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Diagonal and Monomial Solutions of the Matrix Equation $AXB = C$

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Abstract. In this article, we consider the matrix equation $AXB = C$, where A, B, C are given matrices and give new necessary and sufficient conditions for the existence of the diagonal solutions and monomial solutions to this equation. We also present a general form of such solutions. Moreover, we consider the least squares problem $\min_X \|C - AXB\|_F$ where X is a diagonal or monomial matrix. The explicit expressions of the optimal solution and the minimum norm solution are both provided.

Keywords: Matrix equation, Diagonal matrix, Monomial matrix, Least squares problem.

2000 Mathematics subject classification: 15A24, 15A06.

1. Introduction

Let $\mathcal{R}^{m\times n}$ be the set of all $m \times n$ real matrices. For $A \in \mathcal{R}^{m\times n}$, let A^T, A^-, A^L, A^+ and $\|A\|_F$ be the transpose, the generalized inverse(g-inverse), least squares g-inverse, the Moore-Penrose g-inverse and the Frobenius norm of A, respectively. We denote by I_n and $O_{m\times n}$ the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively. $A \otimes B$ and $vec(A)$ stand for the Kronecker product of matrices A and B and the vec notation of matrix A, respectively (see [14], [1] or [8]). For $v \in \mathcal{R}^n$, $\|v\|_2$ represents the Euclidean norm of v.

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Some authors have studied the problem general solution to the matrix equation

$$AXB = C \tag{1.1}$$

where $A \in \mathcal{R}^{m\times n}$, $B \in \mathcal{R}^{n\times p}$ and $C \in \mathcal{R}^{m\times p}$ are known matrices. For instance, Dai[4], Magnus[10], Khatri and Mitra[6], Zhang[18] and Cvetković-Ilić [3] have derived the general symmetric, L-structured, hermitian and nonnegative definite, hermitian nonnegative definite and positive definite and reflexive solutions to the matrix equation (1.1), respectively. The Re-nonnegative definite solutions for the equation (1.1) were investigated by Wang and Yang [17] and Cvetković-Ilić [2]. Hou et al.[7] gave iteration methods for computing the least squares symmetric solution of the equation (1.1), in which the authors presented an algorithm to solve the minimum Frobenius norm residual problem: min $\|AXB - C\|_F$ with unknown symmetric matrix X. Tian[16] gave the maximal and minimal ranks of solutions of the equation (1.1).

Matrix equations play an important role in applied mathematics and engineering sciences. For example, the matrix equation (1.1) arises in the multistatic antenna array processing problem in the scattering theory(see [12], [15] and [9]). In this problem, we obtain a matrix equation in the form $AXB = C$ whose diagonal and monomial solutions are feasible solutions of the problem. Lev-Ari in [9] compute a solution to the multistatic antenna array processing problem by using Kronecker, Khatri-Rao and Schur-Hadamard matrix products. Also, matrix equations appear naturally in solving the differential equations or the controllable linear time-invariant system(see [5] and [11]). For approximating of solutions for matrix equations with unknown matrix $X \in \mathbb{R}^{m\times n}$, we can choose $N = n(m-1)$ or $M = m(n-1)$ unknowns arbitrarily and convert the matrix equations to new one whose diagonal or monomial solutions are feasible [5].

In this article we use the singular value decomposition(SVD) and examine necessary and sufficient conditions for the existence of diagonal solutions and monomial solutions to the matrix equation (1.1). Also, we derive representation of the general diagonal solutions and monomial solutions to this matrix equation. Moreover, we consider the least squares problem $\min_X \|C - AXB\|_F$ subject to the constraint that X is a diagonal or monomial matrix and, compute the general solution and the unique solution of minimum 2-norm to this problem.

Now, we state some well known results which are used in the next section.

Lemma 1.1. *Let $A \in \mathcal{R}^{m\times n}$ and $b \in \mathcal{R}^m$ be given. Then,*

(i) *(see* [1, P. 53] *and* [14]*) the linear system $Ax = b$ is consistent if and only if for some A^-,*

$$AA^-b = b$$

in which case the general solution is

$$x = A^{-}b + (I_n - A^{-}A)h$$

for arbitrary $h \in \mathcal{R}^n$.

