\section*{R-parts in hyperrings}

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\textbf{Abstract.} In this article, first we generalize the concept of complete parts in hyperrings by introducing the concept \( R \)-parts in hyperrings and then we study \( R \)-closures in hyperrings. Finally we characterize \( R \)-closures in hyperfields.

\textbf{Keywords:} hyperrings, (semi)hypergroups, complete parts.


\section{Introduction}

The theory of hyperstructures was introduced in 1934 by Marty [9] at the 8th Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini and Leoreanu [1, 2], Mittas [11, 12], Stratigopoulos [16], and by various authors. Basic definitions and propositions about the hyperstructures are found in [1, 2, 17]. Krasner [8] has studied the notion of hyperfields, hyperrings, and then some researchers. Hyperrings are essentially rings with approximately modified axioms. There are different notions of hyperrings. If the addition + is a hyperoperation and the multiplication is a binary operation, then the hyperring is called Krasner (additive) hyperring [8].

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Rota [14] introduced a multiplicative hyperring, where + is a binary operation and the multiplication is a hyperoperation. De Salvo [15] studied hyperrings in which the additions and the multiplications were hyperoperations. In 2007, Davvaz and Leoreanu-Fotea [4] published a book titled Hyperring Theory and Applications. Complete parts were introduced by Koskas [7] and studied then by Miglirato [10], Corsini and Sureau [1, 2]. Mousavi et al. [13] introduced the notion of $\mathbb{R}$-parts in hypergroups as a generalization of complete parts in hypergroups. In this article we generalize the notion of complete parts by introducing left and right $\mathbb{R}$-parts in hyperrings and we will study $\mathbb{R}$-closures in hyperrings. Finally we characterize $\mathbb{R}$-closures in hyperfields.

2. Preliminaries

A hypergroupoid $(H, \circ)$ is a non-empty set $H$ together with a hyperoperation $\circ$ defined on $H$, that is a mapping of $H \times H$ into the family of non-empty subsets of $H$. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$ and for simplicity by $xy$. If $A, B$ are non-empty subsets of $H$ then $A \circ B$ is given by $A \circ B = \bigcup\{xy \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ (resp. $A \circ x$). A hypergroupoid $(H, \circ)$ is called a hypergroup in the sense of [9] if for all $x, y, z \in H$ the following two conditions hold: (i) $x(yz) = (xy)z$, (ii) $xH = Hx = H$, means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in xu$ and $y \in vx$. If $(H, \circ)$ satisfies only the first axiom, then it is called a semi-hypergroup.

A hyperring [17] is a triple $(R, +, \circ)$ which satisfies the ring-like axioms in the following way: (i) $(R, +)$ is a hypergroup, (ii) $(R, \circ)$ is a semi-hypergroup, (iii) the multiplication is distributive with respect to the hyperoperation $+$. The hyperrings were studied by many authors, for example see [6], [3], [17], [5] and [19]. In [20] and [18] Vougiouklis defines the relation $\Gamma$ on hyperring as follows: $x\Gamma y$ if and only if $x, y \subseteq u$, where $u$ is a finite sum of finite products of elements of $R$, in fact there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$ such that $u = \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}$.

He proved that the quotient $R/\Gamma^*$, where $\Gamma^*$ is the transitive closure of $\Gamma$, is a ring and also $\Gamma^*$ is the smallest equivalent relation on $R$ such that the quotient $R/\Gamma^*$ is a fundamental ring. The both $\oplus$ and $\circ$ on $R/\Gamma^*$ are defined as follow:

\[ \forall z \in \Gamma^*(x) + \Gamma^*(y), 2\Gamma^*(x) \oplus \Gamma^*(y) = \Gamma^*(z); \]
\[ \forall z \in \Gamma^*(x) \circ \Gamma^*(y), 2\Gamma^*(x) \odot \Gamma^*(y) = \Gamma^*(z). \]

Let $M$ be a non-empty subset of $R$. We say that $M$ is a complete part if for every $n \in \mathbb{N}$, $i = 1, 2, ..., n$, $\forall k_i \in \mathbb{N}$, $\forall (z_{i1}, ..., z_{ik_i}) \in R^{k_i}$, we have:

\[ \sum_{i=1}^{n} \prod_{j=1}^{k_i} z_{ij} \cap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} \prod_{j=1}^{k_i} z_{ij} \subseteq M. \]
3. \( R \)-parts

Let \( \mathcal{U} \) be the set of finite sums of finite products of elements of \( R \) and \( \mathcal{R} \) be a relation on \( \mathcal{U} \). In this section first we generalize the notion of complete parts by introducing the notion of \( R \)-parts and then we study \( R \)-closures.

