Chromaticity of Turán Graphs with At Most Three Edges Deleted

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Abstract. Let $P(G, \lambda)$ be the chromatic polynomial of a graph $G$. A graph $G$ is chromatically unique if for any graph $H$, $P(H, \lambda) = P(G, \lambda)$ implies $H$ is isomorphic to $G$. In this paper, we determine the chromaticity of all Turán graphs with at most three edges deleted. As a by product, we found many families of chromatically unique graphs and chromatic equivalence classes of graphs.

Keywords: Chromatic polynomial, Chromatic uniqueness, Turán graph.

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1. Introduction

All graphs considered in this paper are finite and simple. Graph polynomials are a well-developed area useful for analyzing properties of graphs. Chromatic polynomial, characteristic polynomial and domination polynomial are some examples of these polynomials (see [1, 2, 6]). For a graph $G$, we denote by $P(G; \lambda)$ (or $P(G)$), the chromatic polynomial of $G$. Two graphs $G$ and $H$ are said to be chromatically equivalent (simply $\chi$-equivalent), denoted $G \sim H$ if $P(G) = P(H)$. Let $[G]$ denote the equivalence class determined by the graph $G$. A graph $G$ is said to be chromatically unique (simply $\chi$-unique) if $[G] = \{G\}$. A family $\mathcal{G}$ of graphs is said to be chromatically closed (simply $\chi$-closed) if for any graph $G \in \mathcal{G}$, $P(H) = P(G)$ implies that $H \in \mathcal{G}$. For two families of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, if $P(G_1) \neq P(G_2)$ for each $G_1 \in \mathcal{G}_1$ and each $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint (simply $\chi$-disjoint). Many families of $\chi$-unique graphs are known (see [7, 8, 11, 12, 13]).

For a graph $G$, let $e(G)$, $v(G)$, $t(G)$, $Q(G)$, $K(G)$ and $\chi(G)$ respectively be the number of vertices, edges, triangles, induced 4-cycle subgraph, complete subgraph $K_4$ and chromatic number of $G$. By $\overline{G}$, we denote the complement of $G$. Let $O_n$ be an edgeless graph with $n$ vertices. Suppose $S$ is a set of $s$ edges of $G$. Denote by $G - S$ the graph obtained from $G$ by deleting all edges in $S$ and by $(S)$ the graph induced by $S$. For $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \cdots \leq p_t$, let $F = K(p_1, p_2, \ldots, p_t)$ be a complete $t$-partite graph with partition sets $V_i$ such that $|V_i| = p_i$ for $i = 1, 2, \ldots, t$. The Turán graph, denoted $T = K(t_1 \times p, t_2 \times (p+1))$, is the unique complete $t$-partite graph having $t_1 \geq 1$ partite sets of size $p$ and $t_2$ partite sets of size $p+1$. In this paper, we determine the chromaticity of all Turán graphs with at most $s$ ($\leq 3$) edges deleted for $p \geq s+2$. As a by product, we found many families of chromatically unique graphs and equivalent classes of graphs.
2. Preliminary results and notations

Let $K^{-s}(p_1, p_2, \ldots, p_t)$ be the family $\{K(p_1, p_2, \ldots, p_t) - S | S \subset E(G) \text{ and } |S| = s\}$. For $p_1 \geq s + 1$, we denote by $K^{-K(1,s)}_{i,j}(p_1, p_2, \ldots, p_t)$ the graph in $K^{-s}(p_1, p_2, \ldots, p_t)$ where the $s$ edges in $S$ induced a $K(1,s)$ with center in $V_i$ and all the end-vertices in $V_j$, and by $K^{-sK_2}_{i,j}(p_1, p_2, \ldots, p_t)$ the graph in $K^{-s}(p_1, p_2, \ldots, p_t)$ where the $s$ edges in $S$ induced a matching with one end-vertex in $V_i$ and another end-vertex in $V_j$. For convenience, we let $T^{-s} = K^{-s}(t_1 \times p, t_2 \times (p + 1)), T^{-K(1,s)}_{i,j} = K^{-K(1,s)}_{i,j}(t_1 \times p, t_2 \times (p + 1))$ and $T^{-sK_2}_{i,j} = K^{-sK_2}_{i,j}(t_1 \times p, t_2 \times (p + 1))$. Hence, each graph in $T^{-s}$ has $t_1 \geq 1$ partite sets of size $p$ and $t_2$ partite sets of size $p + 1$.

For a graph $G$ and a positive integer $k$, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a $k$-independent partition in $G$ if each $A_i$ is a non-empty independent set of $G$. Let $\alpha(G, k)$ denote the number of $k$-independent partitions in $G$. If $G$ is of order $n$, then $P(G, \lambda) = \sum_{k=1}^{n} \alpha(G, k)(\lambda)^k$ where $(\lambda)^k = \lambda(\lambda - 1) \cdots (\lambda - k + 1)$ (see [14]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots$, if $G \sim H$.

For a graph $G$ with $p$ vertices, the polynomial $\sigma(G, x) = \sum_{k=1}^{p} \alpha(G, k)x^k$ is called the $\sigma$-polynomial of $G$ (see [3]), and the polynomial $h(G, x) = \sum_{k=1}^{p} \alpha(\overline{G}, k)x^k$ is called the adjoint polynomial of $G$ (see [9]). If $h(G, x) = h(H, x)$ then we say $G$ is adjointly equivalent to $H$, denoted $G \sim_h H$. A family $\mathcal{G}$ of graphs is said to be adjoint closed (simply $\chi_k$-closed) if for any graph $G \in \mathcal{G}$, $h(H, x) = h(G, x)$ implies that $H \in \mathcal{G}$. Clearly, the conditions $P(G, \lambda) = P(H, \lambda)$, $\sigma(G, x) = \sigma(H, x)$ and $h(G, x) = h(H, x)$ are equivalent for any graphs $G$ and $H$.

For disjoint graphs $G$ and $H$, $G + H$ denotes the disjoint union of $G$ and $H$, and $mG$ the disjoint union of $m$ copies of $G$; $G \vee H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy | x \in V(G) \text{ and } y \in V(H)\} \cup E(G) \cup E(H)$. Throughout this paper, all the $t$-partite graphs $G$ under consideration are 2-connected with $\chi(G) = t$. For terms used but not defined here we refer to [16].
Lemma 2.1. (see [7]) Let $G$ and $H$ be two graphs with $H \sim G$, then $v(G) = v(H)$, $e(G) = e(H)$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,k) = \alpha(H,k)$ for each $k = 1, 2, \ldots$, and

$$-Q(G) + 2K(G) = -Q(H) + 2K(H).$$

Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.

