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On Graded Weakly Classical Prime Submodules

Rashid Abu-Dawwas\textsuperscript{a,\,*}, Khaldoun Al-Zoubi\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Yarmouk University, Jordan.
\textsuperscript{b}Department of Mathematics and Statistics, Jordan University of Science and Technology, Jordan.

E-mail: rrashid@yu.edu.jo
E-mail: kfzoubi@just.edu.jo

Abstract. Let $R$ be a $G$-graded ring and $M$ be a $G$-gr-$R$-module. In this article, we introduce the concept of graded weakly classical prime submodules and give some properties of such a submodule.

Keywords: Graded prime submodules, Graded weakly classical prime submodules, Graded classical prime submodules.

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1. Introduction

Gr-prime ideals of a commutative graded ring have been introduced and studied by Refai and Al-Zoubi in [14]. Gr-weakly prime ideals of a commutative graded ring have been introduced and studied by Atani in [4]. Gr-prime and gr-weakly prime submodules of graded modules over graded commutative rings have been studied by various authors; (see, for example [5, 6, 7, 12]). Gr-2-absorbing and gr-weakly 2-absorbing submodules have been studied by Al-Zoubi and Abu-Dawwas in [2]. Also, gr-classical prime submodules of graded modules over graded commutative rings have been introduced and studied by various authors; (see [3, 8] ). Here we introduce the concept of graded weakly classical prime (gr-weakly classical prime) submodules. A number of results

\*Corresponding Author

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concerning of gr-weakly classical prime submodules are given (see sec. 2). First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [9] and [10] for these basic properties and more information on graded rings and modules. Let $G$ be a group with identity $e$. A ring $R$ is said to be $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of $R_g$ are called homogeneous of degree $g$ and $R_e$ (the identity component of $R$) is a subring of $R$ and $1 \in R_e$. For $x \in R$, $x$ can be written uniquely as $\sum_{g \in G} x_g$ where $x_g$ is the component of $x$ in $R_g$. Also we write $h(R) = \bigcup_{g \in G} R_g$ and $\text{supp}(R,G) = \{g \in G : R_g \neq 0\}$. Let $M$ be a left $R$-module. Then $M$ is a $G$-graded $R$-module (shortly, $M$ is $gr$-$R$-module) if there exist additive subgroups $M_g$ of $M$ indexed by the elements $g \in G$ such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of $M_g$ are called homogeneous of degree $g$. If $x \in M$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $M_g$. Clearly, $M_g$ is $R_e$-submodule of $M$ for all $g \in G$. Also we write $h(M) = \bigcup_{g \in G} M_g$. and $\text{supp}(M,G) = \{g \in G : M_g \neq 0\}$. Let $R$ be a $G$-graded ring and $I$ be an ideal of $R$. Then $I$ is called $G$-graded ideal if $I = \bigoplus_{g \in G} (I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g \in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a $G$-graded ring need not be $G$-graded.

Let $M$ be a $G$-$gr$-$R$-module and $N$ be an $R$-submodule of $M$. Then $N$ is called $G$-$gr$-$R$-submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$, i.e., if $x \in N$ and $x = \sum_{g \in G} x_g$, then $x_g \in N$ for all $g \in G$. Also, an $R$-submodule of a $G$-graded $R$-module need not be $G$-graded. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. A proper graded ideal $P$ of $R$ is said to be gr-prime (resp. gr-weakly prime) ideal if whenever $r,s \in h(R)$ with $rs \in P$ (resp. $0 \neq rs \in P$), then either $r \in P$ or $s \in P$. A proper graded submodule $N$ of a graded module $M$ is said to be gr-prime (resp. gr-weakly prime) submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$ (resp. $0 \neq rm \in N$), then either $r \in (N :_R M)$ or $m \in N$. A proper graded submodule $N$ of $M$ is called a gr-classical prime submodule if whenever $r,s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rsm \in N$ or $sm \in N$. Of course, every gr-prime submodule is a gr-classical prime submodule, but the converse is not true in general (see [3, Example 2.3]). The annihilator of graded $R$-module $M$ which is denoted by $\text{Ann}_G(M)$ is $(0 : M)$. Furthermore, for every $m \in h(M)$, $(0 : m)$ is denoted by $\text{Ann}_G(m)$.

