لینک های مفید

- عضویت در خبرنامه
- کارگاه های آموزشی
- سرویس ترجمه تخصصی STRS
- فیلم های آموزشی
- بلاگ
- مرکز اطلاعات علمی

40% تخفیف به مناسبت سالروز تاسیس مرکز اطلاعات علمی
Domination and Signed Domination Number of Cayley Graphs

Ebrahim Vatandoost, Fatemeh Ramezani*

Department of Basic Science, Imam Khomeini International University, Qazvin, Iran.
E-mail: vatandoost@sci.ikiu.ac.ir
E-mail: ramezani@ikiu.ac.ir

Abstract. In this paper, we investigate domination number as well as signed domination numbers of Cayley(G : S) for all cyclic group G of order n, where \( n \in \{p^m, pq\} \) and \( S = \{k < n : \gcd(k, n) = 1\} \). We also introduce some families of connected regular graphs \( \Gamma \) such that \( \gamma_s(\Gamma) \in \{2, 3, 4, 5\} \).

Keywords: Cayley graph, Cyclic group, Domination number, Signed domination number.

2000 Mathematics subject classification: 05C69, 05C25

1. Introduction

By a graph \( \Gamma \) we mean a simple graph with vertex set \( V(\Gamma) \) and edge set \( E(\Gamma) \). A graph is said to be connected if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining \( v_i \) and \( v_j \) is called the distance between \( v_i \) and \( v_j \) and denoted by \( d(v_i, v_j) \). A graph \( \Gamma \) is said to be regular of degree \( k \) or, \( k \)-regular if every vertex has degree \( k \). A subset \( P \) of vertices of \( \Gamma \) is a \( k \)-packing if \( d(x, y) > k \) for all pairs of distinct vertices \( x \) and \( y \) of \( P \) [9].

*Corresponding Author

Received 20 April 2016; Accepted 14 January 2017
©2019 Academic Center for Education, Culture and Research TMU
Let $G$ be a non-trivial group, $S$ be an inverse closed subset of $G$ which does not contain the identity element of $G$, i.e. $S = S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph of $G$ denoted by $\text{Cay}(G : S)$, is a graph with vertex set $G$ and two vertices $a$ and $b$ are adjacent if and only if $ab^{-1} \in S$. The Cayley graph $\text{Cay}(G : S)$ is connected if and only if $S$ generates $G$.

A set $D \subseteq V$ of vertices in a graph $\Gamma$ is a dominating set if every vertex $v \in V$ is an element of $D$ or adjacent to an element of $D$. The domination number $\gamma(\Gamma)$ of a graph $\Gamma$ is the minimum cardinality of a dominating set of $\Gamma$.

For a vertex $v \in V(\Gamma)$, the closed neighborhood $N[v]$ of $v$ is the set consisting $v$ and all of its neighbors. For a function $f : V(\Gamma) \to \{-1, 1\}$ and a subset $W$ of $V$ we define $f(W) = \sum_{u \in W} f(u)$. A signed dominating function of $\Gamma$ is a function $f : V(\Gamma) \to \{-1, 1\}$ such that $f(N[v]) > 0$ for all $v \in V(\Gamma)$. The weight of a function $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The signed domination number $\gamma_s(\Gamma)$ is the minimum weight of a signed dominating function of $\Gamma$. A signed dominating function of weight $\gamma_s(\Gamma)$ is called a $\gamma_s(\Gamma)$-function. We denote $f(N[v])$ by $f[v]$. Also for $A \subseteq V(\Gamma)$ and signed dominating function $f$, set $\{v \in A : f(v) = -1\}$ is denoted by $A^{-}_f$.

Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance [2, 3, 5, 6, 11, 13])

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders $p^n, pq$, where $p$ and $q$ are prime numbers.

2. Cayley Graphs of Order $p^n$

In this section $p$ is a prime number and $B(1, n) = \{k < n : \gcd(k, n) = 1\}$.

**Lemma 2.1.** Let $G$ be a group and $H$ be a proper subgroup of $G$ such that $[G : H] = t$. If $S = G \setminus H$, then $\text{Cay}(G : S)$ is a complete $t$-partite graph.

