\varphi}(B - B_n) \right\|_{B(\ell^2)} = \|B - B_n\| = 0.
\]
Since \( A \) is assumed to be in \( \mathcal{M} \), this shows that \( A = A_N \in \mathcal{K} \), and hence
\[
\|A - A_{\nu}\| = \|A - A_{\infty}\| \to 0 \quad \text{as } \nu \to \infty.
\]
For the case \( A = A_N\), we see as above that \( C = A^T\) satisfies \( C = C_N \in \mathcal{K}\), and hence
\[
\|A - A_{\nu}\| = \| (A - A_{\nu})^T \| = \| C - C_{\nu}\| \rightarrow 0 \quad \text{as} \quad \nu \rightarrow \infty.
\]
\[\blacksquare\]

For each \( A = \begin{bmatrix} a_{jk} \end{bmatrix} \in \mathcal{M}, \ A^* \) is defined by
\[
(A^*)_{jk} = a^*_{kj} \quad \text{for all} \quad j, k \in \mathbb{N}.
\]
It is easy to see that \( A^* \in \mathcal{M} \) whenever \( A \in \mathcal{M} \).

**Proposition 2.4.** Let \( A = \begin{bmatrix} a_{jk} \end{bmatrix} \) be a matrix over \( \mathcal{A} \).

1. \( A \in \mathcal{K} \) iff the map \( \varphi \mapsto \tilde{\varphi}(A) \) is continuous from \( s(A) \) with the weak* topology to \( \mathcal{K}(\ell^2) \) with the operator norm topology.
2. \( A \in \mathcal{K} \) iff \( A^* = \begin{bmatrix} a_{jk}^* \end{bmatrix}^T \in \mathcal{K} \).
3. If \( A \in \mathcal{M}, \) then \( A \in \mathcal{K} \) iff
\[
\|A - A_n\| = \| A_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

**Proof.**
(1) \( \Rightarrow \) Suppose \( A \in \mathcal{K} \). Then \( A \in \mathcal{M} \). Thus \( \varphi \mapsto \tilde{\varphi}(A) \) is continuous from \( s(A) \) with weak* topology to \( \mathcal{B}(\ell^2) \) with norm topology. It suffices to show that \( \tilde{\varphi}(A) \in \mathcal{K}(\ell^2) \) for all \( \varphi \in s(A) \). Let \( \varphi \in s(A) \). We have
\[
\|\tilde{\varphi}(A) - [\tilde{\varphi}(A)]_n\|_{\mathcal{B}(\ell^2)} = \|\tilde{\varphi}(A - A_n)\|_{\mathcal{B}(\ell^2)} \leq \|A - A_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]
and hence \( \tilde{\varphi}(A) \in \mathcal{K}(\ell^2) \).

(1) \( \Leftarrow \) Let \( \epsilon > 0 \). By continuity, for each \( \varphi \in s(A) \), there is a weak* open set \( V_\varphi \subseteq s(A) \) such that
\[
\varphi \in V_\varphi \quad \text{and} \quad \|\tilde{\varphi}(A) - \tilde{\psi}(A)\|_{\mathcal{K}(\ell^2)} = \|\tilde{\varphi}(A) - \tilde{\psi}(A)\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{4} \quad \forall \quad \psi \in V_\varphi.
\]
Since \( s(A) \) with the weak* topology is a compact Hausdorff space [4, p. 257], and
\[
s(A) \subseteq \bigcup_{\varphi \in s(A)} V_\varphi,
\]
there are \( \varphi_1, \ldots, \varphi_k \in s(A) \) such that
\[
s(A) \subseteq \bigcup_{j=1}^k V_{\varphi_j}.
\]
For each \( j = 1, \ldots, k \), since \( \tilde{\varphi}_j(A) \in \mathcal{K}(\ell^2) \), there is an \( N_j \in \mathbb{N} \) such that
\[
\|\tilde{\varphi}_j(A) - [\tilde{\varphi}_j(A)]_n\|_{\mathcal{B}(\ell^2)} = \|\tilde{\varphi}_j(A) - [\tilde{\varphi}_j(A)]_n\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{4} \quad \text{for all} \quad n \geq N_j.
\]
Put $N = \max \left\{ N_j : j = 1, \ldots, k \right\}$. Then for $n \geq N$ and $\varphi \in s(A)$, we have $\varphi \in V_{\epsilon_j}$ for some $j = 1, \ldots, k$, and thus