2. Results

Definition 2.1. Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a proper graded submodule of $M$. $N$ is said to be graded weakly classical prime (gr-weakly classical prime) if whenever $a,b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$, then either $am \in N$ or $bm \in N$. 

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Proposition 2.2. Let $M$ be a gr-$R$-module and $N$ be a gr-$R$-submodule of $M$. If $(N : m)$ is a gr-weakly prime ideal of $R$ for every $m \in h(M) - N$, then $N$ is a gr-weakly classical prime $R$-submodule of $M$.

Proof. Let $a, b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$. If $m \in N$, then we are done. Suppose $m \notin N$. Then $0 \neq ab \in (N : m)$ and since $(N : m)$ is a gr-weakly prime ideal, either $a \in (N : m)$ or $b \in (N : m)$ and then either $am \in N$ or $bm \in N$ and hence $N$ is a gr-weakly classical prime $R$-submodule of $M$. □

Proposition 2.3. Let $M$ be a gr-$R$-module and $N$ be a gr-$R$-submodule of $M$. If $N$ is a gr-weakly classical prime $R$-submodule of $M$ and $m \in h(M) - N$ such that $Ann_G(m) = 0$, then $(N : m)$ is a gr-weakly prime ideal of $R$.

Proof. By [5, Lemma 2.1], $(N : m)$ is a graded ideal of $R$. Let $a, b \in h(R)$ such that $0 \neq ab \in (N : m)$. Then since $Ann_G(m) = 0$, $0 \neq abm \in N$ and since $N$ is gr-weakly classical prime, either $am \in N$ or $bm \in N$ and then either $a \in (N : m)$ or $b \in (N : m)$. Hence, $(N : m)$ is a gr-weakly prime ideal of $R$. □

Let $M$ and $L$ be two gr-$R$-modules. A homomorphism of gr-$R$-module $\phi : M \rightarrow L$ is a homomorphism of $R$-modules satisfying $\phi(M_g) \subseteq L_g$ for every $g \in G$(see [10]).

Theorem 2.4. Let $R$ be a G-graded ring and $M, L$ be two gr-$R$-modules and $\phi : M \rightarrow L$ be an epimorphism of gr-modules. If $N$ is a gr-weakly classical prime $R$-submodule of $M$ containing $\text{Ker}(\phi)$, then $f(N)$ is a gr-weakly classical prime $R$-submodule of $L$.