**Proof.** One can see $G = \langle S \rangle$ and $e \notin S = S^{-1}$. Let $a \in G$. If $x, y \in Ha$, then $x = h_1a$, $y = h_2a$. Since $xy^{-1} \in H$, $xy \notin E(\text{Cay}(G : S))$. So induced subgraph on every coset of $H$ is empty. Let $Ha$ and $Hb$ two disjoint cosets of $H$ and $x \in Ha$, $y \in Hb$. Hence, $xy^{-1} \in S$. So $xy \notin E(\text{Cay}(G : S))$. Therefore, $\text{Cay}(G : S) = K_{|H|,|H|,\ldots,|H|}$. □

**Lemma 2.2.** Let $G$ be a group of order $n$ and $G = \langle S \rangle$, where $S = S^{-1}$ and $0 \notin S$. Then $\gamma(\text{Cay}(G : S)) = 1$ if and only if $S = G \setminus \{0\}$.

**Proof.** The proof is straightforward. □
Theorem 2.3. [13] Let \( K_{a,b} \) be a complete bipartite graph with \( b \leq a \). Then
\[
\gamma_S(K_{a,b}) = \begin{cases} 
  a + 1 & \text{if } b = 1, \\
  b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even,} \\
  b + 1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd,} \\
  4 & \text{if } b \geq 4 \text{ and } a, b \text{ are both even,} \\
  6 & \text{if } b \geq 4 \text{ and } a, b \text{ are both odd,} \\
  5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity.}
\end{cases}
\]

Theorem 2.4. Let \( \mathbb{Z}_{2^n} = \langle S \rangle \) and \( S = B(1, 2^n) \). Then
i. \( \text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^n-1, 2^{n-1}} \)
ii. \( \gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2 \).
iii. \( \gamma_s(\text{Cay}(\mathbb{Z}_{2^n} : S)) = \begin{cases} 
  2 & \text{if } n = 1, 2, \\
  4 & \text{if } n \geq 3.
\end{cases} \)

Proof. i. Let \( H = \mathbb{Z}_{2^n} \setminus S \). Then \( H = \{ i : 2 \mid i \} \). It is not hard to see that \( H \) is a subgroup of \( \mathbb{Z}_{2^n} \) and \( [\mathbb{Z}_{2^n} : H] = 2 \). Hence, by Lemma 2.1, \( \text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^n-1, 2^{n-1}} \).
ii. By part i. \( \text{Cay}(\mathbb{Z}_{2^n} : S) \) is a complete bipartite graph. So \( \gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2 \).
iii. The proof is straightforward by Theorem 2.3.

Corollary 2.5. For any integer \( n > 2 \), there is a \( 2^{n-1} \)-regular graph \( \Gamma \) with \( 2^n \) vertices such that \( \gamma_S(\Gamma) = 4 \).

Theorem 2.6. Let \( \mathbb{Z}_{p^n} = \langle S \rangle \) (\( p \) odd prime) and \( S = B(1, p^n) \). Then following statements hold:

i. \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph.
ii. \( \gamma(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 2 \).
iii. \( \gamma_s(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 3 \).