Lemma 2.2. (Brenti [3]) Let $G$ and $H$ be two disjoint graphs. Then

$$\sigma(G \lor H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \ldots, n_t), x) = \prod_{i=1}^{t} \sigma(K_n, x).$$

The above lemma is equivalent to the following:

Remark. Let $G$ and $H$ be two disjoint graphs. Then

$$h(G \lor H, x) = h(G, x)h(H, x).$$

In particular,

$$h(K(n_1, n_2, \ldots, n_t), x) = \prod_{i=1}^{t} h(K_{n_i}, x).$$

For an edge $e = v_1v_2$ of a graph $G$, then $G * e$ is defined as follows: the vertex set of $G * e$ is $(V(G) \setminus \{v_1, v_2\}) \cup \{u\}$, and the edge set of $G * e$ is $\{e' \mid e' \in E(G), e' \text{ is not incident with } v_1 \text{ or } v_2\} \cup \{uv \mid v \in N_G(v_1) \cap N_G(v_2)\}$. For example, let $e_1$ be an edge of $C_4$ and $e_2$ an edge of $K_4$. Then $C_4 * e_1 = K_1 + K_2$ and $K_4 * e_2 = K_3$.

Lemma 2.3. (Liu [10]) Let $G$ be a graph with $e \in E(G)$. Then

$$h(G, x) = h(G - e, x) + h(G * e, x),$$

In particular, if $e = uv$ does not belong to any triangle of $G$, then

$$h(G, x) = h(G - e, x) + xh(G - \{u, v\}, x),$$

where $G - e$ (respectively $G - \{u, v\}$) denote the graph obtained by deleting the edge $e$ (respectively the vertices $u$ and $v$) from $G$. 

Denote by $\beta(G)$ the minimum real root of $h(G)$.

**Lemma 2.4.** (Zhao [17]) Let $G$ be a connected graph such that $G$ contains $H$ as a proper subgraph. Then

$$\beta(G) < \beta(H).$$

**Lemma 2.5.** (Teo and Koh [15]) The graph $K(p, q)$ is $\chi$-unique for all $q \geq p \geq 2$.

Suppose $F = K(p_1, p_2, \ldots, p_t)$ and $G = F - S$ for a set $S$ of $s$ edges of $G$. A non-empty independent set of $G$ is improper if it is not a subset of a partite set of $G$. Otherwise, it is proper. A $k$-independent partition in $G$ is improper if it contains at least one improper independent set. Define $\alpha_k(G) = \alpha(G, k) - \alpha(F, k)$ for $k \geq t + 1$. Hence, $\alpha_k(G)$ is the number of $k$-independent partitions in $G$ that contains at least one improper independent set.

**Lemma 2.6.** (Zhao [17]) Let $F = K(p_1, p_2, \ldots, p_t)$ and $G = F - S$. If $p_1 \geq s + 1$, then

$$s \leq \alpha_{t+1}(G) = \alpha(G, t + 1) - \alpha(F, t + 1) \leq 2^s - 1,$$

$\alpha_{t+1}(G) = s$ if and only if the subgraph induced by any $r \geq 2$ edges in $S$ is not a complete multipartite graph, and $\alpha_{t+1}(G) = 2^s - 1$ if and only if all $s$ edges in $S$ share a common end-vertex and the other end-vertices belong to the same $V_i$ for some $i$.

**Lemma 2.7.** (Dong et al. [5]) Let $p_1, p_2$ and $s$ be positive integers with $3 \leq p_1 \leq p_2$, then

(i) $K_{1, 2}^{K(1, s)}(p_1, p_2)$ is $\chi$-unique for $1 \leq s \leq p_2 - 2$,

(ii) $K_{2, 1}^{K(1, s)}(p_1, p_2)$ is $\chi$-unique for $1 \leq s \leq p_1 - 2$, and

(iii) $K^{sK_2}(p_1, p_2)$ is $\chi$-unique for $1 \leq s \leq p_2 - 1$.

**Lemma 2.8.** (Zhao [17]) Let $s \geq 1$, $p \geq 2$ and $t_1 \geq 1$. If $p \geq s + 2$, then $T^{-s}$ is $\chi$-closed.

**Lemma 2.9.** (Zhao [17]) Suppose $s \geq 1$ and $p \geq s + 2$, then
(i) every \(T^{−K(1,s)}_{i,j}\) is \(\chi\)-unique for any \((i,j)\) where \(1 \leq i \neq j \leq t\) and
\[
|V_i| = |V_j| = p, \text{ or } |V_i| = p, |V_j| = p + 1, \text{ or } |V_i| = p + 1, |V_j| = p, \text{ or }
|V_i| = |V_j| = p + 1.
\]
(ii) \(T^{−K(2)}_{1,2}\) is \(\chi\)-unique if \(t_1 = 2\).

3. Graphs in \(\mathcal{T}^{−s}\) for \(|S| = s \leq 2\) and \(p \geq s + 2\)

It is well known that the Turán graph is \(\chi\)-unique [4]. By Lemma 2.9, all graphs in \(\mathcal{T}^{−1}\) are \(\chi\)-unique for \(p \geq 3\). We shall now determine the chromaticity of all graphs in \(\mathcal{T}^{−2}\) for \(p \geq 4\). Let \(G\) be a graph in \(\mathcal{T}^{−2}\). Denote by \(G\) the (disjoint union of all) non-complete component(s) of \(G\). It is then easy to verify by exhaustive construction that \(\mathcal{T}^{−2}\) contains 25 non-isomorphic graphs, named \(G_{2,i}\) \((1 \leq i \leq 25)\), with the graph \(G_{2,i}'\) shown in Table 1. Note that each “circle” associated with \(G_{2,i}'\) is a complete graph of order \(p\) or \(p + 1\) as indicated.

