Weakly $g(x)$-Clean Rings

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Abstract. A ring $R$ with identity is called “clean” if for every element $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = u + e$.

Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. An element $r \in R$ is called “$g(x)$-clean” if $r = u + s$ where $g(s) = 0$ and $u$ is a unit of $R$ and $R$ is $g(x)$-clean if every element is $g(x)$-clean. In this paper we define a ring to be weakly $g(x)$-clean if each element of $R$ can be written as either the sum or difference of a unit and a root of $g(x)$.

Keywords: Clean ring, $g(x)$-clean ring, Weakly $g(x)$-clean ring.


1. INTRODUCTION

Throughout this note, $R$ is an associative ring with identity. A ring $R$ is called clean if for every element $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = e + u$ [9] and $R$ is called strongly clean if, in addition, $eu = ue$ [10].

Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. Following Camillo and Simon [2], an element $r \in R$ is called $g(x)$-clean if $r = u + s$ where $g(s) = 0$ and $u$ is a unit of $R$, and $R$ is $g(x)$-clean if every element in $R$ is $g(x)$-clean. It is clear that the $(x^2 - x)$-clean rings are precisely

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the clean rings.
Camillo and Simon [2] proved that if $V$ is a countable dimensional vector space over a division ring $D$ and $g(x)$ is any polynomial with coefficients in $K = C(D)$ and two distinct roots in $K$, then $\text{End}(V_D)$ is $g(x)$-clean. Nicholson and Zhou [11] generalized Camillo and Simon’s result by proving that $\text{End}(R,M)$ is $g(x)$-clean where $R,M$ is a semisimple left $R$-module and $g(x) \in (x-a)(x-b)C(R)[x]$ with $a,b \in C(R)$ and $b, b-a \in U(R)$. $g(x)$-clean rings have also been studied in [3], [7] and [6].

It is easy to see that a ring $R$ is $g(x)$-clean if and only if each $x \in R$ can be written in the form $x = u - s$ where $u \in U(R)$ and $g(s) = 0$. This raises the question of whether a ring with the property that, for each $x \in R$, either $x = u + s$ or $x = u - s$ for some $u \in U(R)$ and $g(s) = 0$ must be cleaned. Let us call rings with this property weakly $g(x)$-clean. Here we study weakly $g(x)$-clean rings and also investigate the general properties of weakly $g(x)$-clean rings which are similar to those of $g(x)$-clean rings. For example we prove the following results:

**Proposition 1.1.** Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then
\[
\prod_{i \in I} R_i \text{ is weakly } g(x)\text{-clean if and only if for all } i \in I, R_i \text{ is weakly } g(x)\text{-clean.}
\]

**Theorem 1.2.** Let $R$ be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then $R$ is weakly $g(x)$-clean if and only if the upper triangular matrix ring $T_n(R)$ is weakly $g(x)$-clean.

**Theorem 1.3.** Let $R$ be a commutative ring and $M$ an $R$-module. Let $g(x) \in C(R)[x]$. If $R$ is weakly $g(x)$-clean, then the idealization $R(M)$ of $R$ and $M$ is also weakly $g(x)$-clean.

In section 3 we consider the weakly $(x^n - x)$-clean rings and weakly 2-clean rings.

An usual, $T_n(R)$ denotes the upper triangular matrix ring of order $n$ over $R$; $GL_n(R)$ denotes the general linear group over $R$; and $gcd(m,n)$ means the greatest common divisor of the integers $m$ and $n$. All polynomials are in the polynomial ring $C(R)[x]$ and $U(R)$ denotes the multiplicative unit group of $R$.

## 2. Weakly $g(x)$-Clean Rings

In this section first we define the weakly $g(x)$-clean rings, then we explain the relation between weakly $g(x)$-clean and $g(x)$-clean rings.

**Definition 2.1.** Let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called weakly $g(x)$-clean if $r = u + s$ or $r = u - s$ where $g(s) = 0$ and
$u \in U(R)$. We say that $R$ is weakly $g(x)$-clean if every element is weakly $g(x)$-clean.