Proof. i. Let \( H = \mathbb{Z}_{p^n} \setminus S \). Then \( H = \{ i : p \mid i \} \). \( H \) is a subgroup of \( \mathbb{Z}_{p^n} \) and \( |H| = p^n - \Phi(p^n) = p^{n-1} \). So \( [\mathbb{Z}_{p^n} : H] = p \). Hence, by Lemma 2.1, \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph of size \( p^{n-1} \).
ii. Since \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph, \( D = \{ a, b \} \) is a minimal dominating set where \( a, b \) are not in the same partition.
iii. Let \( \Gamma = \text{Cay}(\mathbb{Z}_{p^n} : S) \). Let \( V(\Gamma) = \bigcup_{i=1}^{p} A_i \) where \( A_i = \{ v_{ij} : 1 \leq j \leq p^{n-1} \} \). Define \( f : V(\Gamma) \to \{-1, 1\} \)
\[
f(v_{ij}) = \begin{cases} 
  -1 & \text{if } 1 \leq i \leq \frac{p}{2} \text{ and } 1 \leq j \leq \frac{p^{n-1}}{2}, \\
  -1 & \text{if } \frac{p}{2} \leq i \leq p \text{ and } 1 \leq j \leq \frac{p^{n-1}}{2}, \\
  1 & \text{otherwise.}
\end{cases}
\]
Let $v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor} A_i$. So $|N(v) \cap V_f^-| = \frac{1}{2}(p^n - p^{n-1} - 4)$. So $f[v] = f(v) + 4 \geq 3$. If $v \in \bigcup_{i=1}^{p} A_i$, then $|N(v) \cap V_f^-| = \frac{1}{2}(p^n - p^{n-1} - 2)$. So $f[v] = f(v) + 2 \geq 1$. Hence, $f$ is a signed dominating function. Since $|V_f^-| = \frac{1}{2}(p^n - 3)$, $\omega_f = 3$. So $\gamma_{S}(\Gamma) \leq 3$. On the contrary, suppose $\gamma_{S}(\Gamma) < 3$. So there is a $\gamma_{S}$-function $g$ such that $\omega(g) < 3$. So $|V_g^-| > \frac{1}{2}(p^n - 3)$. Let $|V_g^-| = \frac{1}{2}(p^n - 1)$. If $A_i \cap V_g^- = \emptyset$ for some $1 \leq i \leq p$, then $g[v] = 1 - p^{n-1}$ for every $v \in A_i$. Hence, $A_i \cap V_g^- \neq \emptyset$ for every $1 \leq i \leq p$. If $|A_i \cap V_g^-| \geq \lceil \frac{p^n}{2} \rceil$ for every $1 \leq i \leq p$, then $|V_g^-| \geq \frac{1}{2}(p^n + p)$. This is impossible. So there is $j \in \{1, 2, \ldots, p\}$ such that $|A_j \cap V_g^-| \leq \lceil \frac{p^n}{2} \rceil$. Let $u \in A_j \cap V_g^-$. So $g[u] = deg(u) + 1 - 2|N(u) \cap V_g^-| < 0$. This is contradiction. Therefore $\gamma_{S}(\Gamma) = 3$.

\[ \square \]

Corollary 2.7. For every integer $n$, there is a $(p^n - p^{n-1})$-regular graph $\Gamma$ with $p^n$ vertices such that $\gamma_{S}(\Gamma) = 3$.

3. Cayley Graphs of Order $pq$

In this section $p$ and $q$ are distinct prime numbers where $p < q$. Let $B(1, pq)$ be a generator of $\mathbb{Z}_{pq}$. For $1 \leq i \leq p$ and $1 \leq j \leq q$, set

$A_i = \{i + kp : 0 \leq k \leq q - 1\}$

and

$B_j = \{j + kq' : 0 \leq k' \leq p - 1\}.$

With these notations in mind we will prove the following results.

Lemma 3.1. Let $\mathbb{Z}_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then following statements hold.

i. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$ and $Cay(\mathbb{Z}_{pq} : S)$ is a $p$-partite graph.

ii. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{j=1}^{q} B_j$ and $Cay(\mathbb{Z}_{pq} : S)$ is a $q$-partite graph.

iii. Let $1 \leq i \leq p$. For any $x \in A_i$ there is some $1 \leq j \leq q$ such that $x \in B_j$.

iv. $|A_i \cap B_j| = 1$ for every $i, j$.

Proof. i. Let $s \in V(Cay(\mathbb{Z}_{pq} : S))$. If $p \mid s$, then $s \in A_p$. Otherwise, $s \in A_i$ where $s = kp + i$ for some $1 \leq k \leq (p - 1)$. Thus $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$. Since $1 \leq i \neq j \leq p$, $A_i \cap A_j = \emptyset$. We show that the
induced subgraph on $A_l$ is empty. Let $l + t \in E(Cay(Z_{pq} : S))$. If $l, t \in A_l$ for some $1 \leq s \leq p$, then $l = s + kp, t = s + k'p$. So $p \mid (l - t)$. This is impossible.

ii. The proof is likewise part i.

iii. Let $1 \leq i \leq p$ and let $x \in A_i$. If $x \leq q$, then $x \in B_s$. If not, $x = i + kp > q$ such that $1 \leq k \leq q - 1$. Hence, $x \equiv t \mod q$ where $1 \leq t \leq q$, and so $x \in B_t$.

iv. By Case iii and since $|A_i| = q$ and also for every $j \neq j'$, $B_j \cap B_{j'} = \emptyset$, the result reaches.

\[ \square \]

**Theorem 3.2.** [6] For any graph $\Gamma$, \[ \frac{n}{1 + \Delta(\Gamma)} \leq \gamma(\Gamma) \leq n - \Delta(\Gamma) \] where $\Delta(\Gamma)$ is the maximum degree of $\Gamma$.