Proof. Firstly, we prove that $f(N)$ is a graded $R$-submodule of $L$. Clearly, $f(N)$ is an $R$-submodule of $L$. Let $y \in f(N)$. Then there exists $x \in N$ such that $f(x) = y$. Let $x = \sum_{i=1}^{n} y_{xi}$, where $x_{gi} \in M_{gi}$, $0$, $y_i \neq y_j$ for $i \neq j$. Then $y = \sum_{i=1}^{n} f(x_{gi})$. For each $1 \leq i \leq n$, there exists $h_i \in \text{supp}(L, G)$ with $f(x_{gi}) \in L_{h_i} - 0$ and $h_i \neq h_j$ for $i \neq j$. If $h_i = h_j$ for all $1 \leq i \leq n$, then $y_h = 0 = f(0) \in f(N)$. If $h = h_i$ for some $1 \leq i \leq n$, then $y_h = f(x_{gi})$. Since $x \in N$ and $N$ is graded, $x_{gi} \in N$ and then $y_h \in f(N)$. Hence, $f(N)$ is a graded $R$-submodule of $L$. Secondly, we prove that $f(M_g) = L_g$ for all $g \in G$. Let $g \in G$ and let $r_g \in L_g$. If $r_g = 0$, then $r_g = 0 = f(0) \in f(M_g)$. Suppose $r_g \neq 0$. Since $f$ is onto, there exists $x \in M - 0$ such that $f(x) = r_g$. Suppose $x = \sum_{i=1}^{n} y_{xi}$, where $x_{gi} \in M_{gi} - 0$, $y_i \neq y_j$ for $i \neq j$. Then $r_g = \sum_{i=1}^{n} f(x_{gi}) = \sum_{i=1}^{k} f(x_{gi})$ where $1 \leq i \leq n$ and $f(x_{gi}) \neq 0$ for all $1 \leq i \leq k$. Since $f(x_{gi}) \in L_{g_{ti}}$, $r_g \in L_g \cap \sum_{i=1}^{k} L_{g_{ti}}$. Thus, $g = g_{t_{i}} = ...... = g_{t_{k}}$ and hence $k = 1$ and $f(x_{gi}) = f(x_{g}) = r_g$. So, $r_g \in f(M_g)$ and hence $L_g \subseteq f(M_g)$ and as $f(M_g) \subseteq L_g$, $L_g = f(M_g)$. Now, let $a, b \in h(R)$ and $s \in h(L)$ such that $0 \neq abs \in f(N)$. Since $s \in h(L), s \in L_g$ for some $g \in G$ and since $L_g = f(M_g)$,
there exists \( m \in M_g \subseteq h(M) \) such that \( f(m) = s \) and then \( 0 \neq f(abm) \in f(N) \), it follows that there exists \( n \in N \cap h(M) \) such that \( f(abm) = f(n) \) and then \( f(abm - n) = 0 \), so \( abm - n \in Ker(f) \subseteq N \) and as \( n \in N \), \( 0 \neq abm \in N \). Since \( N \) is gr-weakly classical prime, either \( am \in N \) or \( bm \in N \) and then either \( as \in f(N) \) or \( bs \in f(N) \). Hence, \( f(N) \) is a gr-weakly classical prime \( R \)-submodule of \( L \).

Let \( M \) be a \( G \)-graded \( R \)-module and \( K \) be an \( R \)-submodule of \( M \). Then \( M/K \) is a graded \( R \)-module by putting \((M/K)_g = (M_g + K)/K\).

**Proposition 2.5.** Let \( K \) and \( N \) be two graded proper \( R \)-submodules of a gr-\( R \)-module \( M \) such that \( K \subseteq N \). If \( K \) is a gr-weakly classical prime \( R \)-submodule of \( M \) and \( N/K \) is a gr-weakly classical prime \( R \)-submodule of \( M/K \), then \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \).

**Proof.** Let \( a, b \in h(R) \) and \( m \in h(M) \) such that \( 0 \neq abm \in N \). If \( abm \in K \), then as \( K \) is gr-weakly classical prime, either \( am \in K \subseteq N \) or \( bm \in K \subseteq N \) and then we are done. Suppose \( abm \notin K \). Since \( m \in h(M) \), \( m \in M_g \) for some \( g \in G \) and then \( m + K \in (M_g + K)/K = (M/K)_g \subseteq h(M/K) \). Now, \( 0 \neq ab(m + K) \in N/K \) and since \( N/K \) is gr-weakly classical prime, either \( am + K \in N/K \) or \( bm + K \in N/K \) and then either \( am \in N \) or \( bm \in N \). Hence, \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \).

**Proposition 2.6.** Let \( N \) be a graded \( R \)-submodule of a gr-\( R \)-module \( M \). If \( N \) is a gr-weakly prime \( R \)-submodule of \( M \), then \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \).

**Proof.** Let \( a, b \in h(R) \) and \( m \in h(M) \) such that \( 0 \neq abm \in N \). Then since \( N \) is gr-weakly prime, either \( bm \in N \) or \( a \in (N : M) \). If \( bm \in N \), then we are done. If \( a \in (N : M) \), then \( am \in N \). Hence, \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \).