Obviously, $g(x)$-clean rings are weakly $g(x)$-clean and also if $g(x)$ is an odd or an even polynomial (i.e., $g(-x) = -g(x)$ or $g(-x) = g(x)$), then the concepts $g(x)$-clean and weakly $g(x)$-clean coincide, that is, if $R$ is a weakly $g(x)$-clean ring then $R$ is also $g(x)$-clean. So the interesting case is when $g(x)$ is neither an even nor an odd polynomial. In [1, Proposition 16] it was shown that if $R$ has exactly two maximal ideals and $2 \in U(R)$, then each $x \in R$ has the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in \{0, 1\}$. Thus $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is weakly clean but is not clean since an indecomposable clean ring is quasi-local [1, Theorem 3]. But since weakly $(x^2 - x)$-clean rings are precisely the weakly clean rings, we can say that for $g(x) = x^2 - x$, the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is weakly $g(x)$-clean, but it is not $g(x)$-clean.

The following two examples explain the relations between weakly $g(x)$-clean rings and weakly clean rings.

**Example 2.2.** Let $R = \mathbb{Z}_p = \{ \frac{a}{n} : \gcd(p, n) = 1 \text{ and } p \ prime\}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ and $g(x) = (x - a)(x^2 + 1) \in C(R)[x]$. Then $R$ is a weakly clean ring, because local rings are strongly clean, thus $R$ is clean (it is of course weakly clean). But as $a$ is the single root of $g(x)$, $R$ is not a weakly $g(x)$-clean ring.

**Example 2.3.** Let $R$ be a Boolean ring with the number of elements $|R| > 2$ and $c \in R$ with $0 \neq c \neq 1$. Define $g(x) = (x + 1)(x + c)$. Then $R$ is not weakly $g(x)$-clean.

Because if $c = u \pm s$ where $u \in U(R)$ and $g(s) = 0$, then it must be that $u = 1$ and $s = \pm(c \pm u)$. But, clearly, $g(c + 1) \neq 0$. However, $R$ is certainly weakly clean.

Let $R$ and $S$ be rings and $\theta : C(R) \to C(S)$ be a ring homomorphism with $\theta(1) = 1$. Then $\theta$ induces a map $\theta'$ from $C(R)[x]$ to $C(S)[x]$ such that For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x], \theta'(g(x)) := \sum_{i=0}^{n} \theta(a_i) x^i \in C(S)[x]$. Clearly, if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta'(g(x)) = g(x)$. We give some properties of weakly $g(x)$-clean rings which are similar to those of weakly clean rings.

**Proposition 2.4.** Let $\theta : R \to S$ be a ring epimorphism. If $R$ is weakly $g(x)$-clean, then $S$ is weakly $\theta'(g(x))$-clean.

**Proof.** Let $g(x) = a_0 + a_1 x + \ldots + a_n x^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1) x + \ldots + \theta(a_n) x^n \in C(S)[x]$. As $\theta$ is a ring epimorphism so for any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since $R$ is weakly $g(x)$-clean, there...
exist \( u \in U(R) \) and \( s_0 \in R \) such that \( r = u \pm s_0 \) and \( g(s_0) = 0 \). Then 
\[ s = \theta(r) = \theta(u \pm s_0) = \theta(u) \pm \theta(s_0) \] with \( \theta(u) \in U(S) \). But 
\[ \theta'(g(\theta(s_0))) = \theta(a_0) + \theta(a_1)\theta(s_0) + \ldots + \theta(a_n)\theta(s_0^n) = \theta(a_0 + a_1s_0 + \ldots + a_ns_0^n) = \theta(g(s_0)) = \theta(0) = 0, \] we have \( s \) is weakly \( \theta'(g(x)) \)-clean. Therefore \( S \) is weakly \( \theta'(g(x)) \)-clean. 

\[ \square \]

**Corollary 2.5.** If \( R \) is weakly \( g(x) \)-clean, then for any ideal \( I \) of \( R \), \( R/I \) is weakly \( \check{g}(x) \)-clean where \( \check{g}(x) \in C(R/I)[x] \).