(ii) *(see* [1, P. 105] *and* [14, Section 6.5]*) the general solution to the least squares problem*

$$\min_{x\in\mathcal{R}^n} \|b - Ax\|_2$$

is of the form

$$x = A^{L}b + (I_n - A^{-}A)y$$

where A^L *is a least squares g-inverse and* A^- *is a g-inverse of* A *and* $y \in \mathcal{R}^n$ *is arbitrary.*

(iii) *(see* [8, P. 66]*) the unique solution of the least squares problem*

$$\min_{x\in\mathcal{R}^n} \|b - Ax\|_2^2 + \|x\|_2^2$$

is $x = A^{+}b$.

2. The diagonal solutions

In this section, we present the general diagonal solutions to the matrix equation (1.1).

Suppose that the diagonal $D = diag(d_1, d_2, ..., d_n)$ is a solution to the equation (1.1), then

$$ADB = C.$$

This relation can be written using the *vec* notation and the Kronecker product in the equivalent form

$$\sum_{i=1}^{n}(b_{i*}^T \otimes a_{*i})d_i = vec(C)$$

where b_{i*} and a_{*i} are the $i-$th row of B and the $i-$th column of A, respectively. This shows that the diagonal $D = diag(d_1, d_2, ..., d_n)$ is a solution to the equation (1.1) if and only if the vector $d = (d_1, d_2, ..., d_n)^T$ satisfies the linear system

$$Vd = vec(C) \tag{2.1}$$

where

$$V = [\ b_{1*}^T \otimes a_{*1}\ ,\ b_{2*}^T \otimes a_{*2}\ ,\ \dots\ ,\ b_{n*}^T \otimes a_{*n}\ ]. \tag{2.2}$$

Therefore, the matrix equation (1.1) has a diagonal solution if and only if the linear system (2.1) is consistent. By Lemma 1.1(i), (2.1) is consistent if and only if

$$VV^{-}vec(C) = vec(C).$$

The following theorem follows from these observations.

Theorem 2.1. *Let $A \in \mathcal{R}^{m\times n}$, $B \in \mathcal{R}^{n\times p}$ and $C \in \mathcal{R}^{m\times p}$ be given. Then equation (1.1) has a diagonal solution if and only if*

$$VV^{-}vec(C) = vec(C) \tag{2.3}$$

where V is defined by (2.2), in which case the general diagonal solution is $D = diag(d^T)$ where the vector d satisfies

$$d = V^{-}vec(C) + (I_n - V^{-}V)h \tag{2.4}$$

for arbitrary $h \in \mathcal{R}^n$.

In the following, we rewrite the matrix V and the solution to (2.1) with more details. First, we give the following lemma.

Lemma 2.2. *Given matrices $A \in \mathcal{R}^{m\times n}$ and $B \in \mathcal{R}^{n\times p}$. Then the matrix V in (2.2) satisfies*

$$V = (B^T \otimes A)PE \tag{2.5}$$

where P is a symmetric permutation matrix of size $n^2 \times n^2$ and E is an $n^2 \times n$ matrix of the form

$$E = \begin{bmatrix} I_n \\ O \end{bmatrix} \tag{2.6}$$

where O is an $(n^2 - n) \times n$ zero matrix.

Proof. The $i-$th column of V satisfies

$$b_{i*}^T \otimes a_{*i} = B^T e_i \otimes Ae_i = (B^T \otimes A)(e_i \otimes e_i),$$

where e_i is the $i-$th column of the $n \times n$ identity matrix. Substituting this in (2.2) yields,

$$V = (B^T \otimes A)F,$$

where

$$F = [\ e_1 \otimes e_1\ ,\ e_2 \otimes e_2\ ,\ \dots\ ,\ e_n \otimes e_n\ ].$$

It is easy to see that the matrix F can be transformed into the matrix E in (2.6) by reordering of its rows. This operation can be achieved by a premultiplication F by permutation matrix