The concept of gr-2-absorbing submodules (respectively, gr-weakly 2-absorbing submodules) of a graded module over a commutative graded ring is studied in [2]. A graded proper \( R \)-submodule \( N \) of a gr-\( R \)-module \( M \) is said to be gr-2-absorbing (gr-weakly 2-absorbing) if whenever \( a, b \in h(R) \) and \( m \in h(M) \) such that \( abm \in N \) \((0 \neq abm \in N)\), then either \( am \in N \), \( bm \in N \) or \( ab \in (N : M) \).

It is clear that if \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \), then \( N \) is a gr-weakly 2-absorbing \( R \)-submodule of \( M \). We introduce the following:

**Proposition 2.7.** If \( N \) is a gr-weakly 2-absorbing \( R \)-submodule of \( M \) and \((N : M)\) is a gr-weakly prime ideal of \( R \), then \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \).

**Proof.** Let \( a, b \in h(R) \) and \( m \in h(M) \) such that \( 0 \neq abm \in N \). Then since \( N \) is gr-weakly 2-absorbing, \( am \in N \), \( bm \in N \) or \( ab \in (N : M) \). If \( am \in N \)
or \( bm \in N \), then we are done. Suppose \( ab \in (N : M) \). If \( ab = 0 \), then \( abm = 0 \) a contradiction. So, \( 0 \neq ab \in (N : M) \) and since \( (N : M) \) is gr-weakly prime, either \( a \in (N : M) \) or \( b \in (N : M) \) and then either \( am \in aM \subseteq N \) or \( bm \in bM \subseteq N \). Hence, \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \).

**Proposition 2.8.** Let \( N \) be a graded \( R \)-submodule of a gr-R-module \( M \). If \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \), then \( N_g \) is a weakly classical prime \( R_g \)-submodule of \( M_g \) for all \( g \in G \).

**Proof.** Let \( g \in G \). Let \( a, b \in R_e \) and \( m \in M_g \) such that \( 0 \neq abm \in N_g \). Since \( R_e \subseteq h(R) \) and \( M_g \subseteq h(M) \), \( a, b \in h(R) \) and \( m \in h(M) \). Since \( N_g \subseteq N \), \( 0 \neq abm \in N \) and since \( N \) is gr-weakly classical prime, either \( am \in N \) or \( bm \in N \). If \( am \in N \), then \( am \in R_eM_g \) \( \bigcap N \subseteq M_g \) \( \bigcap N = N_g \). Similarly, if \( bm \in N \), then \( bm \in N_g \). Hence, \( N_g \) is a weakly classical prime \( R_g \)-submodule of \( M_g \).  

Let \( M \) be an \( R \)-module and \( N \) be an \( R \)-submodule of \( M \). Then for every \( a \in R \), we define \( (N : M) a = \{ m \in M : am \subseteq N \} \). it is easy to prove that \((N : M) a\) is an \( R \)-submodule of \( M \) containing \( N \). Moreover, it is easy to prove that if \( N \) is a graded \( R \)-submodule of a gr-R-module \( M \), then \((N : M) a\) is a graded \( R \)-submodule of \( M \).

The next proposition gives a characterization for gr-weakly classical prime submodules.

**Proposition 2.9.** Let \( M \) be a gr-R-module and \( N \) be a graded \( R \)-submodule of \( M \). Then \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \) if and only if \((N : h(M)) ab = (0 : h(M)) ab \bigcup (N : h(M)) a \bigcup (N : h(M)) b \) for all \( a, b \in h(R) \).

**Proof.** Suppose \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \). Let \( a, b \in h(R) \) and \( m \in (N : h(M)) ab \). Then \( abm \in N \). If \( abm = 0 \), then \( m \in (0 : h(M)) ab \). Suppose \( abm \neq 0 \). Since \( N \) is gr-weakly classical prime, either \( am \in N \) or \( bm \in N \) and then either \( m \in (N : h(M)) a \) or \( (N : h(M)) b \). Conversely, Let \( a, b \in h(R) \) and \( m \in h(M) \) such that \( 0 \neq abm \in N \). Then \( m \in (N : h(M)) ab \) and then by assumption, either \( m \in (N : h(M)) a \) or \( m \in (N : h(M)) b \) that is either \( am \in N \) or \( bm \in N \). Hence, \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \).