Suppose \(F = K(p_1,p_2,\ldots,p_t)\). For \(G = F - S\), denote by \(t_i(G)\) the number of triangles in \(G\) that contains \(i\) deleted edges in \(S\) for \(i = 1, 2, 3\). Suppose \(G \in \mathcal{T}^{−s}\). An edge \(e = uv\) in \(S\) is of Type A (respectively, Type B and Type C) if \(u \in V_i, v \in V_j\) for \(1 \leq i < j \leq t_1\) (respectively, for \(1 \leq i \leq t_1, t_1 + 1 \leq j \leq t\), and for \(t_1 + 1 \leq i < j \leq t\)). Denote by \(s_1(G)\) (respectively, \(s_2(G)\) and \(s_3(G)\)) the number of Type A (respectively, Type B and Type C) edges in \(S\).

By Lemma 2.6, we have \(2 \leq \alpha_{t+1}(G) \leq 3\). Note that for \(1 \leq i \leq 25\), \(t(G_{2,i}) = t(F) - 2(t_1p + t_2(p+1)) + 4p + k\) where \(0 \leq k = s_2(G_{2,i}) + 2s_3(G_{2,i}) + t_2(G_{2,i}) \leq 5\). We compute the ordered pair \((\alpha_{t+1}(G_{2,i}), k)\) for each \(G_{2,i}\) \((1 \leq i \leq 25)\), and the results are shown in Table 1.

We then partition the family \(\mathcal{T}^{−2}\) according to the value \((\alpha_{t+1}(G_{2,i}), k)\), and each part of the partition is denoted by \(\mathcal{G}(\alpha_{t+1}(G_{2,i}), k)\). Hence, we have the following classification of the graphs.
Table 1 (1 of 2): Non-complete component(s) of $G'_{2,i}$ (1 ≤ $i$ ≤ 25)
for $G_{2,i} \in \mathcal{K}^{-2}(t_1 \times p, t_2 \times (p + 1))$

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<th>$G'_{2,1}$</th>
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<th>$G'_{2,9}$</th>
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<td>$p$</td>
<td>$p+1$</td>
<td>$p$</td>
<td>$p+1$</td>
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<td>(2,1)</td>
<td>(2,2)</td>
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Table 1 (2 of 2): Non-complete component(s) of $G_{2,i}$ $(1 \leq i \leq 25)$ for $G_{2,i} \in \mathcal{K}^{-2} (t_1 \times p, t_2 \times (p + 1))$

$G_1 = G(2, 0) = \{G_{2,1}; G_{2,8}; G_{2,20}\};$
$G_2 = G(2, 1) = \{G_{2,9}; G_{2,14}; G_{2,21}\};$
$G_3 = G(2, 2) = \{G_{2,2}; G_{2,10}; G_{2,11}; G_{2,15}; G_{2,22}; G_{2,23}\};$
$G_4 = G(2, 3) = \{G_{2,12}; G_{2,16}; G_{2,17}; G_{2,24}\};$
$G_5 = G(2, 4) = \{G_{2,13}; G_{2,18}; G_{2,25}\};$
$G_6 = G(2, 5) = \{G_{2,19}\};$
$G_7 = G(3, 0) = \{G_{2,4}\};$
$G_8 = G(3, 2) = \{G_{2,5}; G_{2,6}\};$
$G_9 = G(3, 4) = \{G_{2,7}\}.$

Note that for an edgeless graph $O_n$, $\alpha(O_n, 2) = 2^{n-1} - 1$ and $\alpha(O_n, 3) = \frac{1}{3!} (3^n - 3 \cdot 2^n + 3)$. We now present our main theorem of this section.
Theorem 3.1. For integers \( p \geq 4, t_1 \geq 1 \) and \( t_1 + t_2 \geq 3 \), all the graphs in \( T^{-2} \) are \( \chi \)-unique except that \( \{G_{2,16}, G_{2,17} \mid t_1, t_2 \geq 2 \} \) is a \( \chi \)-equivalence class.

Proof. By Lemma 2.8, \( T^{-2} \) is \( \chi \)-closed. Observe that for \( 1 \leq i < j \leq 9 \) and each graph \( H' \in G_i \) and each graph \( H'' \in G_j \), either \( \alpha_{t+1}(H') \neq \alpha_{t+1}(H'') \) or \( t(H') \neq t(H'') \). By Lemma 2.1, \( H' \neq H'' \). Hence, each \( G_i \) and \( G_j \) \( (1 \leq i < j \leq 9) \) are \( \chi \)-disjoint. Since \( T^{-2} \) is \( \chi \)-closed, we conclude that each \( G_i \) \( (1 \leq i \leq 9) \) is \( \chi \)-closed. It follows immediately that \( G_{2,4}, G_{2,7} \) and \( G_{2,19} \) are \( \chi \)-unique. By Lemma 2.9(i), we also know that \( G_{2,5} \) and \( G_{2,6} \) are \( \chi \)-unique.

We now determine the chromaticity of the graphs in \( G_i \) for \( 1 \leq i \leq 5 \). It suffices to show that for any two graphs \( G' \) and \( G'' \) in \( G_i \), either \( \alpha_k(G') \neq \alpha_k(G'') \) for \( k = t + 2 \) or \( t + 3 \), or \( \beta(G') \neq \beta(G'') \). Otherwise, we shall show that \( h(G', x) = h(G'', x) \). Since the proofs in determining the chromaticity of each graph in \( G_i \) are similar for \( 1 \leq i \leq 5 \), we shall only elaborate in more details for graphs in \( G_2 \) and \( G_3 \).

(1). Graphs in \( G_1 \). We first note that by Lemma 2.9(ii), \( G_{2,1} \) is \( \chi \)-unique. Now observe that if \( G_{2,8} \sim G_{2,20} \), then Lemma 2.2 implies that \( h(K_p + G'_{2,8}, x) = h(G_{2,20}, x) \). However, Lemma 2.4 implies that \( \beta(K_p + G'_{2,8}) = \beta(G'_{2,8}) < \beta(G_{2,20}) \), a contradiction. Hence, all the graphs in \( G_1 \) are \( \chi \)-unique.

(2). Graphs in \( G_2 \). We compute \( \alpha_{t+2}(G) \) for each \( G \in G_2 \).