**Definition 3.1.** Let \( R \) be a hyperring and \( \mathcal{U} \) be the set of finite sum of finite products of elements of \( R \) and \( \mathcal{R} \) be a relation on \( \mathcal{U} \). For a non-empty subset \( A \) of \( R \) we say:

(i) \( A \) is a left \( R \)-part of \( R \) with respect to \( \mathcal{U} \) (or briefly is \( \mathcal{L}_{\mathcal{R}_u} \)-part) if for all \( \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \) and \( \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} \) in \( \mathcal{U} \) the following implication is valid:

\[
\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \bigcap A \neq \emptyset \text{ and } 2 \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} 2R1 \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \Rightarrow \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} 1 \subseteq A;
\]

(ii) \( A \) is a right \( R \)-part of \( R \) with respect to \( \mathcal{U} \) (or briefly is \( \mathcal{R}_{\mathcal{R}_u} \)-part) if for all \( \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \) and \( \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} \) in \( \mathcal{U} \) the following implication is valid:

\[
\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \bigcap A \neq \emptyset \text{ and } 2 \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} 2R1 \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \Rightarrow \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} 1 \subseteq A;
\]

(iii) \( A \) is a \( R \)-part of \( R \) with respect to \( \mathcal{U} \) (or briefly is \( \mathcal{R}_u \)-part) if it is \( \mathcal{L}_{\mathcal{R}_u} \)-part and \( \mathcal{R}_{\mathcal{R}_u} \)-part.

**Proposition 3.2.** Let \( \mathcal{R} \) be a relation on \( \mathcal{U} \) and \( \mathcal{R}^{-1} \) be the inverse of \( \mathcal{R} \) then

(i) \( A \) is \( \mathcal{L}_{\mathcal{R}_u}^{-1} \)-part if and only if it is \( \mathcal{R}_{\mathcal{R}_u} \)-part;

(ii) \( A \) is \( \mathcal{R}_{\mathcal{R}_u}^{-1} \)-part if and only if it is \( \mathcal{L}_{\mathcal{R}_u} \)-part.

**Definition 3.3.** The intersection of \( \mathcal{L}_{\mathcal{R}_u} \)-parts (or \( \mathcal{R}_{\mathcal{R}_u} \)-parts, \( \mathcal{R} \)-parts) which contain \( A \) is called \( \mathcal{L}_{\mathcal{R}_u} \)-closure (or \( \mathcal{R}_{\mathcal{R}_u} \)-closure, \( \mathcal{R} \)-closure) of \( A \) in \( R \) and it will be denoted by \( \mathcal{L}_{\mathcal{R}_u}^{-1}(A) \) (or \( \mathcal{R}_{\mathcal{R}_u}^{-1}(A) \), \( \mathcal{R}_u^{-1}(A) \)).

From now on \( R \) is a hyperring, \( \mathcal{U} \) is the set of finite sum of finite products of elements of \( R \), \( u \in \mathcal{U} \) means \( u = \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \) and \( A \) is a non-empty subset of \( R \).

**Proposition 3.4.** For a non-empty subset \( A \) of \( R \) we have:

(i) \( \mathcal{L}_{\mathcal{R}_u}^{-1}(A) = \mathcal{R}_{\mathcal{R}_u}^{-1}(A) \);

(ii) \( \mathcal{R}_{\mathcal{R}_u}^{-1}(A) = \mathcal{L}_{\mathcal{R}_u}^{-1}(A) \).

Proof. Follows from Proposition 3.2. \( \square \)

**Lemma 3.5.** For a non-empty subset \( A \) of \( R \) define:

\[
\sum_{A}^{\mathcal{U}} \overset{\text{def}}{=} \{ \mathcal{R}1 \subseteq \mathcal{U} \times \mathcal{U} \mid \mathcal{L}_{\mathcal{R}_u}(A) = A \} \text{ and } \sum_{A}^{\mathcal{U}} \overset{\text{def}}{=} \{ \mathcal{R}1 \subseteq \mathcal{U} \times \mathcal{U} \mid \mathcal{R}_{\mathcal{R}_u}(A) = A \}.
\]
If \( A \sum_{i=1}^{d} \neq \emptyset \) (resp. \( \sum_{A}^{d} \neq \emptyset \)), then \( (A \sum_{i=1}^{d}, \circ) \) (resp. \( (\sum_{A}^{d}, \circ) \)) is a semigroup, where \( \circ \) is the operation of relation composition.