Similarly, we introduce the following:

**Proposition 2.10.** Let \( M \) be a gr-R-module and \( N \) be a graded \( R \)-submodule of \( M \). If \( N \) is a gr-weakly classical prime \( R \)-submodule of \( M \), then \((N : h(R)) abm = (0 : h(R)) abm \bigcup (N : h(R)) am \bigcup (N : h(R)) bm \) for all \( a, b \in h(R) \) and \( m \in h(M) \).

**Proof.** Let \( a, b \in h(R) \) and \( m \in h(M) \). Assume that \( r \in (N : h(R)) abm \). Then \( rabm \in N \). If \( rabm = 0 \), then \( r \in (0 : h(R)) abm \). Suppose \( rabm \neq 0 \). Then
0 \neq ab(rm) \in N \text{ and since } N \text{ is gr-weakly classical prime, either } arm \in N \text{ or } brm \in N \text{ and then either } r \in (N : h(R) am) \text{ or } r \in (N : h(R) bm). \qed

**Theorem 2.11.** Let $M_1$, $M_2$ be two graded $R$-modules and $N_1$ be a proper graded $R$-submodule of $M_1$. Then the following conditions are equivalent:

1. $N = N_1 \times M_2$ is a gr-weakly classical prime submodule of $M = M_1 \times M_2$.
2. $N_1$ is a gr-weakly classical prime submodule of $M_1$ and for each $a, b \in h(R)$ and $m_1 \in h(M_1)$ we have $abm_1 = 0$, $am_1 \notin N_1$, $bm_1 \notin N_1 \Rightarrow ab \in \text{Ann}_{G}(M_2)$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $N = N_1 \times M_2$ is a gr-weakly classical prime submodule of $M = M_1 \times M_2$. Let $a, b \in h(R)$ and $m_1 \in h(M_1)$ be such that $0 \neq abm_1 \in N_1$. Then $(0, 0) \neq ab(m_1, 0) \in N$. Thus $a(m_1, 0) \in N$ or $b(m_1, 0) \in N$, and so $am_1 \in N_1$ or $bm_1 \in N_1$. Consequently $N_1$ is a gr-weakly classical prime submodule of $M_1$. Now, assume that $abm_1 = 0$ for some $a, b \in h(R)$ and $m_1 \in h(M_1)$ such that $am_1 \notin N_1$ and $bm_1 \notin N_1$. Suppose that $ab \notin \text{Ann}_{G}(M_2)$. Then there exists $m_2 \in h(M_2)$ such that $abm_2 \neq 0$. Hence $(0, 0) \neq ab(m_1, m_2) \in N$, and so $a(m_1, m_2) \in N$ or $b(m_1, m_2) \in N$. Thus $am_1 \in N_1$ or $bm_1 \in N_1$ which is a contradiction. Consequently $ab \in \text{Ann}_{G}(M_2)$.

(2) $\Rightarrow$ (1) Let $a, b \in h(R)$ and $(m_1, m_2) \in h(M) = h(M_1 \times M_2)$ be such that $(0, 0) \neq ab(m_1, m_2) \in N = N_1 \times M_2$. First assume that $abm_1 \neq 0$. Then by part (2), $am_1 \in N_1$ or $bm_1 \in N_1$. So $a(m_1, m_2) \in N$ or $b(m_1, m_2) \in N$, and thus we are done. If $abm_1 = 0$, then $abm_2 \neq 0$. Therefore $ab \notin \text{Ann}_{G}(M_2)$, and so part (2) implies that either $am_1 \in N_1$ or $bm_1 \in N_1$. Again we have that $a(m_1, m_2) \in N$ or $b(m_1, m_2) \in N$ which shows $N$ is a gr-weakly classical prime submodule of $M$. \qed

The following two propositions have easy verifications.