$$\|\tilde{\varphi}(A) - \tilde{\varphi}(A_n)\|_{b(\ell^2)} \leq \|\tilde{\varphi}(A) - \tilde{\varphi}_j(A)\|_{b(\ell^2)} + \|\tilde{\varphi}_j(A) - \tilde{\varphi}(A_n)\|_{b(\ell^2)} < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \|\tilde{\varphi}_j(A) - \tilde{\varphi}(A)\|_{b(\ell^2)} < \frac{3\epsilon}{4}.$$  

Since $\varphi \in s(A)$ is arbitrary,

$$\|A - A_n\| = \sup_{\varphi \in s(A)} \|\tilde{\varphi}(A - A_n)\|_{b(\ell^2)} = \sup_{\varphi \in s(A)} \|\tilde{\varphi}(A) - \tilde{\varphi}(A_n)\|_{b(\ell^2)} \leq \frac{3\epsilon}{4} < \epsilon$$

for all $\psi \in U_{\epsilon}$, $\tilde{\psi}(A) \in K$ and $\|\tilde{\varphi}(A) - \tilde{\psi}(A)\|_{b(\ell^2)} < \epsilon$.

Since $\tilde{\psi}$ is a positive linear functional, $\tilde{\psi}(\alpha^*) = \overline{\tilde{\psi}(\alpha)}$ for all $\alpha \in A$ [1, p. 255]. From $\tilde{\psi}(A) \in K(\ell^2)$, we have $\tilde{\psi}(A^*) = [\tilde{\psi}(A)]^* \in K(\ell^2)$, and

$$\|\tilde{\varphi}(A^*) - \tilde{\psi}(A^*)\|_{b(\ell^2)} = \|\tilde{\varphi}(A)^* - [\tilde{\psi}(A)]^*\|_{b(\ell^2)} = \|\tilde{\varphi}(A) - \tilde{\psi}(A)\|_{b(\ell^2)} < \epsilon.$$  

Thus the map $\varphi \mapsto \tilde{\varphi}(A^*)$ is continuous from $s(A)$ with weak* topology to $K(\ell^2)$ with norm topology. Hence $A^* \in K$ by part (1).

(2) $\Rightarrow$ Suppose that $A^* \in K$. Then $A = (A^*)^* \in K$.

(3) $\Rightarrow$ Suppose $A \in K$. Then $A^* \in K$ and hence

$$\|A^* - (A_n^*)_n\| \to 0 \quad \text{as} \quad n \to \infty.$$  

Thus

$$\|A - A_n\| = \|(A - A_n)^*\| = \|A^* - (A^*)_n\| \to 0 \quad \text{as} \quad n \to \infty.$$  

(3) $\Leftarrow$ Suppose $\|A - A_n\| \to 0$. Since each $A_n \in K$ by Proposition 2.3, and since $K$ is closed under the operator norm, $A \in K$. $\Box$

3. The Dual of $K$

In this section we will obtain a functional matrix representation of the dual $K^*$ of $K$. First note that for $A = [a_{jk}] \in M$, and each $j, k \in \mathbb{N}$, we have

$$\|a_{jk}\| \leq 2\|a_{jk}\|_{s} = 2 \sup_{\varphi \in s(A)} |\varphi(a_{jk})| \leq 2 \sup_{\varphi \in s(A)} \|\tilde{\varphi}(A)\|_{b(\ell^2)} = 2\|A\|.$$  

We will need the following lemma in the proofs of Propositions 3.2 and 3.3.
Lemma 3.1. Let $\{f_n\}$ be a sequence in the dual space $X^*$ of a Banach space $X$ such that $f(x) = \sum_{k=1}^{\infty} f_k(x)$ converges for all $x \in X$. Then $f \in X^*$.