**Theorem 3.3.** Let $Z_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then the following is hold.

\[ \gamma(Cay(Z_{pq} : S)) = \begin{cases} 2 & p = 2; \\ 3 & p > 2. \end{cases} \]

**Proof.** Let $p = 2$. By Lemma 3.1, $D = \{i, i + q\}$ is a dominating set. Since $Cay(Z_{pq} : S)$ is a $(q - 1)$-regular graph, by Theorem 3.2, $\gamma(Cay(Z_{pq} : S)) \geq 2$. Thus $\gamma(Cay(Z_{pq} : S)) = 2$.

Let $p > 2$. We define $D = \{1, 2, s\}$ where $s \in A_1 \setminus N(2)$. Since $1, 2$ are adjacent, $N(1) \cup N(2) = V(Cay(Z_{pq} : S)) \setminus D$. Thus $D$ is a dominating set. As a consequence, $\gamma(Cay(Z_{pq} : S)) \leq 2$. It is easy to show that $\gamma(Cay(Z_{pq} : S)) \neq 2$. Let $D' = \{x, y\}$. We show that $D'$ is not a dominating set. If $x, y \in A_i$ for some $1 \leq i \leq p$, then for every $z \in A_i \setminus D'$, $z \notin N(D')$. If not, $x \in A_i$, and $y \in A_j$ for some $1 \leq i \neq j \leq p$. If $x, y$ are adjacent, then there is $x' \in A_i \setminus \{x\}$ such that $x' \notin N(y)$. Thus $D'$ is not dominating set. If $x$ and $y$ are not adjacent, then there is $z \in A_l, l \neq i, j$, such that the induced subgraph on $\{x, y, z\}$ is empty. Hence, $D'$ is not a dominating set and the proof is completed.

\[ \square \]

**Theorem 3.4.** Let $Z_{pq} = \langle S \rangle$ where $p \in \{2, 3, 5\}$ and $S = B(1, pq)$. Then

\[ \gamma_s(Cay(Z_{pq} : S)) = p. \]

**Proof.** Let $A = \{1, 1 + p, \ldots, 1 + \left(\frac{q}{2}\right) - 1\}$ and $B = \{i + tq : i \in A$ and $1 \leq t \leq p - 1\}$. We define $f : V(Cay(Z_{pq} : S)) \to \{-1, 1\}$ such that

\[ f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise}. \end{cases} \]

Let $v \in V(Cay(Z_{pq} : S))$. If $f(v) = -1$, then

\[ f[v] = -1 + (p - 1)(q - 1) - 2 \left(\left(\frac{q}{2}\right) - 1\right) (p - 1) = 2p - 3. \]
Otherwise,

\[ f[v] = 1 + (p - 1)(q - 1) - 2 \left\lfloor \frac{q}{2} \right\rfloor \left( p - 1 \right) = 1. \]

Hence, \( f \) is a dominating function. Also

\[ \omega(f) = pq - 2(\lvert A \rvert + \lvert B \rvert) = pq - 2 \left( \left\lfloor \frac{q}{2} \right\rfloor + (p - 1) \left\lfloor \frac{q}{2} \right\rfloor \right) = p. \]

It is enough to show that \( f \) has the minimal wait. Let, to the contrary, \( g \) be a dominating function and \( \omega(g) < \omega(f) \). So \( |V_g^-| > |V_f^-| \). Without lose of generality, suppose that \( |V_g^-| = p \left( \frac{q}{2} \right) + 1 \). Let \( A_i^- = A_i \cap V_g^- \), \( A_i^+ = A_i \backslash A_i^- \) and \( B_j^- = B_j \cap V_g^- \). We will reach the contradiction by three steps.

Step 1. For every \( 1 \leq i \leq p \), \( A_i^- \neq \emptyset \).

On the contrary, let \( A_i^- = \emptyset \) for some \( 1 \leq s \leq p \). Let \( u \in A_s \). Then by Lemma 3.1, \( u \in A_s \cap B_t \) for some \( 1 \leq t \leq q \). So

\[ g[u] = (p - 1)(q - 1) + 2(|V_g^-| - |A_s|^t) \geq 1. \]

Thus \( |B_t^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \). Hence, \( |V_g^-| \geq \left\lvert A_s \right\rvert + \left\lfloor \frac{q}{2} \right\rfloor \). This implies \( q + (q - p) \left\lfloor \frac{q}{2} \right\rfloor < 1 \). This is a contradiction. Hence, \( A_i^- \neq \emptyset \).