**Proposition 2.12.** Let $M_1, M_2$ be two graded $R$-modules and $N_1$ be a proper graded $R$-submodule of $M_1$. Then $N = N_1 \times M_2$ is a gr-classical prime submodule of $M = M_1 \times M_2$ if and only if $N_1$ is a gr-classical prime submodule of $M_1$.

**Proposition 2.13.** Let $M_1, M_2$ be two graded $R$-modules and $N_1, N_2$ be two proper graded $R$-submodules of $M_1, M_2$, respectively. If $N = N_1 \times N_2$ is a gr-weakly classical prime (resp. gr-classical prime) submodule of $M = M_1 \times M_2$, then $N_1$ is a gr-weakly classical prime (resp. gr-classical prime) submodule of $M_1$ and $N_2$ is a gr-weakly classical prime (resp. gr-classical prime) submodule of $M_2$.

Let $R_i$ be a commutative graded ring with unity and $M_i$ be a graded $R_i$-module, for $i = 1, 2$. Consider the graded ring $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a graded $R$-module and each graded submodule of $M$ is in the form of $N = N_1 \times N_2$ for some graded submodules $N_1$ of $M_1$ and $N_2$ of $M_2$. 

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Theorem 2.14. Let $R = R_1 \times R_2$ be a graded ring and $M = M_1 \times M_2$ be a graded $R$-module where $M_1$ is a graded $R_1$-module and $M_2$ is a graded $R_2$-module. Suppose that $N = \bigcap S = N_1 \times M_2$ is a proper graded submodule of $M$. Then the following conditions are equivalent:

1. $N_1$ is a gr-classical prime submodule of $M_1$;
2. $N$ is a gr-classical prime submodule of $M$;
3. $N$ is a gr-weakly classical prime submodule of $M$.

Proof. (1) \Rightarrow (2) Let $(r_1, r_2)(s_1, s_2)(m_1, m_2) \in N$ for some $(r_1, r_2), (s_1, s_2) \in h(R)$ and $(m_1, m_2) \in h(M)$. Then $r_1s_1m_1 \in N_1$ so either $r_1m_1 \in N_1$ or $s_1m_1 \in N_1$ which shows that either $(r_1, r_2)(m_1, m_2) \in N$ or $(s_1, s_2)(m_1, m_2) \in N$. Consequently $N$ is a gr-classical prime submodule of $M$.

(2) \Rightarrow (3) It is clear that every gr-classical prime submodule is a gr-weakly classical prime submodule.

(3) \Rightarrow (1) Let $rsm \in N_1$ for some $r, s \in h(R_1)$ and $m \in h(M_1)$. We may assume that $0 \neq m' \in h(M_2)$. Therefore $0 \neq (r, 1)(s, 1)(m, m') \in N$. So either $(r, 1)(m, m') \in N$ or $(s, 1)(m, m') \in N$. Therefore $rm \in N_1$ or $sm \in N_1$. Hence $N_1$ is a gr-classical prime submodule of $M_1$.

Let $R$ be a $G$-graded ring, $M$ be a graded $R$-module and $S \subseteq h(R)$ be a multiplicatively closed subset of $R$. Then the ring of fraction $S^{-1}R$ is a graded ring which is called graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \text{ and } g = (\deg s)^{-1}(\deg r)\}$. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called module of fractions, if $S^{-1}M = \bigoplus_{g \in G} (S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S \text{ and } g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \bigcup_{g \in G} (S^{-1}R)_g$ and $h(S^{-1}M) = \bigcup_{g \in G} (S^{-1}M)_g$, (see[10]).

A graded zero-divisor on a graded $R$-module $M$ is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $rm = 0$. The set of all graded zero-divisors on $M$ is denoted by $G = Zdv_R(M)$.

The following result studies the behavior of gr-weakly classical prime submodules under localization.