Proof. A routine argument shows that $f$ is linear. For the boundedness of $f$, let $g_n = \sum_{k=1}^{n} f_k$ for each $n \in \mathbb{N}$. Then $g_n \in X^*$. For each $x \in X$, since $\sum_{k=1}^{\infty} f_k(x)$ converges, there is an $\alpha_x \geq 0$ such that $|g_n(x)| \leq \alpha_x$ for all $n \in \mathbb{N}$. So $\{g_n\}$ is a sequence in $X^*$ that is pointwise bounded. The uniform boundedness principle implies that $\{g_n\}$ is uniformly bounded; i.e., there is a $\beta$ such that $\|g_n\| \leq \beta$ for all $n \in \mathbb{N}$. For each $x \in X$, we have

$$|f(x)| = \lim_{n \to \infty} \left| \sum_{k=1}^{n} f_k(x) \right| = \lim_{n \to \infty} |g_n(x)| \leq \limsup_{n \to \infty} \|g_n\| \|x\| \leq \beta \|x\|.$$ 

Thus $f \in X^*$ with $\|f\| \leq \beta$. □

Proposition 3.2. For each $f \in K^*$, there exists a unique matrix $[f_{jk}]$, with $f_{jk} \in A^*$, such that

$$f(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{for all} \quad A = [a_{jk}] \in K.$$ 

Conversely, each matrix $[g_{jk}]$ over $A^*$ with the property that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \quad \text{converges for every} \quad A = [a_{jk}] \in K,$$

defines a bounded linear functional

$$g(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \quad (A = [a_{jk}] \in K) \quad \text{on} \quad K.$$ 

Moreover, in this case,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \quad \text{converges, and},$$

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{jk}(a_{jk}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \quad \text{for all} \quad [a_{jk}] \in K.$$ 

Thus $K^*$ is identified with the space of all such matrices. The norm of such a matrix is defined to be the norm of the bounded linear functional it represents, i.e., $\|[f_{jk}]\| = \|f\|$ if $[f_{jk}]$ represents $f \in K^*$.

Proof. Let $f \in K^*$. For each $(j,k) \in \mathbb{N} \times \mathbb{N}$ and each $a \in A$, since the matrix $E_{jk}(a)$ with $(j,k)$ entry $a$ and all others 0 is easily seen from Proposition 2.3 to be in $K$ with

$$\|E_{jk}(a)\| = \|a\|_s \leq \|a\|,$$
we define $f_{jk}$ by

$$f_{jk}(a) = f(E_{jk}(a)) \quad \text{for all } a \in A.$$ 

It is readily seen that $f_{jk}$ is linear, and

$$|f_{jk}(a)| = |f(E_{jk}(a))| \leq ||f|| \|E_{jk}(a)\| \leq ||f|| \|a\|.$$ 

Hence $f_{jk} \in \mathcal{A}^\#$ with $||f_{jk}|| \leq ||f||$. Let $A = [a_{jk}] \in \mathcal{K}$. For each $n \in \mathbb{N}$, $A_n \in \mathcal{K}$, and, by Proposition 2.3,

$$\|A_n - [A_n]_{\nu} \| \to 0 \quad \text{as } \nu \to \infty.$$ 

Thus, by linearity,

$$\sum_{j=1}^{n} \sum_{k=1}^{\nu} f_{jk}(a_{jk}) = f([A_n]_{\nu}) \to f(A_n) \quad \text{as } \nu \to \infty.$$ 

That is

$$f(A_n) = \sum_{j=1}^{n} \sum_{k=1}^{\infty} f_{jk}(a_{jk}).$$ 

Since $\|A - A_n\| \to 0$ as $n \to \infty$, $f(A_n) \to f(A)$, and hence

$$f(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}).$$ 

Now suppose $[g_{jk}]$ is a matrix over $\mathcal{A}^\#$ such that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \text{ converges for every } A = [a_{jk}] \in \mathcal{K}.$$ 