**Proposition 2.6.** Let \( g(x) \in \mathbb{Z}[x] \) and \( \{R_i\}_{i \in I} \) be a family of rings. Then 
\[ \prod_{i \in I} R_i \] is weakly \( g(x) \)-clean if and only if for all \( i \in I \), \( R_i \) is weakly \( g(x) \)-clean.

**Proof.** Let \( \prod_{i \in I} R_i \) be a weakly \( g(x) \)-clean. Define \( \pi_j : \prod_{i \in I} R_i \rightarrow R_j \) by \( \pi_j(\{a_i\}_{i \in I}) = a_j \). Since for all \( j \in I \), \( \pi_j \) is a ring epimorphism, so by Proposition 2, for every \( i \in I \), each \( R_i \) is a weakly \( g(x) \)-clean ring.

For the converse, let \( x = \{x_i\}_{i \in I} \in R = \prod_{i \in I} R_i \). In \( R_{i_0} \), we can write 
\[ x_i = u_{i_0} + s_{i_0} \] or \( x_i = u_{i_0} - s_{i_0} \) where \( u_{i_0} \in U(R_{i_0}) \) and \( g(s_{i_0}) = 0 \). If 
\[ x_{i_0} = u_{i_0} + s_{i_0} \] for \( i \neq i_0 \), let \( x_i = u_i + s_i \) where \( u_i \in U(R_i) \), \( g(s_i) = 0 \); while if 
\[ x_{i_0} = u_{i_0} - s_{i_0} \] for \( i \neq i_0 \), let \( x_i = u_i - s_i \) where \( u_i \in U(R_i) \), \( g(s_i) = 0 \). Then 
\[ u = \{u_i\}_{i \in I} \in U(R) \]
and 
\[ g(s = \{s_i\}_{i \in I}) = a_0\{1_{R_i}\}_{i \in I} + a_1\{s_i\}_{i \in I} + \ldots + a_n\{s_i^n\}_{i \in I} \]
\[ = \{a_0\}_{i \in I} + \{a_1s_i\}_{i \in I} + \ldots + \{a_ns_i^n\}_{i \in I} \]
\[ = \{g(s_i)\}_{i \in I} = 0 \]
That is, \( \prod_{i \in I} R_i \) is weakly \( g(x) \)-clean. 

Define \( \pi_n : C(R) \rightarrow M_n(R) \) by \( a \mapsto aI_n \) with \( I_n \) being the identity matrix of \( M_n(R) \) and \( a \in C(R) \). Then \( M_n(R) \) is a \( C(R) \)-algebra.

**Theorem 2.7.** Let \( R \) be a ring, \( g(x) \in C(R)[x] \), and \( n \in \mathbb{N} \). Then \( R \) is weakly \( g(x) \)-clean if and only if the upper triangular matrix ring \( T_n(R) \) is weakly \( g(x) \)-clean.

**Proof.** Let \( R \) be weakly \( g(x) \)-clean and \( A = (a_{ij}) \in T_n(R) \) with \( a_{ij} = 0 \) for 
\[ 1 \leq j < i \leq n. \] Since \( R \) is weakly \( g(x) \)-clean, for any \( 1 \leq i \leq n \), there exist 
\( s_{ii} \in R \) and \( u_{ii} \in U(R) \) such that \( a_{ii} = u_{ii} \pm s_{ii} \) with \( g(s_{ii}) = 0 \). So we have
\[ A = \begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    0 & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & a_{nn}
\end{bmatrix} = \begin{bmatrix}
    u_{11} + s_{11} & a_{12} & \ldots & a_{1n} \\
    0 & u_{22} + s_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & u_{nn} + s_{nn}
\end{bmatrix} \]