$$P = \prod_{i=2}^{n} G_{i,((i-1)n+i)}, \tag{2.7}$$

where $G_{i,((i-1)n+i)}$ is an interchange matrix (the identity matrix with its $i-$th row and $((i-1)n+i)-$th row interchanged). Thus we have

$$E = PF.$$

Note that the matrix P is unitary and also symmetric, since there is no overlap between the interchange matrices(see [13, P. 73]). Therefore we obtain

$$F = PE.$$

This completes the proof. □

Lemma 2.3. *Let $A \in \mathcal{R}^{m\times n}$ and $B \in \mathcal{R}^{n\times p}$ be given. Let*

$$U_A^T \begin{bmatrix} \Sigma_A & O \\ O & O \end{bmatrix} W_A \text{ and } U_B^T \begin{bmatrix} \Sigma_B & O \\ O & O \end{bmatrix} W_B \tag{2.8}$$

be singular value decompositions of A and B, respectively, where Σ_A and Σ_B are diagonal and nonsingular. Then there are two symmetric permutation matrices P_1 and P_2 such that

$$(W_B^T \otimes U_A^T)P_1 \begin{bmatrix} \Sigma_A \otimes \Sigma_B & O \\ O & O \end{bmatrix} P_2(U_B \otimes W_A) \tag{2.9}$$

is a singular value decomposition of $(B^T \otimes A)$.

In the above lemma, two permutation matrices P_1 and P_2 have been used to permute the rows and columns of the diagonal matrix

$$\begin{bmatrix} \Sigma_A & O \\ O & O \end{bmatrix} \otimes \begin{bmatrix} \Sigma_B & O \\ O & O \end{bmatrix},$$

to convert this matrix to the following matrix.

$$\begin{bmatrix} \Sigma_A \otimes \Sigma_B & O \\ O & O \end{bmatrix}.$$

On the other hand, These permutations have not any overlapping. Therefore, P_1 and P_2 are symmetric. This shows that the matrices P_1 and P_2 satisfied in the above lemma can be computed, easily.

Theorem 2.4. *Let $A \in \mathcal{R}^{m\times n}$, $B \in \mathcal{R}^{n\times p}$ and $C \in \mathcal{R}^{m\times p}$ be given. Let $rank(A) = r_1$ and $rank(B) = r_2$, and let the matrices $U_A, U_B, W_A, W_B, \Sigma_A, \Sigma_B$, P_1 and P_2 satisfy (2.8) and (2.9). Then equation (1.1) has a diagonal solution if and only if*

$$(\widetilde{U}_1^T \Sigma \widetilde{W}_{11} \widetilde{W}_{11}^- \Sigma^{-1} \widetilde{U}_1 + \widetilde{U}_1^T \Sigma \widetilde{W}_{11} H \widetilde{U}_2) vec(C) = vec(C) \tag{2.10}$$

for arbitrary matrix $H \in \mathcal{R}^{n\times(mp-r_1r_2)}$, in which case the general diagonal solution is $D = diag(d^T)$ where the vector d satisfies

$$d = (\widetilde{W}_{11}^- \Sigma^{-1} \widetilde{U}_1 + H\widetilde{U}_2)vec(C) + (I_n - \widetilde{W}_{11}^- \widetilde{W}_{11})h \tag{2.11}$$

for arbitrary $h \in \mathcal{R}^n$ where $\Sigma = \Sigma_A \otimes \Sigma_B$, $\widetilde{U}_1$ is an $r_1r_2 \times mp$ matrix consisting of the first r_1r_2 rows of the matrix $P_1(W_B \otimes U_A)$, $\widetilde{U}_2$ is an $(mp - r_1r_2) \times mp$ matrix consisting of the last $(mp - r_1r_2)$ rows of the matrix $P_1(W_B \otimes U_A)$ and $\widetilde{W}_{11}$ is an $r_1r_2 \times n$ matrix whose $(i, j)-th$ entry is equal to the $(i, ((j-1)n + j))-th$ entry of the matrix $P_2(U_B \otimes W_A)$.