Similar argument applies for \( B_j \). Therefore, \( B_j^- \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \).

On the contrary, Let \( |A_i^-| < \left\lfloor \frac{q}{2} \right\rfloor \) for some \( 1 \leq l \leq p \). Without lose of generality suppose that \( |A_l^-| = \left\lfloor \frac{q}{2} \right\rfloor - 1 \). Let \( v \in A_l \). By Lemma 3.1, \( v \in A_l \cap B_k \) for some \( 1 \leq k \leq q \). If \( g(v) = -1 \), then \( g[v] = (p - 1)(q - 1) - 2(|V_g^-| - |A_l^-| - |B_k^-| + 2) \geq 1 \). Then \( |B_k^- \setminus \{v\}| \geq 2 \). Hence, \( |V_g^-| \geq 4|A_i^-| + |A_i^+| + 2|A_i^+| \).

As a consequence \( p > 8 \). This is impossible.

Therefore, for every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \) and since \( |V_g^-| = p \left( \frac{q}{2} \right) + 1 \), we may suppose that \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rfloor \) and \( |A_i^+| = \left\lfloor \frac{q}{2} \right\rfloor \) for \( 2 \leq i \leq p \).

Step 3. For every \( 1 \leq j \leq q \), \( |B_j^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \).

On the contrary, let \( |B_h^-| < \left\lfloor \frac{q}{2} \right\rfloor \) for some \( 1 \leq h \leq q \). Suppose that \( |B_h^-| = \left\lfloor \frac{q}{2} \right\rfloor \). By Lemma 3.1, \( B_h \cap A_i \neq \emptyset \) for any \( 1 \leq i \leq p \). Let \( z \in B_h \cap A_i \). Thus

\[ g[z] = -1 + (p - 1)(q - 1) - 2(\lvert V_g^- \rvert - |A_i^-| - |B_h^-| + 2) \]

\[ \leq -1 + (p - 1)(q - 1) - 2(p \left\lfloor \frac{q}{2} \right\rfloor + 1 - \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{q}{2} \right\rfloor + 2) \]

\[ \leq p - 6 \]

Since \( p \in \{2, 3, 5\} \), \( g[z] \leq -1 \). This is a contradiction.

By Step 3, \( |V_g^-| \geq q \left( \frac{q}{2} \right) \). Hence, \( p \left( \frac{q}{2} \right) + 1 \geq q \left( \frac{q}{2} \right) \). So \( p + q \leq 2 \). This is impossible. Therefore \( \gamma_s(Cay(G : S)) = \omega(f) = p. \)

\[ \square \]

**Theorem 3.5.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) where \( p \geq 7 \) and \( S = B(1, pq) \). Then

\[ \gamma_s(Cay(\mathbb{Z}_{pq} : S)) = 5. \]
Proof. We define \( f : V(Cay(Z_{pq} : S)) \to \{-1, 1\} \) such that \( f(i) = -1 \) if and only if \( i \in \{1, 2, \ldots, \frac{pq-3}{2}\} \). It is easily seen that \( \left\lfloor \frac{q}{2} \right\rfloor \leq |A_i^-| \leq \left\lfloor \frac{q}{2} \right\rfloor \) for every \( 1 \leq i \leq p \). Also \( \left\lfloor \frac{q}{2} \right\rfloor \leq |B_j^+| \leq \left\lfloor \frac{q}{2} \right\rfloor \) for any \( 1 \leq j \leq q \). Let \( v \in A_i \cap B_s \) such that \( 1 \leq t \leq p \) and \( 1 \leq s \leq q \). In the worst situation, \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rfloor \) and \( |B_s^+| = \left\lfloor \frac{q}{2} \right\rfloor \). In this case \( 1 \leq f(v) \leq 5 \). Hence, \( f \) is a signed dominating function. Also \( \omega(f) = pq - 2|V_j^-| = 5 \). Thus \( \gamma_s(Cay(Z_{pq} : S)) \leq 5 \). What is left is to show that if \( g \) is a \( \gamma_s \)-function, then \( \omega(g) \geq 5 \). On the contrary, suppose that \( g \) be a \( \gamma_s \)-function and \( \omega(g) < \omega(f) \). Hence, \( |V_j^-| < |V_j^-| \). There is no loss of generality in assuming \( |V_j^-| = \frac{pq-3}{2} \). Let \( A_i^- = A_i \cap V_g^- \) and \( B_j^- = B_j \cap V_g^- \). In order to reach the contradiction we use two following steps:

Step 1. \( A_i^- \neq \emptyset \) for every \( 1 \leq i \leq p \).