For each fixed $m, n \in \mathbb{N}$, define $\hat{g}_{mn} : \mathcal{K} \to \mathbb{C}$ by

$$\hat{g}_{mn}(A) = g_{mn}(a_{mn}) \quad \text{for each } A = [a_{jk}] \in \mathcal{K}.$$ 

Then

$$|\hat{g}_{mn}(A)| \leq \|g_{mn}\| \|a_{mn}\| \leq 2 \|A\| \|g_{mn}\|$$

i.e., $\hat{g}_{mn} \in \mathcal{K}^\#$. Since by assumption

$$g_m(A) := \sum_{k=1}^{\infty} \hat{g}_{mk}(A) = \sum_{k=1}^{\infty} g_{mk}(a_{mk}) \text{ converges for every } A = [a_{jk}] \in \mathcal{K},$$

by Lemma 3.1, $g_m \in \mathcal{K}^\#$. Since we also assume that

$$g(A) := \sum_{m=1}^{\infty} g_m(A) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} g_{mk}(a_{mk}) \text{ converges for every } A = [a_{jk}] \in \mathcal{K},$$

by Lemma 3.1 again, the functional $g$ is bounded, i.e., $g \in \mathcal{K}^\#$.

For each $A = [a_{jk}] \in \mathcal{K}$, since the matrix $A_{[kj]} = A_{[kj]} - A_{[k-1]j}$, with the $k$-th column the same as that of $A$ and all others 0, is in $\mathcal{K}$,

$$\sum_{j=1}^{\infty} g_{jk}(a_{jk}) = g(A_{[kj]}) \text{ converges, for all } k \in \mathbb{N}. $$
Since \( \|A - A_m\| \to 0 \) as \( m \to \infty \),
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) = g(A) = \lim_{m \to \infty} g(A_m) = \lim_{m \to \infty} \left[ \sum_{k=1}^{m} g(A_k) \right]
\]
\[
= \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \sum_{j=1}^{\infty} g_{jk}(a_{jk}) \right] = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{jk}(a_{jk}).
\]

Next we show that if \( \left[ f_{jk} \right] \in \mathcal{K}^\# \), then the two double sums both converge and are equal for each \( A = [a_{jk}] \in \mathcal{M} \), not just for elements in \( \mathcal{K} \).

**Proposition 3.3.** For each \( f = \left[ f_{jk} \right] \in \mathcal{K}^\# \) and each \( A = [a_{jk}] \in \mathcal{M} \), both
\[
\hat{f}(A) := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{and} \quad g(A) := \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f_{jk}(a_{jk})
\]
converge, and they have the same sum. Furthermore \( \hat{f} \) is a bounded linear functional on \( \mathcal{M} \) with norm \( \| \hat{f} \|_\mathcal{M}^\# = \| f \|_{\mathcal{K}^\#} \).

**Proof.** Let \( A = [a_{jk}] \in \mathcal{M} \). Then for each \( j \in \mathbb{N} \), the row \( j \) matrix \( A_j = A_{j-1} \in \mathcal{K} \). Thus
\[
\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ converges for every } j \in \mathbb{N}.
\]

Suppose
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ does not converge.}
\]

Then there are an \( \epsilon > 0 \) and two sequences \( \{j_\nu\}, \{l_\nu\} \) in \( \mathbb{N} \) such that
\[
1 \leq j_1 < l_1 < j_2 < l_2 < \ldots < j_\nu < l_\nu < \ldots, \quad \text{and}
\]
\[
\left| \sum_{j=j_\nu}^{l_\nu} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \right| > \epsilon \quad \text{for all } \nu \in \mathbb{N}.
\]