Proof. Denote by $\widetilde{U}$ the matrix $P_1(W_B \otimes U_A)$ and by $\widetilde{W}$ the matrix $P_2(U_B \otimes W_A)$. Substituting (2.9) into (2.5), we obtain

$$V = \widetilde{U}^T \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} \widetilde{W} P \begin{bmatrix} I_n \\ O \end{bmatrix}. \tag{2.12}$$

We note that the postmultiplication of $\widetilde{W}$ by the permutation matrix P in (2.7) is equivalent to interchanging the columns i and $((i-1)n+i)$ of the matrix $\widetilde{W}$ for $i = 2, 3, ..., n$. Therefore the product

$$\widetilde{W} P \begin{bmatrix} I_n \\ O \end{bmatrix}$$

is equal to an $n^2 \times n$ matrix $\widetilde{W}_1$ which consists of the n columns $\widetilde{w}_{*((i-1)n+i)}, i = 1, 2, ..., n$ of $\widetilde{W}$. Now partition $\widetilde{W}_1$ as

$$\begin{bmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \end{bmatrix} \tag{2.13}$$

where $\widetilde{W}_{11}$ is of size $r_1 r_2 \times n$, then (2.12) becomes

$$\begin{aligned} V &= \widetilde{U}^T \begin{bmatrix} \Sigma & O_{r_1 r_2 \times (n^2 - r_1 r_2)} \\ O_{(mp - r_1 r_2) \times r_1 r_2} & O_{(mp - r_1 r_2) \times (n^2 - r_1 r_2)} \end{bmatrix} \begin{bmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \end{bmatrix} \\ &= \widetilde{U}^T \begin{bmatrix} \Sigma \widetilde{W}_{11} \\ O_{(mp - r_1 r_2) \times n} \end{bmatrix} \end{aligned}$$

$$= \widetilde{U}^T \begin{bmatrix} \Sigma & O_{r_1 r_2 \times (mp - r_1 r_2)} \\ O_{(mp - r_1 r_2) \times r_1 r_2} & I_{(mp - r_1 r_2)} \end{bmatrix} \begin{bmatrix} \widetilde{W}_{11} \\ O_{(mp - r_1 r_2) \times n} \end{bmatrix}. \tag{2.14}$$

We now compute a g-inverse of V in terms of a g-inverse of $\widetilde{W}_{11}$. To do this, we know that

$$\begin{bmatrix} \widetilde{W}_{11} \\ O \end{bmatrix}^- \begin{bmatrix} \Sigma^{-1} & O \\ O & I \end{bmatrix} \widetilde{U}$$

is a g-inverse of V(see [1, P. 43]). On the other hand, it is easy to verify that

$$\begin{bmatrix} \widetilde{W}_{11} \\ O \end{bmatrix}^- = [\widetilde{W}_{11}^- \ , \ H]$$

for arbitrary matrix $H \in \mathcal{R}^{n \times (mp - r_1 r_2)}$. Then we have

$$V^- = [\widetilde{W}_{11}^- \ , \ H] \begin{bmatrix} \Sigma^{-1} & O \\ O & I \end{bmatrix} \widetilde{U}. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.3) gives

$$\widetilde{U}^T \begin{bmatrix} \Sigma \widetilde{W}_{11} \widetilde{W}_{11}^- \Sigma^{-1} & \Sigma \widetilde{W}_{11} H \\ O & O \end{bmatrix} \widetilde{U} vec(C) = vec(C)$$

and Substituting (2.14) and (2.15) into (2.4) gives

$$d = [\widetilde{W}_{11}^- \Sigma^{-1} \ , \ H] \widetilde{U} vec(C) + (I_n - \widetilde{W}_{11}^- \widetilde{W}_{11}) h.$$

By partitioning the matrix $\widetilde{U}$ as

$$\begin{bmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \end{bmatrix}$$

where $\widetilde{U}_1$ consists of the r_1r_2 first rows of $\widetilde{U}$, and substituting it into the two above relations, we obtain (3.8) and (3.9).