In \( R \) for any \( 0 \leq i \leq n \), we can write \( a_{ii} = u_{ii} + s_{ii} \) or \( a_{ii} = u_{ii} - s_{ii} \) where \( u_{ii} \in U(R) \) and \( g(s_{ii}) = 0 \). If \( a_{ii} = u_{ii} + s_{ii} \) for \( j \neq i \), let \( a_{jj} = u_{jj} + s_{jj} \) where \( u_{jj} \in U(R), g(s_{jj}) = 0 \); while if \( a_{ii} = u_{ii} - s_{ii} \), for \( j \neq i \), let \( a_{jj} = u_{jj} - s_{jj} \) such that \( u_{jj} \in U(R), g(s_{jj}) = 0 \). Then by elementary row and column operations we can see that,

\[
U = \begin{bmatrix}
    u_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
    0 & u_{22} & a_{23} & \ldots & a_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & u_{nn}
\end{bmatrix} \in GL_n(R).
\]

Suppose \( g(x) = \sum_{i=0}^{m} a_i x^i \in \mathbb{C}([R][x]) \), then

\[
g(S) = \begin{bmatrix}
    s_{11} & 0 & \ldots & 0 \\
    0 & s_{22} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & s_{nn}
\end{bmatrix} = a_0 I_n + a_1 S + \ldots + a_n S^n
\]

\[
= \begin{bmatrix}
    a_0 & 0 & \ldots & 0 \\
    0 & a_0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & a_0
\end{bmatrix} + \begin{bmatrix}
    a_1 s_{11} & 0 & \ldots & 0 \\
    0 & a_1 s_{22} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & a_1 s_{nn}
\end{bmatrix} + \ldots
\]

\[
= \begin{bmatrix}
    a_m s_{11}^m & 0 & \ldots & 0 \\
    0 & a_m s_{22}^m & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & a_m s_{nn}^m
\end{bmatrix} + \begin{bmatrix}
    g(s_{11}) & 0 & \ldots & 0 \\
    0 & g(s_{22}) & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & g(s_{nn})
\end{bmatrix} = 0.
\]

So \( T_n(R) \) is weakly \( g(x) \)-clean.

Now let \( T_n(R) \) be weakly \( g(x) \)-clean. Define \( \theta : T_n(R) \rightarrow R \) by \( \theta(A) = a_{11} \).
Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}$. Then $\theta$ is a ring epimorphism. For any $a \in R$, let $B$ be the diagonal matrix $\text{diag}(a, \ldots a)$. Then $a = \theta(B) = \theta(U \pm S) = \theta(U) \pm \theta(S)$ where $U \in GL_n(R)$ and

$$g(\theta(S)) = a_0 + a_1 \theta(S) + \ldots + a_n \theta(S^n)$$

$$= \theta(B_0) + \theta(B_1) \theta(S) + \ldots + \theta(B_n) \theta(S^n)$$

$$= \theta(B_0 + B_1 S + \ldots + B_n S^n)$$

$$= \theta(a_0 I_n + (a_1 I_n) S + \ldots + (a_n I_n) S^n)$$

$$= \theta(g(S)) = 0.$$

Thus $a$ is weakly $g(x)$-clean, i.e., $R$ is a weakly $g(x)$-clean ring.

Remark 2.8. Let $R$ be a ring with identity, then the following hold:

1. $f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a unit if and only if $a_0$ is a unit of $R$.
2. $U(R[t]) = \{r_0 + r_1 t + \ldots + r_n t^n \mid r_0 \in U(R), r_i \in \sqrt{0} \text{ for } i = 0, 1, \ldots, n\}$

Proposition 2.9. Let $R$ be a ring and $g(x) \in C(R)[x]$. Then the formal power series ring $R[[t]]$ is weakly $g(x)$-clean if and only if $R$ is weakly $g(x)$-clean.

Proof. Let $R$ be weakly $g(x)$-clean and $f = \sum_{i \geq 0} a_i t^i \in R[[t]]$. Since $R$ is weakly $g(x)$-clean, $a_0 = u \pm s$ for some $s \in R$ and $u \in U(R)$ and $g(s) = 0$. Then $f = (u + \sum_{i \geq 1} a_i t^i) \pm s$. By Remark 6, $u + \sum_{i \geq 1} a_i t^i \in U(R[[t]])$. So $f$ is weakly $g(x)$-clean, i.e., $R[[t]]$ is weakly $g(x)$-clean.