Let \( A_\nu = A_{l_\nu} - A_{j_\nu-1} \), the matrix whose rows from \( j_\nu \)-th through \( l_\nu \)-th coincide with that of \( A \) and all others are 0; let
\[
\alpha_\nu = \frac{1}{\nu} \text{ sgn} \left[ \sum_{k=1}^{l_\nu} \sum_{j=j_\nu}^{l_\nu} f_{jk}(a_{jk}) \right] ; \quad \text{and} \quad B = \sum_{\nu=1}^{\infty} \alpha_\nu A_\nu.
\]

We show that \( B \in \mathcal{K} \) but the sum for \( f(B) \) diverges. Let \( \eta > 0 \). There is a \( \nu_0 \in \mathbb{N} \) such that
\[
\sum_{\nu=\nu_0}^{\infty} \frac{\|A\|^2}{\nu^2} < \frac{\eta^2}{4}.
\]
For \( n \geq j_{\nu_0}, \varphi \in s(A) \), and \( x = \{x_k\} \in \ell^2 \), let \( \nu_i \) be the largest \( \nu \) such that \( j_{\nu} \leq n \). Thus \( \nu_1 \geq \nu_0 \), and hence,

\[
\| \tilde{\varphi}(B - B_{\nu_0}) x \|_{\ell^2}^2 = \| [\tilde{\varphi}(B) - \tilde{\varphi}(B_{\nu_0})] x \|_{\ell^2}^2
\]

\[
= \sum_{j=n+1}^{\nu_1} \left| \alpha_{\nu} \sum_{k=1}^{\infty} \varphi(a_{jk}) x_k \right|^2 + \sum_{\nu=\nu_1+1}^{\nu_2} \| \alpha_{\nu} \sum_{j=j_{\nu}}^{\infty} \sum_{k=1}^{\infty} \varphi(a_{jk}) x_k \|^2
\]

\[
\leq \frac{1}{\nu_1} \sum_{j=1}^{\nu_1} \left| \sum_{k=1}^{\infty} \varphi(a_{jk}) x_k \right|^2 + \sum_{\nu=\nu_1+1}^{\nu_2} \frac{1}{\nu_2} \sum_{j=1}^{\nu_2} \left| \sum_{k=1}^{\infty} \varphi(a_{jk}) x_k \right|^2
\]

\[
\leq \left\| A \right\|^2 \| x \|_{\ell^2}^2 + \sum_{\nu=\nu_1+1}^{\nu_2} \frac{\| A \|^2}{\nu_2} \| x \|_{\ell^2}^2 < \frac{\eta^2}{4} \| x \|_{\ell^2}^2.
\]

Since this is true for all \( x \in \ell^2 \), we see that

\[
\| \tilde{\varphi}(B - B_{\nu_0}) \|_{\ell^2(\ell^2)} \leq \frac{\eta}{2}.
\]

But \( \varphi \in s(A) \) is also arbitrary,

\[
\| B - B_{\nu_0} \| \leq \frac{\eta}{2} < \eta.
\]

Since this is true for all \( n \geq j_{\nu_0} \), we conclude that \( B \in \mathcal{K} \).

On the other hand we also have

\[
f(B) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{\nu=1}^{\infty} \alpha_{\nu} \sum_{j=j_{\nu}}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk})
\]

\[
= \sum_{\nu=1}^{\infty} \frac{1}{\nu} \left| \sum_{j=j_{\nu}}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \right| \geq \sum_{\nu=1}^{\infty} \frac{\epsilon}{\nu} = \infty,
\]

contradicting \( B \in \mathcal{K} \) and \( f \in \mathcal{K}^\# \). Therefore

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ converges.}
\]

A similar argument shows that the sum in the other order for \( g \) also converges.

Uniform boundedness arguments similar to that used in the proof of Proposition 3.2 show that \( \hat{f} \) and \( g \) are both bounded linear functionals on \( \mathcal{M} \).

For \( A \in \mathcal{M} \), since \( A_n \in \mathcal{K} \), for each \( n \in \mathbb{N} \), by last part of the preceding proposition,

\[
|g(A)| = \lim_{n \to \infty} |g(A_n)| = \lim_{n \to \infty} |f(A_n)| \leq \lim_{n \to \infty} \| f \| \| A_n \| \leq \| f \| \| A \|,
\]

\[
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\]
thus \( \|g\| \leq \|f\| \). Also \( g|_{\mathcal{K}} = f \), we see that \( \|g\| \geq \|f\| \), and thus \( \|f\| = \|g\| \). Similarly \( \|\hat{f}\| = \|f\| \).