The proof is complete. $\square$

Proposition 2.5. *Let $A \in \mathcal{R}^{m\times n}$, $B \in \mathcal{R}^{n\times p}$ and $C \in \mathcal{R}^{m\times p}$ be given. If $D = diag(d_1, d_2, ..., d_n)$ is a diagonal solution to (1.1) ,then the number of nonzero diagonal entries of D is at least r_c where r_c is the rank of the matrix C.*

Proof. The proof is an immediate consequence of the fact that

$$rank(ADB) \leq min\{rank(A), rank(D), rank(B)\}.$$

$\square$

Now, we consider the more general case where the linear system (2.1) may be inconsistent. In this case, we solve the least squares problem

$$\begin{array}{ll} \min_X & \|C - AXB\|_F \\ subject\ to: & X \in \mathcal{R}^{n\times n} \text{ and } X \text{ is diagonal,} \end{array} \tag{2.16}$$

where A, B and C are given matrices, or equivalently, the least squares problem

$$d_o = arg \min_{d\in\mathcal{R}^n} \|vec(C) - Vd\|_2 \tag{2.17}$$

where V is defined by (2.2). It is evident that d_o is a solution of the problem (2.17) if and only if $X_o = diag(d_o^T)$ is a solution of the problem (2.16). To solve the above least squares problems, we need the following algorithm.

Algorithm 1.

1. Input $\widetilde{W}_{11} \in \mathcal{R}^{r\times n}$ and nonsingular diagonal matrix Σ of size $r \times r$.
2. Calculate the integer r_w according to
$$r_w = rank(\widetilde{W}_{11}).$$
3. Find permutation matrices $Q_1 \in \mathcal{R}^{r\times r}$ and $Q_1 \in \mathcal{R}^{n\times n}$ satisfying the relations
$$\widetilde{W}_{11} = Q_1 \begin{bmatrix} \widehat{W}_1 & \widehat{W}_2 \\ \widehat{W}_3 & \widehat{W}_4 \end{bmatrix} Q_2$$
where the $r_w \times r_w$ matrix $\widehat{W}_1$ is full rank.
4. Find the diagonal matrix $\widehat{\Sigma}$ satisfying
$$\Sigma Q_1 = Q_1\widehat{\Sigma}.$$

5. Partition $\widehat{\Sigma}$ as
$$\widehat{\Sigma} = \begin{bmatrix} \widehat{\Sigma}_1 & O \\ O & \widehat{\Sigma}_2 \end{bmatrix}$$
where the diagonal matrices $\widehat{\Sigma}_1$ and $\widehat{\Sigma}_2$ are of sizes $r_w \times r_w$ and $(r - r_w) \times (r - r_w)$, respectively.
6. Solve the matrix equation
$$Y(\widehat{\Sigma}_1 \widehat{W}_1) = \widehat{\Sigma}_2 \widehat{W}_3.$$
7. Calculate the matrix L_w according to
$$L_w = Q_2^T \begin{bmatrix} L_1 & L_2 \\ O & O \end{bmatrix} Q_1^T$$
where $L_1 = \widehat{W}_1^{-1} \widehat{\Sigma}_1^{-1} (I_{r_w} + Y^T Y)^{-1}$ and $L_2 = L_1 Y^T$.
8. Solve the matrix equation
$$\widehat{W}_1 Y' = \widehat{W}_2$$
9. Calculate the matrix M_w according to
$$M_w = Q_2^T \begin{bmatrix} I_{r_w} \\ {Y'}^T \end{bmatrix} (I_{r_w} + Y'{Y'}^T)^{-1} L_1 [I_{r_w} \;\; Y^T] Q_1^T.$$

□

The matrices $(I_{r_w} + Y^T Y)$ and $(I_{r_w} + Y'{Y'}^T)$ in the above algorithm are nonsingular, since these matrices are symmetric positive definite.