**Proof.** Suppose that \( \mathcal{R}, \mathcal{R}' \in A \sum^{d} \) and \( (\sum_{i=1}^{n} k_{i} \prod_{j=1}^{m} y_{ij}, \sum_{i=1}^{n} l_{i} \prod_{j=1}^{m} x_{ij}) \in \mathcal{U} \times \mathcal{U} \) are given. Let \( \sum_{i=1}^{m} \prod_{j=1}^{t_{i}} x_{ij} \cap A \neq \emptyset \) and \( \sum_{i=1}^{n} k_{i} \prod_{j=1}^{m} y_{ij} \circ \mathcal{R}' \sum_{i=1}^{k_{i}} x_{ij} \). So there exists \( \sum_{i=1}^{k} \prod_{j=1}^{s_{i}} z_{ij} \) such that \( \sum_{i=1}^{k} \prod_{j=1}^{s_{i}} z_{ij} \mathcal{R} \sum_{i=1}^{m} \prod_{j=1}^{t_{i}} x_{ij} \) and \( \sum_{i=1}^{n} l_{i} \prod_{j=1}^{m} x_{ij} \). From \( \sum_{i=1}^{k} \prod_{j=1}^{s_{i}} z_{ij} \mathcal{R} \sum_{i=1}^{m} \prod_{j=1}^{t_{i}} x_{ij} \) and \( \mathcal{R} \in A \sum^{d} \), we have \( \sum_{i=1}^{k} \prod_{j=1}^{s_{i}} z_{ij} \subseteq A \). Since \( \mathcal{R}' \in A \sum^{d} \) and \( \sum_{i=1}^{m} \prod_{j=1}^{t_{i}} x_{ij} \), we have \( \sum_{i=1}^{k} \prod_{j=1}^{s_{i}} z_{ij} \subseteq A \).

**Theorem 3.6.** If \( \mathcal{R} \) is a permutation of finite order in \( S_{a} \) (the symmetric group on the set \( A \)), then the following are equivalent:

(i) \( A \) is \( L \mathcal{R}_{a} \)-part;
(ii) \( A \) is \( R \mathcal{R}_{a} \)-part;
(iii) \( A \) is \( R' \mathcal{R}_{a} \)-part.

**Proof.** (i)\( \Rightarrow \) (ii). For this reason suppose that \( A \) is \( L \mathcal{R}_{a} \)-part. So \( L \mathcal{R}_{a}^{-1}(A) = A \) and hence \( \mathcal{R} \in A \sum^{d} \). Since \( \mathcal{R} \) is a permutation of finite order in \( S_{a} \), \( \langle \mathcal{R} \rangle = \{ \mathcal{R}^{n} \mid n \in \mathbb{N} \} \) is a subgroup of \( A \sum^{d} \) and so \( \mathcal{R}^{-1} \in A \sum^{d} \). Therefore by Proposition 3.4 we have \( A = \overline{L \mathcal{R}_{a}^{-1}}(A) = \overline{R \mathcal{R}_{a}}(A) \), thus \( \mathcal{R} \in A \sum^{d} \) and hence \( A \) is \( R \mathcal{R}_{a} \)-part.

**Theorem 3.7.** Suppose that \( \mathcal{R} \subseteq \mathcal{U} \times \mathcal{U} \).

(i) We pose \( K_{1,0}^{L}(A) = A \) and

\[
K_{n+1,0}^{L}(A) = \{ x \in \mathcal{R} \mid \exists (u, v) \in \mathcal{R}, x \in u \text{ and } v \cap K_{n,0}^{L}(A) \neq \emptyset \},
\]

if we consider \( K_{n}^{L}(A) = \cup_{n-1} K_{n,0}^{L}(A) \), then \( K_{n}^{L}(A) = \overline{L \mathcal{R}_{a}}(A) \) and \( K_{n}^{L}(A) \) is the smallest \( L \mathcal{R}_{a} \)-part containing \( A \);

(ii) We pose \( K_{1,0}^{R}(A) = A \) and

\[
K_{n+1,0}^{R}(A) = \{ x \in \mathcal{R} \mid \exists (v, u) \in \mathcal{R}, x \in u \text{ and } v \cap K_{n,0}^{R}(A) \neq \emptyset \},
\]

if we consider \( K_{n}^{R}(A) = \cup_{n-1} K_{n,0}^{R}(A) \), then \( K_{n}^{R}(A) = \overline{R \mathcal{R}_{a}}(A) \) and \( K_{n}^{R}(A) \) is the smallest \( R \mathcal{R}_{a} \)-part containing \( A \);