Let \( \mathcal{V} = \{V_1, V_2, \ldots, V_t\} \) be the unique \( t \)-independent partition of \( G \in T^{-2} \). Suppose \( \mathcal{V}' = \{V'_1, V'_2, \ldots, V'_t, V'_{t+1}, V'_{t+2}\} \) is an improper \((t+2)\)-independent partition in \( G \). Since \( p \geq 4 \), \( \mathcal{V}' \) has at most two improper independent sets. If \( \mathcal{V}' \) contains exactly two improper independent sets, say \( V'_{t+1} \) and \( V'_{t+2} \), then \( V_i \setminus (V'_{t+1} \cup V'_{t+2}) \) is a proper independent set in \( \mathcal{V}' \) for each \( 1 \leq i \leq t \). If \( \mathcal{V}' \) contains exactly one improper independent set, say \( V'_{t+1} \), then we may assume that \( V_i \cap V'_{t+2} = \emptyset \) for \( i = 1, 2, \ldots, t - 2 \), and \( V_i \cap V'_{t+2} \neq \emptyset \) for \( i = t - 1, t \). Hence, there exist exactly two proper independent sets in \( \mathcal{V}' \), say \( V'_i \) and \( V'_{i+1} \).
such that $V'_t \cup V'_{t+1} = V$ for $V \in \{V_1, V_2, \ldots, V_{t-2}, V_{t-1} \setminus V'_t, V_t \setminus V'_{t+2}\}$. Define
\[
    f_1 = (t_1 + 2)(2^p - 1) + (t_2 - 2)(2^p - 1),
\]
\[
    f_2 = (2^p - 1) + t_1(2^p - 1) + (t_2 - 1)(2^p - 1),
\]
\[
    f_3 = 2(2^p - 1) + (t_1 - 2)(2^p - 1) + t_2(2^p - 1).
\]

Hence,
\[
    \alpha_{t+2}(G_{2, i}) = 1 + f_2 + f_3 \quad \text{for} \quad i = 9, 21,
\]
\[
    \alpha_{t+2}(G_{2, 14}) = 2f_3.
\]

Since $p \geq 4$, $\alpha_{t+2}(G_{2, 14}) - \alpha_{t+2}(G_{2, i}) = 2^{p-2} - 1 > 0$ for $i = 9, 21$. We now show that $G_{2, 9} \not\sim G_{2, 21}$. By Lemma 2.4, $\beta(G_{2, 9}) = \beta(G'_{2, 9}) < \beta(G'_{2, 21}) = \beta(G_{2, 21})$, a contradiction. Hence, all the graphs in $G_2$ are $\chi$-unique.

(3). Graphs in $G_3$. By a similar argument as in (2) above, we have for each graph $G \in G_3$,
\[
    \alpha_{t+2}(G_{2, i}) = 1 + 2f_2 \quad \text{for} \quad i = 2, 10, 11, 23,
\]
\[
    \alpha_{t+2}(G_{2, 15}) = f_2 + f_3,
\]
\[
    \alpha_{t+2}(G_{2, 22}) = 1 + f_1 + f_3.
\]

Since $p \geq 4$, $\alpha_{t+2}(G_{2, 15}) - \alpha_{t+2}(G_{2, i}) = 2^{p-2} - 1 > 0$ for $i = 2, 10, 11, 22, 23$. We need to compute $\alpha_{t+3}(G_{2, i})$ for $i = 2, 10, 11, 22, 23$. Suppose $V' = \{V'_t, V'_{t+1}, V'_1, V'_{t+2}, V'_{t+3}\}$ is an improper $(t+3)$-independent partition in $G$. If $V'$ contains exactly two improper independent sets, say $V'_{t+2}$ and $V'_{t+3}$, then exactly one of $V'_i \setminus (V'_{t+2} \cup V'_{t+3})$ ($1 \leq i \leq t$) needs to be partitioned into two proper independent sets. Note that $V'$ has at most two improper independent sets. If $V'$ contains exactly one improper independent set, say $V'_{t+3}$, then either exactly one of $V'_i \setminus V'_{t+3}$ ($1 \leq i \leq t$) needs to be partitioned into three proper independent sets, or else exactly two of $V'_i \setminus V'_{t+3}$ ($1 \leq i \leq t$)
need to be partitioned into two proper independent sets respectively. Define
g_1 = \left[ \frac{t_1 + 2}{3!} (3^p - 3 \cdot 2^p + 3) + \frac{t_2 - 2}{3!} (3^{p+1} - 3 \cdot 2^{p+1} + 3) \right] +
\left[ \left( \frac{t_1 + 2}{2} \right) (2^{p-1} - 1)^2 + (t_1 + 2)(t_2 - 2)(2^{p-1} - 1)(2^p - 1) + \right.
\left( \frac{t_2 - 2}{2} \right) (2^p - 1)^2 \right],

g_2 = \left[ \frac{1}{3!} (3^{p-1} - 3 \cdot 2^{p-1} + 3) + \frac{t_1}{3!} (3^p - 3 \cdot 2^p + 3) + \right.
\frac{t_2 - 1}{3!} (3^{p+1} - 3 \cdot 2^{p+1} + 3) \left] + \left[ t_1 (2^{p-2} - 1)(2^{p-1} - 1) + 
(t_2 - 1)(2^{p-2} - 1)(2^p - 1) + \frac{t_1}{2} (2^{p-1} - 1)^2 + 
\frac{t_1 (t_2 - 1)(2^{p-1} - 1)(2^p - 1) + \frac{t_2 - 1}{2} (2^p - 1)^2 \right], \right.

\left[ \frac{2}{3!} (3^{p-1} - 3 \cdot 2^{p-1} + 3) + \frac{t_1 - 2}{3!} (3^p - 3 \cdot 2^p + 3) + \right.
\frac{t_2}{3!} (3^{p+1} - 3 \cdot 2^{p+1} + 3) \left] + \left[ (2^{p-2} - 1)^2 + 
2(t_1 - 2)(2^{p-2} - 1)(2^{p-1} - 1) + 2t_2(2^{p-2} - 1)(2^p - 1) + 
\frac{t_1 - 2}{2} (2^{p-1} - 1)^2 + (t_1 - 2)t_2(2^{p-1} - 1)(2^p - 1) + \right.
\left( \frac{t_2}{2} \right) (2^p - 1)^2 \right]. \right.