Lemma 2.6. *Let $\widetilde{W}_{11} \in \mathcal{R}^{r \times n}$ and nonsingular diagonal matrix $\Sigma \in \mathcal{R}^{r \times r}$ be given. Let L_w and M_w be the matrices obtained from Algorithm 1. Then L_w is a least squares g-inverse and M_w is the Moore-Penrose g-inverse of $\Sigma \widetilde{W}_{11}$.*

Proof. By straightforward computations, it can be shown that L_w satisfy
$$\Sigma \widetilde{W}_{11} L_w \Sigma \widetilde{W}_{11} = \Sigma \widetilde{W}_{11} \text{ and } (\Sigma \widetilde{W}_{11} L_w)^T = \Sigma \widetilde{W}_{11} L_w,$$
and M_w satisfy
$$\Sigma \widetilde{W}_{11} M_w \Sigma \widetilde{W}_{11} = \Sigma \widetilde{W}_{11} \; , \; (\Sigma \widetilde{W}_{11} M_w)^T = \Sigma \widetilde{W}_{11} M_w$$
$$M_w \Sigma \widetilde{W}_{11} M_w = M_w \text{ and } (M_w \Sigma \widetilde{W}_{11})^T = M_w \Sigma \widetilde{W}_{11}.$$
Therefore, L_w is a least squares g-inverse of $\Sigma \widetilde{W}_{11}$ and $(\Sigma \widetilde{W}_{11})^+ = M_w$. □

Theorem 2.7. *Let $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{n \times p}$ and $C \in \mathcal{R}^{m \times p}$ be given and let the matrices $U_A, U_B, W_A, W_B, \Sigma_A, \Sigma_B, P_1$ and P_2 satisfy (2.8) and (2.9). Let $r_1, r_2, \Sigma, \widetilde{U}, \widetilde{W}, \widetilde{U}_1, \widetilde{U}_2$ and $\widetilde{W}_{11}$ be as in Theorem 2.4. Then the general solution to the least squares problem (2.17) is of the form*
$$d_o = L_w \widetilde{U}_1 vec(C) + (I - \widetilde{W}_{11}^- \widetilde{W}_{11}) h,$$
for arbitrary $h \in \mathcal{R}^n$ where L_w is a least squares g-inverse of $\Sigma \widetilde{W}_{11}$ obtained from Algorithm 1.

In addition, the unique solution of minimum 2-norm is

$$d_o^m = M_w \widetilde{U}_1 vec(C)$$

where M_w is the Moore-Penrose g-inverse of $\Sigma\widetilde{W}_{11}$ obtained from Algorithm 1.

Proof. From (2.14), above lemma and the definition of least squares g-inverse, it follows that

$$[L_w \ H']\widetilde{U} \tag{2.18}$$

is a least squares g-inverse of V where $H' \in \mathcal{R}^{n\times(mp-r_1r_2)}$ is any solution of the matrix equation

$$\widetilde{W}_{11}H' = O. \tag{2.19}$$

We know that the general solution of (2.19) is

$$H' = (I - \widetilde{W}_{11}^{-}\widetilde{W}_{11})H$$

where $\widetilde{W}_{11}^{-}$ is an arbitrary g-inverse of $\widetilde{W}_{11}$ and H an arbitrary matrix in $\mathcal{R}^{n\times(mp-r_1r_2)}$[14, P. 215]. By substituting this into (2.18), a least squares g-inverse of V is obtained as

$$L_w\widetilde{U}_1 + (I - \widetilde{W}_{11}^{-}\widetilde{W}_{11})H\widetilde{U}_2. \tag{2.20}$$

Similarly, it can be verified that

$$V^+ = M_w\widetilde{U}_1. \tag{2.21}$$

It follows from (2.15), (2.20) and Lemma 1.1(ii) that the general least squares solution of (2.17) is

$$d_o = L_w\widetilde{U}_1 vec(C) + (I - \widetilde{W}_{11}^{-}\widetilde{W}_{11})(H\widetilde{U}_2 vec(C) + y) \tag{2.22}$$

where $H \in \mathcal{R}^{n\times(mp-r_1r_2)}$ and $y \in \mathcal{R}^n$ are arbitrary. It is easy to show that for every $h \in \mathcal{R}^n$, there are $H \in \mathcal{R}^{n\times(mp-r_1r_2)}$ and $y \in \mathcal{R}^n$ such that $h = (H\widetilde{U}_2 vec(C) + y)$. This proves the first part of the theorem.