To see that the two sums are equal, we first show that the sequence \( \{g_n\} \) defined by

\[
   g_n(A) := \sum_{k=1}^{n} \sum_{j=1}^{\infty} f_{jk}(a_{jk}) \quad (A = [a_{jk}] \in \mathcal{K})
\]

is a Cauchy sequence in \( \mathcal{K}^* \). Suppose \( \{g_n\} \) is not a Cauchy sequence in \( \mathcal{K}^* \). Then there exist an \( \epsilon > 0 \) and sequences \( \{k_\nu\}_{\nu \in \mathbb{N}}, \{l_\nu\}_{\nu \in \mathbb{N}} \in \mathbb{N} \) such that

\[
   l_{\nu-1} + 1 \leq k_\nu < l_\nu \quad (\text{where } l_0 = 0), \quad \left\| g_{l_\nu} - g_{k_\nu} \right\| > 2\epsilon \quad \text{for all } \nu \in \mathbb{N}.
\]

Thus there are elements \( A_\nu \in \mathcal{K} \) such that \( \|A_\nu\| = 1 \) and \( \left| g_{l_\nu}(A_\nu) - g_{k_\nu}(A_\nu) \right| > 2\epsilon \).

Let

\[
   \alpha_\nu = \frac{1}{\nu} \text{ sgn} \left[ g_{l_\nu}(A_\nu) - g_{k_\nu}(A_\nu) \right] \quad \text{and} \quad B = \sum_{\nu=1}^{\infty} \alpha_\nu A_\nu.
\]

Then an argument similar to that used above shows that

\[
   B \in \mathcal{K} \quad \text{ but } \quad g(B) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f_{jk}(B(j,k)) \text{ diverges,}
\]

which is a contradiction. Therefore \( \{g_n\} \) is a Cauchy sequence in \( \mathcal{K}^* \). Thus there is an \( h \in \mathcal{K}^* \) such that

\[
   \|g_n - h\|_{\mathcal{K}^*} \to 0.
\]

But since each \( A \in \mathcal{K} \) has \( \|A - A_n\| \to 0 \), also \( g \in \mathcal{K}^* \) and \( g_n(A) = g(A_n) \), we have

\[
   g_n(A) \to g(A) \quad \text{for each } A \in \mathcal{K}.
\]

Thus \( g = h \) and hence

\[
   \|g_n - g\|_{\mathcal{K}^*} \to 0.
\]

For each \( A = [a_{jk}] \in \mathcal{M} \), since

\[
   \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{and} \quad \sum_{j=1}^{n} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ converge for all } n \in \mathbb{N},
\]

\[
(\hat{f} - g_n)(A) = \hat{f}(A) - g_n(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) - \sum_{j=1}^{n} \sum_{k=1}^{\infty} f_{jk}(a_{jk})
\]

\[
= \sum_{j=1}^{\infty} \sum_{k=n+1}^{\infty} f_{jk}(a_{jk}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \hat{f}_{jk}(a_{jk})
\]

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where
\[ \tilde{f}_{jk} = \begin{cases} f_{jk} & \text{for } k > n \\ 0 & \text{otherwise.} \end{cases} \]

Notice that \((\hat{f} - g_n) = \hat{f} - g_n\), and, by Proposition 3.2, that \(f = g\) on \(\mathcal{K}\). Thus, from the first part, we have
\[
\lim_{n \to \infty} \| \hat{f} - g_n \|_{\mathcal{M}^*} = \lim_{n \to \infty} \| (\hat{f} - g_n) \| = \lim_{n \to \infty} \| f - g_n \|_{\mathcal{K}^*} = \lim_{n \to \infty} \| g - g_n \|_{\mathcal{K}^*} = 0.
\]

Therefore
\[
\hat{f}(A) = \lim_{n \to \infty} g_n(A) \quad \text{for all } A \in \mathcal{M},
\]
and hence, for each \(A = [a_{jk}] \in \mathcal{M}\),
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) = \hat{f}(A) = \lim_{n \to \infty} g_n(A) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f_{jk}(a_{jk}).
\]

Note that this proposition corresponds to the fact that the trace functional satisfies \(\text{tr}(AB) = \text{tr}(BA)\) for a trace class \(A\) and bounded \(B\) on a Hilbert space.