Hence,
\alpha_{t+3}(G_{2,2}) = \left[ (2^{p-3} - 1) + (2^{p-2} - 1) + (t_1 - 1)(2^{p-1} - 1) + 
(t_2 - 1)(2^p - 1) \right] + 2g_2,
\alpha_{t+3}(G_{2,10}) = \left[ 3(2^{p-2} - 1) + (t_1 - 2)(2^{p-1} - 1) + (t_2 - 1)(2^p - 1) \right] + 
2g_2.
Clearly, and By Lemmas 2.2 and 2.3, we have

\[
\begin{align*}
\alpha_{t+3}(G_{2,11}) &= [(2^{p-3} - 1) + (t_1 + 1)(2^{p-1} - 1) + (t_2 - 2)(2^p - 1)] + 2g_2, \\
\alpha_{t+3}(G_{2,22}) &= [2(2^{p-2} - 1) + t_1(2^{p-1} - 1) + (t_2 - 2)(2^p - 1)] + g_1 + g_3, \\
\alpha_{t+3}(G_{2,23}) &= 2(2^{p-2} - 1) + t_1(2^{p-1} - 1) + (t_2 - 2)(2^p - 1) + 2g_2.
\end{align*}
\]

Since \( p \geq 4 \), by Software Maple,

\[
\begin{align*}
\alpha_{t+3}(G_{2,22}) - \alpha_{t+3}(G_{2,2}) &= 2^{2p-4} - 3(2^{p-3}) > 0, \\
\alpha_{t+3}(G_{2,2}) - \alpha_{t+3}(G_{2,10}) &= 2^{p-3} > 0, \\
\alpha_{t+3}(G_{2,10}) - \alpha_{t+3}(G_{2,11}) &= 2^{p-3} > 0, \\
\alpha_{t+3}(G_{2,11}) - \alpha_{t+3}(G_{2,23}) &= 2^{p-3} > 0.
\end{align*}
\]

Hence, all the graphs in \( G_3 \) are \( \chi \)-unique.

(4). Graphs in \( G_4 \). For each graph \( G \in G_4 \),

\[
\begin{align*}
\alpha_{t+2}(G_{2,i}) &= 1 + f_1 + f_2 \quad \text{for} \quad i = 12, 24, \\
\alpha_{t+2}(G_{2,j}) &= 2f_2 \quad \text{for} \quad j = 16, 17.
\end{align*}
\]

Since \( p \geq 4 \), \( \alpha_{t+2}(G_{2,i}) - \alpha_{t+2}(G_{2,i}) = 2^{p-2} - 1 > 0 \). By Lemma 2.4, \( \beta(G_{2,12}) = \beta(G_{2,12}) < \beta(G_{2,24}) = \beta(G_{2,24}) \).

By Lemmas 2.2 and 2.3, we have

\[
\begin{align*}
h(G_{16}, x) &= [h(K_p, x)]^{t_1-2}h(K_{p+1}, x)]^{t_2-1}h(G_{2,16}, x) \\
&= [h(K_p, x)]^{t_1-2}[h(K_{p+1}, x)]^{t_2-1}[(h(K_p, x))^2h(K_{p+1}, x) + 2xh(K_p, x)(h(K_p, x))^2]
\end{align*}
\]

and

\[
\begin{align*}
h(G_{17}, x) &= [h(K_p, x)]^{t_1-1}[h(K_{p+1}, x)]^{t_2-2}h(G_{2,17}, x) \\
&= [h(K_p, x)]^{t_1-1}[h(K_{p+1}, x)]^{t_2-2}[(h(K_p, x))^2h(K_{p+1}, x) + 2xh(K_p, x)h(K_p, x)]
\end{align*}
\]

Clearly, \( G_{2,16} \sim G_{2,17} \) for \( t_1, t_2 \geq 2 \). Hence, all the graphs in \( G_4 \) are \( \chi \)-unique except that \( \{G_{2,16}, G_{2,17} | t_1, t_2 \geq 2 \} \) is a \( \chi \)-equivalence class.

(5). Graphs in \( G_5 \). For each \( G \in G_5 \),

\[
\alpha_{t+2}(G_{2,i}) = 1 + 2f_1 \quad \text{for} \quad i = 3, 13, 25,
\]

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\[ \alpha_{t+2}(G_{2,18}) = f_1 + f_2. \]

Since \( p \geq 4 \), \( \alpha_{t+2}(G_{2,18}) > \alpha_{t+2}(G_{2,1}) = 2^{p-2} - 1 > 0 \). We now compare \( \alpha_{t+3}(G_{2,3}) \), \( \alpha_{t+3}(G_{2,13}) \) and \( \alpha_{t+3}(G_{2,25}) \). We have

\[
\begin{align*}
\alpha_{t+3}(G_{2,3}) & = [2(2^{p-2} - 1) + t_1(2^{p-1} - 1) + (t_2 - 2)(2^p - 1)] + 2g_1, \\
\alpha_{t+3}(G_{2,13}) & = [(2^{p-2} - 1) + (t_1 + 2)(2^{p-1} - 1) + (t_2 - 3)(2^p - 1)] + 2g_1, \\
\alpha_{t+3}(G_{2,25}) & = [(t_1 + 4)(2^{p-1} - 1) + (t_2 - 4)(2^p - 1)] + 2g_1.
\end{align*}
\]

Since \( p \geq 4 \), by Software Maple, \( \alpha_{t+3}(G_{2,3}) - \alpha_{t+3}(G_{2,13}) = \alpha_{t+3}(G_{2,13}) - \alpha_{t+3}(G_{2,25}) = 2^{p-2} - 1 > 0 \). Hence, all the graphs in \( G_5 \) are \( \chi \)-unique.

The proof is now complete.

4. Graphs in \( T^{-3} \) for \( p \geq 5 \)

Let \( u, v \) and \( w \) be a vertex of the complete graphs \( K_p, K_q \) and \( K_r \) respectively. Denote by \( K_p \cdot K_q \) the graph obtained from \( K_p + K_q \) by adding the edge \( uv \). For convenience, let \( K_p^+ = K_1 \cdot K_p \). Also let \( K_p \cdot K_q \cdot K_r \) be the graph obtained from \( K_p + K_q + K_r \) by adding two edges \( uv \) and \( vw \).

We now list all the 25 possible “structures” induced by three edges deleted from the Turán graph which can be obtained by brute force construction (with respect to the partite sets) in Figure 1.