Proposition 2.15. Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $S \subseteq h(R)$ a multiplication closed subset of $R$. Then the following hold:

1. If $N$ is a gr-weakly classical prime $R$-submodule of $M$ and $(N : M) \cap S = \phi$, then $S^{-1}N$ is a gr-weakly classical prime $R$-submodule of $S^{-1}M$.
2. If $S^{-1}N$ is a gr-weakly classical prime $R$-submodule of $S^{-1}M$ such that $S \cap G = Zdv_R(N) = \phi$ and $S \cap G = Zdv_R(M/N) = \phi$, then $N$ is a gr-weakly classical prime $R$-submodule of $M$.

Proof. (1) Let $N$ be a gr-weakly classical prime $R$-submodule of $M$ and $(N : M) \cap S = \phi$. Suppose $0 \neq \frac{r \cdot s}{s} \in S^{-1}N$ for some $\frac{r}{s}, \frac{t}{s} \in h(S^{-1}R)$
Theorem 2.16. Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a $gr$-weakly classical prime submodule of $M$. Then for each $g \in M_g$, either $N_g$ is a classical prime $R_e$-submodule of $M_g$ or $(N_g : R_e M_g)^2 N_g = 0$.

Proof. By Proposition 2.8, $N_g$ is a weakly classical prime $R_e$-submodule of $M_g$ for every $g \in M_g$. It is enough to show that if $(N_g : R_e M_g)^2 N_g \neq 0$ for some $g \in G$, then $N_g$ is a classical prime $R_e$-submodule of $M_g$. Let $rs m \in N_g$ where $r, s \in R_e$ and $m \in M_g$. If $rs m \neq 0$, then either $rm \in N_g$ or $sm \in N_g$ since $N_g$ is a weakly classical prime $R_e$-submodule of $M_g$. So suppose that

$$rs n = 0.$$  

If $rs N_g \neq 0$, then there is an element $n \in N_g$ such that $rs n \neq 0$, so $0 \neq rs(m + n) = rs n \in N_g$, so we conclude that $r(m + n) \in N_g$ or $s(m + n) \in N_g$. Thus $rm \in N_g$ or $sm \in N_g$. So we can assume that $rs N_g = 0$. If $r(N_g : R_e M_g)m \neq 0$ then there is an element $w \in (N_g : R_e M_g)$ such that $rwm \neq 0$. Then $r(s + w)m \neq 0$ because $rs m = 0$. Since $wm \in N_g$, $r(s + w)m \in N_g$. Then $rm \in N_g$ or $(s + w)m \in N_g$. Hence $rm \in N_g$ or $sm \in N_g$. So we can assume that $r(N_g : R_e M_g)m = 0$. Similarly, we can assume that $s(N_g : R_e M_g)m = 0$. If $r(N_g : R_e M_g)N_g \neq 0$ then $rka \neq 0$ for some $k \in (N_g : R_e M_g)$ and $a \in N_g$. Since $rs N_g = 0$ and $r(N_g : R_e M_g)m = 0$, we conclude that $0 \neq r(s + k)(m + a) = rka \in N_g$. So $r(m + a) \in N_g$ or $(s + k)(m + a) \in N_g$. Hence $rm \in N_g$ or $sm \in N_g$. So we can assume that $r(N_g : R_e M_g)N_g = 0$. Similarly, we can assume that $s(N_g : R_e M_g)N_g = 0$. Since we assume that $(N_g : R_e M_g)^2 N_g \neq 0$, there are $r_1, r_2 \in (N_g : R_e M_g)$ and $t \in N_g$ such that $r_1 r_2 t \neq 0$. Then $(r + r_1)(s + r_2)(m + t) = r_1 r_2 t \in N_g$. So $(r + r_1)(m + t) \in N_g$ or $(s + r_2)(m + t) \in N_g$. Hence $rm \in N_g$ or $sm \in N_g$. Thus $N_g$ is a classical prime $R_e$-submodule of $M_g$. 

□
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References

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