The second part of the theorem can be easily proved from (2.21) and Lemma 1.1(iii). □

3. The monomial solutions

In this section, we obtain the general monomial solutions to the matrix equation (1.1) by using the general diagonal solution presented in the previous section.

A square matrix is called monomial if each its row and column contains at most one non-zero entry. It is easy to verify that a matrix M is a monomial matrix if and only if it can be written as a product of a diagonal matrix D and a permutation matrix P, i.e.,

$$M = DP.$$

If we denote by $monom(m_{1,j_1}, m_{2,j_2}, ..., m_{n,j_n})$, an $n \times n$ monomial matrix whose all entries are equal to zero except probably its $(i, j_i)-$th entry, $i = 1, 2, ..., n$, which is equal to m_{i,j_i}, then the above relation can be written as

$$monom(m_{1,j_1}, m_{2,j_2}, ..., m_{n,j_n}) = diag(m_{1,j_1}, m_{2,j_2}, ..., m_{n,j_n})P^T \qquad (3.1)$$

in which P is the permutation matrix

$$[e_{j_1}, e_{j_2}, ..., e_{j_n}]$$

where e_{j_i} is the j_i-th column of the $n \times n$ identity matrix.

Assume that the monomial matrix $monom(m_{1,j_1}, m_{2,j_2}, ..., m_{n,j_n})$ is a solution to the matrix equation (1.1). By (3.1) we obtain

$$AD_m B_m = C \qquad (3.2)$$

where $B_m = [e_{j_1}, e_{j_2}, ..., e_{j_n}]^T B$ and $D_m = diag(m_{1,j_1}, m_{2,j_2}, ..., m_{n,j_n})$. This shows that the monomial matrix $monom(m_{1,j_1}, m_{2,j_2}, ..., m_{n,j_n})$ is a solution to (1.1) if and only if the diagonal matrix D_m is a solution to (3.2). Therefore by Lemma 2.2, (3.2) is equivalent to

$$({B_m}^T \otimes A)PEd_m = vec(C) \qquad (3.3)$$

where P and E are defined by (2.7) and (2.6), respectively, and d_m is a column vector whose components are the diagonal entries of D_m(i.e., $D_m = diag({d_m}^T)$). Replacing B_m and A by $[e_{j_1}, e_{j_2}, ..., e_{j_n}]^T B$ and AI_n, respectively, in (3.3) gives

$$(B^T \otimes A)([e_{j_1}, e_{j_2}, ..., e_{j_n}] \otimes I_n)PEd_m = vec(C) \qquad (3.4)$$

Denote by P_m, the permutation matrix $([e_{j_1}, e_{j_2}, ..., e_{j_n}] \otimes I_n)P$ in (3.4). We now show that P_m is not unique. Toward this end, we can rewrite the relation (3.2) by using vec notation as follows:

$$[b_{j_1*}^T \otimes a_{*1} \ , \ b_{j_2*}^T \otimes a_{*2} \ , \ ... \ , \ b_{j_n*}^T \otimes a_{*n}]d_m = vec(C) \qquad (3.5)$$

where b_{i*} and a_{*i} are the $i-$th row of B and the $i-$th column of A, respectively. Then, similarly, we have

$$(B^T \otimes A)[e_{j_1} \otimes e_1 \ , \ e_{j_2} \otimes e_2 \ , \ ... \ , \ e_{j_n} \otimes e_n]d_m = vec(C). \qquad (3.6)$$

To transform the matrix $[e_{j_1} \otimes e_1 \ , \ e_{j_2} \otimes e_2 \ , \ ... \ , \ e_{j_n} \otimes e_n]$ into the matrix E in (2.6), premultiply this matrix by permutation matrix

$$P'_m = \prod_{i=1}^{n} G_{i,((j_i-1)n+i)}.$$

Thus, (3.6) becomes

$$(B^T \otimes A)P'_m Ed_m = vec(C). \qquad (3.7)$$

Comparing this with (3.4) yields the desired result, i.e., the permutation matrix P_m is not unique. But the n first columns of these permutation matrices are the same.