On the contrary, suppose that for some \( 1 \leq m \leq p \), \( A_m^- = \emptyset \). Let \( w \in A_m \). So there is \( 1 \leq \ell \leq q \) such that \( w \in A_m \cap B_r \). Hence, \( g[w] = (p-1)(q-1)+1-2(|V_j^-| - |B_r^-|) \geq 1 \). Thus \( |B_r^-| \geq \frac{pq-3}{2} \). So \( |V_j^-| \geq q(\frac{pq-4}{2}) \). Hence, \( pq - 3 \geq q(pq - 4) \). This makes a contradiction.

By similar argument we have \( B_j^- \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \).

On the contrary, let \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rfloor - 1 \). Let \( v \in A_i \). There is \( 1 \leq \ell' \leq q \) such that \( v \in A_i \cap B_r \). If \( g(v) = -1 \), then \( g[v] = (p-1)(q-1)+1-2(|V_j^-| - |A_k^-| - |B_s^-| + 2) \geq 1 \). Hence, \( |B_r^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \). If \( g(v) = 1 \), then \( |B_r^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \). Therefore, \( |V_j^-| \geq |A_i^-| (\left\lfloor \frac{q}{2} \right\rfloor + 1) + |A_i^-| \left\lfloor \frac{q}{2} \right\rfloor \). This implies that \( q \leq 3 \). This is a contradiction.

Likewise Step 2, \( |B_j^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \) for every \( 1 \leq j \leq q \). Since \( |V_g^-| = \frac{pq-3}{2} \), there is \( 1 \leq k \leq p \) such that \( |A_k^-| = \left\lfloor \frac{q}{2} \right\rfloor \). On the other hand, suppose that for \( 1 \leq t \leq q \), \( |B_k^-| = \left\lfloor \frac{q}{2} \right\rfloor \). Let \( u \in A_i \cap B_s^- \). If \( s \in \{l_1, \ldots, l_t\} \), then

\[
g[u] = -1 + (p-1)(q-1) - 2 \left( |V_j^-| - |A_k^-| - |B_s^-| + 2 \right) \\
= -1 + (p-1)(q-1) - 2 \left( \frac{pq-3}{2} - \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{p}{2} \right\rfloor + 2 \right) \\
= 3.
\]

This is a contradiction by \( g \) is a signed dominating function. Hence, \( s \) is not in \( \{l_1, \ldots, l_t\} \). Since \( |A_k^-| = \left\lfloor \frac{q}{2} \right\rfloor \), \( q - t \geq \left\lfloor \frac{q}{2} \right\rfloor \) and so \( t \leq \left\lfloor \frac{q}{2} \right\rfloor \). As a consequence,

\[
|V_j^-| \geq t|\frac{p}{2}| + (q-t)|\frac{p}{2}| \geq |\frac{t}{2}||\frac{p}{2}| + |\frac{q}{2}||\frac{p}{2}|.
\]

Since \( |V_g^-| = \frac{pq-3}{2} \), this makes a contradiction. Therefore,

\[
\gamma_s(Cay(Z_{pq} : S)) = 5.
\]

\[ \square \]

Corollary 3.6. For any \( k \)-regular graph \( \Gamma \) on \( n \) vertices \( \gamma_s(\Gamma) \geq \frac{n}{k+1} \). Hence, \( \gamma_s(\Gamma) \geq 1 \). It is easy to check that \( \gamma_s(\Gamma) = 1 \) if and only if \( \Gamma \) is a complete
graph and \( n \) is odd. Furthermore, for any prime numbers \( p < q \), there is a 
\((p - 1)(q - 1)\)–regular graph \( \Gamma \) with \( pq \) vertices such that \( \gamma_s(\Gamma) \in \{2, 3, 5\} \).

ACKNOWLEDGMENTS

The author is thankful of referees for their valuable comments.

REFERENCES

11. L. Volkmann, B. Zelinka, Signed Domatic Number of a Graph, Discrete applied mathematics, 150(1), (2005), 261–267.