Since each circle in Figure 1 is a partite set of size \( p \) or \( p + 1 \), we obtain from the “structures” in Figure 1, 213 non-isomorphic graphs in \( T^{-3} \), named \( G_{3,i} \) (\( 1 \leq i \leq 213 \)). A table of the graphs \( G'_{3,i} \) similar to that of Table 1 is available upon request. Note that each “circle” associated with \( G'_{3,i} \) is a complete graph of order \( p \) or \( p + 1 \) as indicated. For example, the top left “structure” in Figure 1 will give us three non-isomorphic graphs \( G_{3,1}, G_{3,2} \) and \( G_{3,3} \) as shown in Figure 2.

By Lemma 2.6, we have \( 3 \leq \alpha_{t+1}(G_{3,i}) \leq 7 \) for \( 1 \leq i \leq 213 \). Also note that for \( 1 \leq i \leq 213 \), \( t(G_{3,i}) = t(F) - 3(t_1p + t_2(p + 1)) + 6p + k \) where
0 \leq k = s_2(G_{3,i}) + 2s_3(G_{3,i}) + t_2(G_{3,i}) - t_3(G_{3,i}) \leq 9. We now compute the ordered pair \((\alpha_{t+1}(G_{3,i}), k)\) for each \(G_{3,i}\) \((1 \leq i \leq 213)\). A table of \(G'(3, i)\) with \((\alpha_{t+1}(G_{3,i}), k)\) is also available upon request. We then partition the family \(T^{-3}\) according to the value \((\alpha_{t+1}(G_{3,i}), k)\), and each part of the partition is denoted by \(G(\alpha_{t+1}(G_{3,i}), k)\).

Figure 1. List of “structures” induced by three edges deleted from \(K(t_1 \times p, t_2 \times (p+1))\)

Figure 2: Graphs obtained from top left structure in Figure 1.

Hence, we have the following classification of the graphs.

\[ G_1 = G(3, 0) = \{G_{3,1}; G_{3,15}; G_{3,35}; G_{3,83}; G_{3,104}; G_{3,140}; G_{3,168}; G_{3,204}\}; \]
\( G_2 = \mathcal{G}(3, 1) = \{ G_{3.19}; G_{3.36}; G_{3.59}; G_{3.84}; G_{3.105}; G_{3.114}; G_{3.141}; G_{3.148}; G_{3.169}; G_{3.170}; G_{3.186}; G_{3.205} \}; \\
G_4 = \mathcal{G}(3, 3) = \{ G_{3.2.2}; G_{3.2.20}; G_{3.2.21}; G_{3.3.8}; G_{3.3.9}; G_{3.6.1}; G_{3.8.7}; G_{3.10.8}; G_{3.10.9}; G_{3.11.7}; G_{3.11.8}; G_{3.11.9}; G_{3.13.1}; G_{3.14.2}; G_{3.14.5}; G_{3.15.2}; G_{3.15.3}; G_{3.16.9}; G_{3.17.4}; G_{3.17.6}; G_{3.17.7}; G_{3.17.9}; G_{3.19.9}; G_{3.19.11}; G_{3.19.13}; G_{3.20.8}; G_{3.20.9}; \}; \\
G_5 = \mathcal{G}(3, 4) = \{ G_{3.1.17}; G_{3.1.26}; G_{3.2.7}; G_{3.4.6}; G_{3.6.22}; G_{3.6.3}; G_{3.8.38}; G_{3.8.9}; G_{3.11.10}; G_{3.11.11}; G_{3.12.0}; G_{3.12.1}; G_{3.12.2}; G_{3.12.3}; G_{3.13.13}; G_{3.14.4}; G_{3.15.1}; G_{3.15.6}; G_{3.16.1}; G_{3.17.5}; G_{3.18.0}; G_{3.18.1}; G_{3.18.2}; G_{3.19.2}; G_{3.19.14}; G_{3.19.5}; G_{3.19.7}; G_{3.21.0}; G_{3.21.1}; \}; \\
G_6 = \mathcal{G}(3, 5) = \{ G_{3.22}; G_{3.23}; G_{3.41}; G_{3.64}; G_{3.90}; G_{3.11.2}; G_{3.12.4}; G_{3.12.5}; G_{3.12.6}; G_{3.13.4}; G_{3.13.5}; G_{3.14.6}; G_{3.15.4}; G_{3.15.5}; G_{3.16.3}; G_{3.18.3}; G_{3.18.4}; G_{3.19.6}; G_{3.19.8}; G_{3.19.9}; G_{3.20.0}; G_{3.21.2}; \}; \\
G_7 = \mathcal{G}(3, 6) = \{ G_{3.3.3}; G_{3.3.15}; G_{3.2.25}; G_{3.2.25.4}; G_{3.2.25.4}; G_{3.6.5}; G_{3.9.1}; G_{3.11.3}; G_{3.12.7}; G_{3.12.8}; G_{3.13.6}; G_{3.13.7}; G_{3.14.7}; G_{3.15.7}; G_{3.15.8}; G_{3.16.2}; G_{3.16.5}; G_{3.18.5}; G_{3.20.11}; G_{3.20.2}; G_{3.21.3}; \}; \\
G_8 = \mathcal{G}(3, 7) = \{ G_{3.24}; G_{3.66}; G_{3.129}; G_{3.138}; G_{3.159}; G_{3.164}; G_{3.203}; \}; \\
G_9 = \mathcal{G}(3.8) = \{ G_{3.30}; G_{3.119}; G_{3.166}; \}; \\
G_{10} = \mathcal{G}(3, 9) = \{ G_{3.167}; \}; \\
G_{11} = \mathcal{G}(4, 0) = \{ G_{3.4}; G_{3.43}; G_{3.51}; G_{3.92}; \}; \\
G_{12} = \mathcal{G}(4, 1) = \{ G_{3.44}; G_{3.52}; G_{3.67}; G_{3.93}; \}; \\
G_{13} = \mathcal{G}(4, 2) = \{ G_{3.31}; G_{3.45}; G_{3.53}; G_{3.68}; G_{3.75}; G_{3.94}; G_{3.95}; G_{3.96}; \}; \\
G_{14} = \mathcal{G}(4, 3) = \{ G_{3.5}; G_{3.6}; G_{3.46}; G_{3.47}; G_{3.54}; G_{3.55}; G_{3.69}; G_{3.76}; G_{3.97}; G_{3.98}; \}; \\
G_{15} = \mathcal{G}(4, 4) = \{ G_{3.32}; G_{3.48}; G_{3.56}; G_{3.76}; G_{3.77}; G_{3.99}; G_{3.100}; G_{3.101}; \}; \\
G_{16} = \mathcal{G}(4, 5) = \{ G_{3.49}; G_{3.57}; G_{3.72}; G_{3.78}; G_{3.79}; G_{3.102}; \}; \\
G_{17} = \mathcal{G}(4, 6) = \{ G_{3.7}; G_{3.33}; G_{3.50}; G_{3.58}; G_{3.73}; G_{3.80}; G_{3.103}; \}; \\
G_{18} = \mathcal{G}(4, 7) = \{ G_{3.74}; G_{3.81}; \}; \\
G_{19} = \mathcal{G}(4, 8) = \{ G_{3.34}; G_{3.82}; \};
For integers