For the converse, let $R[[t]]$ be weakly $g(x)$-clean. Since $\theta : R[[t]] \longrightarrow R$ with $\theta(f) = a_0$ is a ring epimorphism where $f = \sum_{i \geq 0} a_i t^i \in R[[t]]$, by Proposition 2, $R$ is weakly $g(x)$-clean.

Remark 2.10. Generally, the polynomial ring $R[t]$ is not weakly $g(x)$-clean for non-zero polynomial $g(x) \in C(R)[x]$. For example let $R$ be a commutative ring and also let $g(x) = x$, we show that $t$ is not weakly $g(x)$-clean. If $t = u \pm s$ then it must be that $s = 0$ and so $t = u$. But, by Remark 6, clearly $t \notin U(R[t])$, i.e., $R[t]$ is not weakly $g(x)$-clean.
For more examples of weakly \( g(x) \)-clean rings, we consider the method of idealization. Let \( R \) be a commutative ring and \( M \) an \( R \)-module. The idealization of \( R \) and \( M \) is the ring \( R(M) = R \oplus M \) with product \( (r, m)(r', m') = (rr', rm' + r'm) \) and addition \( (r, m)(r', m') = (r + r', m + m') \).

**Theorem 2.11.** Let \( R \) be a commutative ring, \( M \) an \( R \)-module and \( g(x) = \sum_{i=0}^{n} a_i x^i \in R[x] \). If \( R \) is a weakly \( g(x) \)-clean ring, then the idealization \( R(M) \) of \( R \) and \( M \) is weakly \( g(x) \)-clean.

**Proof.** Let \( (r, m) \in R(M) \). Since \( R \) is a weakly \( g(x) \)-clean ring, we have \( r = u \pm s \) where \( u \in U(R) \) and \( g(s) = 0 \). So \( (r, m) = (u \pm s, m) = (u, m) \pm (s, 0) \).

We have \( (u, m)(u^{-1}, -u^{-1}mu^{-1}) = (uu^{-1}, u(-u^{-1}mu^{-1}) + mu^{-1}) = (1, 0) \).

Therefore \( (u, m) \in U(R(M)) \). Also we have

\[
g((s, 0)) = a_0(1, 0) + a_1(s, 0) + \ldots + a_n(s, 0)^n = a_0(1, 0) + a_1(s, 0) + \ldots + a_n(s^n, 0) = (a_0, 0) + (a_1s, 0) + \ldots + (a_ns^n, 0).
\]

Thus \( (r, m) \) is weakly \( g(x) \)-clean and so \( R(M) \) is a weakly \( g(x) \)-clean ring. \( \square \)

3. WEAKLY \((x^n - x)\)-CLEAN RINGS

In this section we consider the weakly \((x^n - x)\)-clean rings and weakly 2-clean rings.

**Theorem 3.1.** Let \( R \) be a ring, \( n \in \mathbb{N} \) and \( a, b \in R \). Then \( R \) is weakly \((ax^{2n} - bx)\)-clean if and only if \( R \) is weakly \((ax^{2n} + bx)\)-clean.

**Proof.** Suppose \( R \) is weakly \((ax^{2n} - bx)\)-clean. Then for any \( r \in R \), \(-r = u \pm s \) where \((as^{2n} - bs) = 0 \) and \( u \in U(R) \). So \( r = (-u) \pm (-s) \) where \((-u) \in U(R) \) and \( a(-s)^{2n} + b(-s) = 0 \). Hence, \( r \) is weakly \((ax^{2n} + bx)\)-clean. Therefore, \( R \) is weakly \( c(ax^{2n} + bx)\)-clean. Now suppose \( R \) is weakly \((ax^{2n} + bx)\)-clean. Let \( r \in R \). Then there exist \( s \) and \( u \) such that \(-r = u \pm s \), \( as^{2n} + bs = 0 \) and \( u \in U(R) \). So \( r = (-u) \pm (-s) \) satisfies \( (as^{2n} - bs) = 0 \). Hence, \( R \) is weakly \((ax^{2n} - bx)\)-clean. \( \square \)

**Proposition 3.2.** Let \( 2 \leq n \in \mathbb{N} \). If for every \( a \in R \), \( a = u \pm v \) where \( u \in U(R) \) and \( v^{n-1} = 1 \), then \( R \) is weakly \((x^n - x)\)-clean.
The following Lemma is well-known.