The proof of this proposition can easily be adapted to a proof of the trace identity. Since each \([f_{jk}] \in \mathcal{K}^*\) defines a bounded linear functional \(\hat{f}_{jk} = f_{jk}|_{\mathcal{K}}\) on \(\mathcal{K}\), then as in the proof of Proposition 3.2 the matrix \([f_{jk}]\) represents a bounded linear functional \(\hat{f} = f|_{\mathcal{K}}\) on \(\mathcal{K}\). By Proposition 3.3, \([f_{jk}]\) defines a bounded linear functional \(g = \hat{f} \) on \(\mathcal{M}\), where
\[
g(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{for all } A = [a_{jk}] \in \mathcal{M},
\]

4. The main theorem

Now we are ready for the main Dixmier’s theorem. Denote by \(\mathcal{K}^\perp\) the subspace of \(\mathcal{M}^*\) consisting of bounded linear functionals on \(\mathcal{M}\) that vanish on \(\mathcal{K}\).

**Theorem 4.1.** For each \(f \in \mathcal{M}^*\), there is a unique pair \(g \in \hat{\mathcal{K}}^*\) and \(h \in \mathcal{K}^\perp\) such that
\[ f = g + h \quad \text{and} \quad \| f \| = \| g \| + \| h \|. \]

**Proof.** For each \((j, k) \in \mathbb{N} \times \mathbb{N}\), define \(f_{jk}\) by \(f_{jk}(a) = f(E_{jk}(a))\) for all \(a \in \mathcal{A}\). Then \(f_{jk} \in \mathcal{A}^*\) with \(\| f_{jk} \| \leq \| f \|\). Then as in the proof of Proposition 3.2 the matrix \([f_{jk}]\) represents a bounded linear functional \(\tilde{f} = f|_{\mathcal{K}}\) on \(\mathcal{K}\). By Proposition 3.3, \([f_{jk}]\) defines a bounded linear functional \(g = \hat{f} \) on \(\mathcal{M}\), where
\[
g(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{for all } A = [a_{jk}] \in \mathcal{M},
\]
and
\[ \| g \|_{\mathcal{M}^*} = \| \tilde{f} \|_{K^*}. \]

Let \( h = f - g \). It is clear that \( h \in K^\perp \). The uniqueness of the decomposition follows from the fact that \( \mathcal{K}^* \oplus K^\perp = \mathcal{M}^* \) is a direct sum.

Since \( \| f \| \leq \| g \| + \| h \| \), it suffices to prove that \( \| f \| \geq \| g \| + \| h \| \). Let \( \epsilon > 0 \).

Since \( \| g \|_{\mathcal{M}^*} = \| g \|_K \), there is an \( A = [a_{jk}] \in \mathcal{K} \) such that
\[ \| A \| = 1 \quad \text{and} \quad g(A) > \| g \| - \frac{\epsilon}{8}. \]

There is also a \( B = [b_{jk}] \in \mathcal{M} \) such that
\[ \| B \| = 1 \quad \text{and} \quad h(B) > \| h \| - \frac{\epsilon}{8}. \]

Form the convergence of the double sum, there is a \( j_0 \) such that
\[ \left| \sum_{j=n}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \right| < \frac{\epsilon}{8} \quad \forall \quad n > j_0. \]

There is also a \( k_0 \) such that
\[ \left| \sum_{j=1}^{j_0} \sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk}) \right| < \frac{\epsilon}{8}. \]