By Lemma 2.8,

are

t

each graph

We now determine the chromaticity of the graphs in

We can now present our main result of this section.

**Theorem 4.1.** For integers \( p \geq 5, t_1 \geq 1 \) and \( t_1 + t_2 \geq 3 \), all the graphs in

\[ K^{-3}(t_1 \times p, t_2 \times (p + 1)) \] are \( \chi \)-unique except that

\[ \{G_{3,19}, G_{3,149} \mid t_1 \geq 4, t_2 \geq 0\}, \{G_{3,20}, G_{3,118} \mid t_1 \geq 3, t_2 \geq 1\}, \]  

\[ \{G_{3,21}, G_{3,61}, G_{3,132} \mid t_1, t_2 \geq 2\}, \{G_{3,22}, G_{3,124}, G_{3,155} \mid t_1, t_2 \geq 2\}, \]  

\[ \{G_{3,24}, G_{3,159} \mid t_1 \geq 1, t_2 \geq 4\}, \{G_{3,26}, G_{3,132} \mid t_1 \geq 3, t_2 \geq 1\}, \]  

\[ \{G_{3,28}, G_{3,136} \mid t_1, t_2 \geq 2\}, \{G_{3,32}, G_{3,77} \mid t_1 \geq 2, t_2 \geq 1\}, \]  

\[ \{G_{3,65}, G_{3,157} \mid t_1 \geq 1, t_2 \geq 3\}, \{G_{3,85}, G_{3,122} \mid t_1 \geq 3, t_2 \geq 1\}, \]  

\[ \{G_{3,90}, G_{3,112} \mid t_1 \geq 1, t_2 \geq 3\}, \{G_{3,117}, G_{3,156}, G_{3,193} \mid t_1 \geq 4, t_2 \geq 2\}, \]  

\[ \{G_{3,125}, G_{3,198} \mid t_1 \geq 2, t_2 \geq 3\}, \{G_{3,127}, G_{3,158} \mid t_1 \geq 4, t_2 \geq 2\}, \]  

\[ \{G_{3,128}, G_{3,202} \mid t_1 \geq 1, t_2 \geq 4\}, \{G_{3,162}, G_{3,165} \mid t_1, t_2 \geq 3\}, \]  

\[ \{G_{3,192}, G_{3,197} \mid t_1, t_2 \geq 3\} \] and \( \{G_{3,196}, G_{3,200} \mid t_1 \geq 2, t_2 \geq 4\} \)

are \( \chi \)-equivalence classes.

**Proof.** By Lemma 2.8, \( T^{-3} \) is \( \chi \)-closed. Observe that for \( 1 \leq i < j \leq 25 \) and each graph \( H' \in G_i \) and each graph \( H'' \in G_j \), either \( \alpha_{t+1}(H') \neq \alpha_{t+1}(H'') \) or \( t(H') \neq t(H'') \). By Lemma 2.1, \( H' \not\sim H'' \). Hence, each \( G_i \) and \( G_j \) \((1 \leq i < j \leq 25)\) are \( \chi \)-disjoint. Since \( T^{-3} \) is \( \chi \)-closed, we conclude that each \( G_i \) \((1 \leq i \leq 25)\) is \( \chi \)-closed. It follows immediately that \( G_{3,8}, G_{3,9}, G_{3,10}, G_{3,11}, G_{3,14} \) and \( G_{2,167} \) are \( \chi \)-unique. By Lemma 2.9(i), we also know that \( G_{3,12} \) and \( G_{3,13} \) are \( \chi \)-unique.

We now determine the chromaticity of the graphs in \( G_i \) for \( i \in \{1, 2, \ldots, 19\} \backslash \{10\} \).

It suffices to show that for any two graphs \( G' \) and \( G'' \) in \( G_i \), either \( \alpha_k(G') \neq \alpha_k(G'') \) for \( k = t+2, t+3 \) or \( t+4 \), or \( \beta(G') \neq \beta(G'') \). Otherwise, we shall show that \( h(G', x) = h(G'', x) \). We use an argument similar to that in the proof of
Theorem 3.1. In what follows, \( f_i \) and \( g_i \) \((i = 1, 2, 3)\) are as defined in the proof of Theorem 3.1.

(1). Graphs in \( G_1 \). Define

\[
\begin{align*}
\delta_1 &= 2(2^p-3) - 1 + (t_1-2)(2^{p-1} - 1) + t_2(2^{p-1} - 1), \\
\delta_2 &= (2^p-3) - 1 + 2(2^p - 1) + (t_1-3)(2^{p-1} - 1) + t_2(2^{p-1} - 1), \\
\delta_3 &= 4(2^{p-2} - 1) + (t_1 - 4)(2^{p-1} - 1) + t_2(2^{p-1} - 1).
\end{align*}
\]

So,

\[
\begin{align*}
\alpha_{t+3}(G_{3,3,1}) &= 1 + 3\delta_1 + 3g_3, \\
\alpha_{t+3}(G_{3,i}) &= 1 + 3\delta_2 + 3g_3 \quad \text{for } i = 15, 140, \\
\alpha_{t+3}(G_{3,3,35}) &= 1 + \delta_1 + 2\delta_2 + 3g_3, \\
\alpha_{t+3}(G_{3,83}) &= 1 + \delta_1 + 2\delta_3 + 3g_3, \\
\alpha_{t+3}(G_{3,104}) &= 1 + 2\delta_2 + \delta_3 + 3g_3, \\
\alpha_{t+3}(G_{3,168}) &= 1 + \delta_2 + 2\delta_3 + 3g_3, \\
\alpha_{t+3}(G_{3,204}) &= 1 + 3\delta_3 + 3g_3.
\end{align*}
\]