**Lemma 3.3.** Let \( a \in R \). The following statements are equivalent for \( n \geq 1 \):

1. \( a = a(ua)^n \) for some \( u \in U(R) \);
2. \( a = ve \) for some \( e^{n+1} = e \) and some \( v \in U(R) \);
3. \( a = fw \) for some \( f^{n+1} = f \) and some \( w \in U(R) \).

**Proof.** See Lemma 4.3 of [3].

**Proposition 3.4.** Let \( R \) be a weakly \((x^n - x)\)-clean ring where \( n \geq 2 \) and \( a \in R \). Then either (i) \( a = u \pm v \) where \( u \in U(R) \) and \( v^{n-1} = 1 \) or (ii) both \( aR \) and \( Ra \) contain nontrivial idempotents.

**Proof.** Since \( R \) is weakly \((x^n - x)\)-clean, write \( a = u \pm e \) where \( u \in U(R) \) and \( e^n = e \). Then \( ae^{n-1} = ue^{n-1} \pm e \). So \( a(1 - e^{n-1}) = u(1 - e^{n-1}) \). Since \( 1 - e^{n-1} \) is an idempotent, by Lemma 12, \( u(1 - e^{n-1}) = fw \) where \( w \in U(R) \) and \( f^2 = f \in R \). So \( f = a(1 - e^{n-1})w^{-1} \in aR \). Suppose (i) does not hold. Then \( 1 - e^{n-1} \neq 0 \), hence \( f \neq 0 \). Thus, \( aR \) contains a nontrivial idempotent. Similarly, \( Ra \) contains a nontrivial idempotent.

**Definition 3.5.** An element \( r \in R \) is called weakly \( n \)-clean if \( r = u_1 + u_2 + ... + u_n \pm e \) with \( e^2 = e \in R \) and \( u_i \in U(R) \) for \( 1 \leq i \leq n \) and \( R \) is called weakly \( n \)-clean if every element of \( R \) is weakly \( n \)-clean.

**Definition 3.6.** An element \( a \in R \) is called right \( \pi \)-regular if it satisfies the following equivalent conditions,

1. \( a^n \in a^{n+1}R \) for some integer \( n \geq 1 \);
2. \( a^nR = a^{n+1}R \) for some integer \( n \geq 1 \);
3. The chain \( aR \supseteq a^2R \supseteq ... \) terminates.

The left \( \pi \)-regular elements are defined analogously. An element \( a \in R \) is called strongly \( \pi \)-regular if it is both left and right \( \pi \)-regular, and \( R \) is called strongly \( \pi \)-regular if every element is strongly \( \pi \)-regular [10].

**Proposition 3.7.** Let \( n \in \mathbb{N} \), if the ring \( R \) is weakly \((x^n - x)\)-clean, then \( R \) is weakly 2-clean.

**Proof.** Let \( r \in R \). Then \( r = u \pm t \) for some \( t^n = t \in R \) and \( u \in U(R) \). Since \( t \) is a strongly \( \pi \)-regular element and strongly \( \pi \)-regular elements are strongly clean [10] (it is of course clean and weakly clean), \( t = v \pm e \) for some \( e^2 = e \) and...
\[ v \in U(R). \] Then \( r = u \pm v \pm e \) is weakly 2-clean. Hence, \( R \) is weakly 2-clean. \hfill \Box

In fact, all weakly \((x^2 - x)\)-clean rings and weakly \((x^2 + cx + d)\)-clean rings with \( d \in U(R) \) discussed above, are weakly 2-clean rings.

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