By Proposition 3.3,
\[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f_{jk}(b_{jk}), \]
thus there is a \( j_1 \geq j_0 \) such that
\[ \left| \sum_{j=j_1+1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(b_{jk}) \right| < \frac{\epsilon}{8}. \]

Put
\[ \hat{f}_{jk} = \begin{cases} 0 & \text{if } 1 \leq j \leq j_1 \\ f_{jk} & \text{if } j_1 < j \end{cases} \]

Then \( [\hat{f}_{jk}] \in \mathcal{K}^* \). Thus
\[ \sum_{j=j_1+1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \hat{f}_{jk}(b_{jk}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \hat{f}_{jk}(b_{jk}) = \sum_{k=1}^{\infty} \sum_{j=j_1+1}^{\infty} f_{jk}(b_{jk}) \]
converges, and hence there is a $k_i \geq k_0$ such that

$$\left| \sum_{k=k_i+1}^{\infty} \sum_{j=j_i+1}^{\infty} f_{jk}(b_{jk}) \right| < \frac{\epsilon}{8}.$$ 

Let

$$A_o(j, k) = \begin{cases} a_{jk} & \text{if } 1 \leq j \leq j_0, \text{ and } 1 \leq k \leq k_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_o(j, k) = \begin{cases} b_{jk} & \text{if } j_1 < j, \text{ and } k_1 < k \\ 0 & \text{otherwise} \end{cases}$$

and let $C = [c_{jk}] = A_o + B_o$. Then $\|C\| = \max \{\|A_o\|, \|B_o\|\} \leq 1$. Since $h \in \mathcal{K}^\perp$, and $A_o, B - B_o \in \mathcal{K}$, we have $h(A_o) = 0$, and hence $h(B) = h(B_o)$. Therefore

$$\|f\| \geq |f(C)| = |g(A_o) + g(B_o) + h(A_o) + h(B_o)|$$

$$\geq |g(A_o) + h(B_o)| - |g(B_o)| > \text{Re}[g(A_o)] + \text{Re}[h(B_o)] - \frac{\epsilon}{8}$$

$$= \text{Re} \left[ g(A) - \sum_{j=j_0+1}^{\infty} \sum_{k=1}^{k_0+1} f_{jk}(a_{jk}) - \sum_{j=1}^{j_0} \sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk}) \right] + h(B) - \frac{\epsilon}{8}$$

$$> \|g\| - \frac{\epsilon}{8} - \left| \sum_{j=j_0+1}^{\infty} \sum_{k=1}^{k_0+1} f_{jk}(a_{jk}) \right| - \left| \sum_{j=1}^{j_0} \sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk}) \right| + \|h\| - \frac{\epsilon}{4}$$

$$> \|g\| + \|h\| - \frac{5\epsilon}{8} > \|g\| + \|h\| - \epsilon.$$ 

Since the preceding argument holds for every $\epsilon > 0$, we conclude that

$$\|f\| \geq \|g\| + \|h\|.$$

We note that when $\mathcal{A}$ is the complex field $\mathbb{C}$, then $s(\mathcal{A})$ consists of the identity map alone. So a matrix $A$ over $\mathbb{C}$ is in $\mathcal{M}$ iff $A$ is in $\mathcal{B}(\ell^2)$ and $A$ is in $\mathcal{K}$ iff $A$ is in $\mathcal{K}(\ell^2)$. A matrix defines a bounded linear functional on $\mathcal{K}(\ell^2)$ iff it is represented by a trace class matrix and hence it is a trace class matrix itself. Thus Dixmier’s Theorem is an immediate consequence of this result.

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References


1 Department of Mathematics, Statistics and Computer, Faculty of Science, Ubon Ratchathani University, Ubon Ratchathani 34190, Thailand.
E-mail address: scittawo@ubu.ac.th, ma_pintto@yahoo.com

2 Department of Mathematics, Central Michigan University, Mt. Pleasant, MI 48859, USA.
E-mail address: ongl@cmich.edu, ong3pf@gmail.com