Now by Theorem 2.1 and Lemma 2.3 and similar to Theorem 2.4 we can write the following theorem.

Theorem 3.1. *Suppose the hypothesis of Theorem 2.4 is satisfied. Then equation (1.1) has a monomial solution of the form (3.1) if and only if*

$$(\widetilde{U}_1^T \Sigma \widetilde{W_{m11}} \widetilde{W_{m11}}^- \Sigma^{-1} \widetilde{U}_1 + \widetilde{U}_1^T \Sigma \widetilde{W_{m11}} H \widetilde{U}_2) vec(C) = vec(C) \tag{3.8}$$

for arbitrary matrix $H \in \mathcal{R}^{n \times (mp - r_1 r_2)}$, in which case the general monomial solution of the form (3.1) is $diag(d_m^T)P^T$ where the vector d_m satisfies

$$d_m = (\widetilde{W_{m11}}^- \Sigma^{-1} \widetilde{U}_1 + H \widetilde{U}_2) vec(C) + (I_n - \widetilde{W_{m11}}^- \widetilde{W_{m11}}) h \tag{3.9}$$

for arbitrary $h \in \mathcal{R}^n$ where $\widetilde{W_{m11}}$ is an $r_1 r_2 \times n$ matrix whose $(i,t)-th$ entry is equal to the $(i, ((j_t - 1)n + t))-th$ entry of the matrix $P_2(U_B \otimes W_A)$.

The above theorem introduces an algorithm for computing the general monomial solution of the matrix equation (1.1). This algorithm depends on a permutation $\{j_1, j_2, ..., j_n\}$ of the set $\{1, 2, ..., n\}$. Then we can compute all of the monomial solutions of (1.1), with implementation these algorithm for all of the permutations of the set $\{1, 2, ..., n\}$.

Now, we consider the least squares problem

$$\begin{array}{ll} \min_X & \|C - AXB\|_F \\ subject\ to: & X \in \mathcal{R}^{n \times n} \text{ and } X = monom(m_{1,j_1}, m_{2,j_2}, ..., m_{n,j_n}), \end{array} \tag{3.10}$$

where A, B and C are given matrices, or equivalently, the least squares problem

$$d_{mo} = arg \min_{d_m \in \mathcal{R}^n} \|vec(C) - V_m d_m\|_2 \tag{3.11}$$

where $V_m = [b_{j_1 *}^T \otimes a_{*1}\ ,\ b_{j_2 *}^T \otimes a_{*2}\ ,\ ...\ ,\ b_{j_n *}^T \otimes a_{*n}]$. It is evident that d_{mo} is a solution of the problem (3.11) if and only if $X_o = diag(d_{mo}^T)P^T$ is a solution of the problem (3.10). By a similar argument to Theorem 2.7, we can easily derive the following theorem.

Theorem 3.2. *Let $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{n \times p}$ and $C \in \mathcal{R}^{m \times p}$ be given and let the matrices $U_A, U_B, W_A, W_B, \Sigma_A, \Sigma_B, P_1$ and P_2 satisfy (2.8) and (2.9). Let $r_1, r_2, \Sigma, \widetilde{U}, \widetilde{W}, \widetilde{U}_1$ and $\widetilde{U}_2$ be as in Theorem 2.4 and let $\widetilde{W_{m11}}$ be as in Theorem 3.1. Then the general solution to the least squares problem (3.11) is of the form*

$$d_{mo} = L_{w_m} \widetilde{U}_1 vec(C) + (I - \widetilde{W_{m11}}^- \widetilde{W_{m11}}) h,$$

for arbitrary $h \in \mathcal{R}^n$ where L_{w_m} is a least squares g-inverse of $\Sigma \widetilde{W_{m11}}$ obtained from Algorithm 1.

In addition, the unique solution of minimum 2-norm is

$$d_{mo}^m = M_{w_m} \widetilde{U}_1 vec(C)$$

where M_{w_m} is the Moore-Penrose g-inverse of $\Sigma \widetilde{W_{m11}}$ obtained from Algorithm 1.

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