Since \( p \geq 5 \),

\[
\begin{align*}
\alpha_{t+3}(G_{3,3,1}) - \alpha_{t+3}(G_{3,3,35}) &= 2^{p-2} > 0, \\
\alpha_{t+3}(G_{3,3,35}) - \alpha_{t+3}(G_{3,3,1}) &= 2^{p-3} > 0 \quad \text{for } i = 15, 140, \\
\alpha_{t+3}(G_{3,i}) - \alpha_{t+3}(G_{3,j}) &= 2^{p-3} > 0 \quad \text{for } j = 83, 104, \\
\alpha_{t+3}(G_{3,3,3}) - \alpha_{t+3}(G_{3,168}) &= 2^{p-3} > 0, \\
\alpha_{t+3}(G_{3,168}) - \alpha_{t+3}(G_{3,204}) &= 2^{p-3} > 0.
\end{align*}
\]

So, \( \alpha_{t+3}(G_{3,3,1}) > \alpha_{t+3}(G_{3,3,35}) > \alpha_{t+3}(G_{3,15}) = \alpha_{t+3}(G_{3,3,140}) > \alpha_{t+3}(G_{3,83}) = \alpha_{t+3}(G_{3,104}) > \alpha_{t+3}(G_{3,168}) > \alpha_{t+3}(G_{3,204}) \).

Let \( I_k^j(G) \) be the number of \( k \)-independent partitions with \( j \geq 1 \) improper independent sets. Then, \( \alpha_k(G) = \sum_{j \geq 1} I_k^j(G) \). We now compare \( \alpha_{t+4}(G_{3,15}) \)
with $\alpha_{t+4}(G_{3,140})$.

\[
\alpha_{t+4}(G_{3,145}) = T_{t+4}^1(G_{3,145}) + T_{t+4}^2(G_{3,145}) + [3(2^{p-3} - 1) + (t_1 - 3)(2^{p-1} - 1) + t_2(2^p - 1)],
\]
\[
\alpha_{t+4}(G_{3,140}) = T_{t+4}^1(G_{3,140}) + T_{t+4}^2(G_{3,140}) + [(2^{p-4} - 1) + 3(2^{p-2} - 1) + (t_1 - 4)(2^{p-1} - 1) + t_2(2^p - 1)].
\]

Note that $T_{t+4}^j(G_{3,145}) = T_{t+4}^j(G_{3,140})$ for $j = 1, 2$. Since $p \geq 5$, $\alpha_{t+4}(G_{3,145}) - \alpha_{t+4}(G_{3,140}) = 2^{p-4} > 0$.

We now show that $h(G_{3,138}, x) \neq h(G_{3,104}, x)$. By Lemmas 2.2 and 2.3,

\[
h(G_{3,138}, x) = \left[ h(K_p, x) \right]^{t_1-4} \left[ h(K_{p+1}, x) \right]^{t_2} \left( h(K_p, K_p, x) \right)^2 + xh(K_p, K_p, x)h(K_{p-1}, K_{p-1}, x),
\]
\[
h(G_{3,104}, x) = \left[ h(K_p, x) \right]^{t_1-4} \left[ h(K_{p+1}, x) \right]^{t_2} \left( h(K_p, K_p, x) \right)^2 + x(h(K_p, K_{p-1}, x))^2.
\]

By Lemma 2.4, $\beta(K_p, K_p) < \beta(K_p, K_{p-1})$. Therefore, $\beta(G_{3,138}) < \beta(G_{3,104})$.

Hence, all the graphs in $G_i$ are $\chi$-unique.

The chromaticity of graphs in $G_i$, $i = \{2, 3, \ldots, 19\}\setminus\{10\}$ can be similarly determined as in (1). Since it is long and rather repetitive, the proofs are omitted. The proof is now complete.  \( \Box \)

A diagram of all the $\chi$-equivalence classes obtained in Theorem 4.1 is presented in Figure 3 for easy reference and comparison.

**Remarks:** Observe that graphs $G_{3,162}$ and $G_{3,165}$ can be obtained from $G_{2,16}$ and $G_{2,17}$ respectively with one more edge deleted. For $t_1, t_2 \geq s \geq 2$, let $G'$ (respectively, $G''$) be a graph in $T^{-s}$ such that $\langle S \rangle$ is a star graph with the central vertex belongs to a partite set of size $p$ (respectively, size $p+1$) and the end-vertices belong to different partite sets of size $p + 1$ (respectively, size $p$).

It is clear that $G' \not\cong G''$. By Lemmas 2.2, 2.3 and mathematical induction on $s$, it is easy to show that $h(G', x) = h(G'', x)$ for all $p \geq 2$.  

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Conjecture 1. The family \( \{G', G''\} \) is a \( \chi \)-equivalence class.

For integers \( m_1, m_2, m_3 \geq 0 \), let \( G \) and \( H \) be graphs in \( T^{-s} \) such that the graph induced by all the non-complete components of \( G \) (respectively, \( H \)) is

\[
G'_{2,16} + m_1(K_p \cdot K_p) + m_2(K_p \cdot K_{p+1}) + m_3(K_{p+1} \cdot K_{p+1}) \quad \text{(respectively)}, \quad G''_{2,17} + m_1(K_p \cdot K_p) + m_2(K_p \cdot K_{p+1}) + m_3(K_{p+1} \cdot K_{p+1})
\]

where \( m_1 + m_2 + m_3 \geq 1 \).

![Graphs in Chromatic Equivalence](image-url)

Figure 3: "Structures" of graphs in Chromatic Equivalence

Classes listed in Theorem 2.
Clearly, $G \sim H$ but $G \not\sim H$. Observe that $\{G, H\}$ is a $\chi$-equivalence class if $m_1 = m_j = 0$ for $j = 2$ or 3. Hence, we end this paper with the following problem.

**Problem 1.** Find all the values of $m_i, i = 1, 2, 3$ such that $\{G, H\}$ is a $\chi$-equivalence class for $G$ and $H$ defined above.

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**References**

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