(iii) We pose \( K_{1,0}(A) = A \) and

\[
K_{n+1,0}(A) = \{ x \in \mathcal{R} \mid \exists (u, v) \in \mathcal{R} \cup \mathcal{R}^{-1}, x \in u \text{ and } v \cap K_{n,0}(A) \neq \emptyset \},
\]

if \( K_{n}(A) = \cup_{n-1} K_{n,0}(A) \), then \( K_{n}(A) = \overline{R \mathcal{R}_{a}}(A) \) and \( K_{n}(A) \) is the smallest \( \mathcal{R}_{a} \)-part containing \( A \).
Proof. (i) It is necessary to prove:

(1) $K^\mathcal{L}_R(A)$ is $\mathcal{L}_R$-part,

(2) if $A \subseteq B$ and $B$ is $\mathcal{L}_R$-part, then $K^\mathcal{L}_R(A) \subseteq B$.

For the proof (1) suppose that $v \cap K^\mathcal{L}_R(A) \neq \emptyset$ and $u1R1v$. Therefore there exists $n \in \mathbb{N}$ such that $v \cap K^\mathcal{L}_n(A) \neq \emptyset$, from which follows $u1 \in 1K^\mathcal{L}_{n+1,R}(A) 1 \subseteq 1K^\mathcal{L}_R(A)$.

Now we prove (2) by induction on $n$. We have $K^\mathcal{L}_{1,R}(A) 1 = 1A1 \subseteq 1B$. Suppose that $K^\mathcal{L}_{n,R}(A) 1 \subseteq 1B$. We prove that $K^\mathcal{L}_{n+1,R}(A) 1 \subseteq 1B$. If $z \in K^\mathcal{L}_{n+1,R}(A)$, then there exists $(u, v) \in \mathcal{U} \times \mathcal{U}$ such that $z \in u$, $u1R1v$ and $v \cap K^\mathcal{L}_{n,R}(A) \neq \emptyset$. Therefore $v \cap B \neq \emptyset$ and hence $z \in u1B$. So $K^\mathcal{L}_{n+1,R}(A) 1 \subseteq 1B$.

(ii) We have

$$K^\mathcal{R}_R(A) = K^\mathcal{L}_{R^{-1}}(A)$$

$$= \overline{\mathcal{L}_R}(A), \text{by part (i)}$$

$$= \overline{\mathcal{R}_R}(A), \text{by Proposition 3.4}.$$

(iii) Follows from (i) and (ii).

Proposition 3.8. Suppose that $B$ is a non-empty subset of $R$ and $\mathcal{R}$ is a relation on $\mathcal{U}$. Then we have:

(i) $\overline{\mathcal{L}_R}(B) = \bigcup_{b \in B} \mathcal{L}_R(b)$;

(ii) $\overline{\mathcal{R}_R}(B) = \bigcup_{b \in B} \mathcal{R}_R(b)$;

(iii) $\mathcal{R}_R(B) = \bigcup_{b \in B} \mathcal{R}_R(b)$.

Proof. (i) It is clear that for all $b \in B$, $\overline{\mathcal{L}_R}(b) \subseteq 1 \overline{\mathcal{L}_R}(B)$. By Theorem 3.7(i), $\overline{\mathcal{L}_R}(B) = \bigcup_{n \geq 1} K^\mathcal{L}_{n,R}(B)$. We follow the proposition by induction on $n$. For $n = 1$, $K^\mathcal{L}_{1,R}(B) = B = \bigcup_{b \in B} \{b\} = \bigcup_{b \in B} K^\mathcal{L}_{1,R}(b)$. Supposing it is true for $n$, we show that $K^\mathcal{L}_{n+1,R}(B) \subseteq 1 \bigcup_{b \in B} K^\mathcal{L}_{n,R}(b)$. If $z \in K^\mathcal{L}_{n+1,R}(B)$, then there exists $(u, v) \in \mathcal{R}$ such that $z \in u$ and $v \cap K^\mathcal{L}_{n,R}(B) \neq \emptyset$. From this it follows, by the hypothesis of induction, $v \cap (\bigcup_{b \in B} K^\mathcal{L}_{n,R}(b)) \neq \emptyset$ and therefore $b' \in B$ exists such that $v \cap K^\mathcal{L}_{n,R}(b') \neq \emptyset$. So $z \in K^\mathcal{L}_{n+1,R}(b')$ and hence $\overline{\mathcal{L}_R}(B) \subseteq 1 \bigcup_{b \in B} \overline{\mathcal{L}_R}(b)$.

Theorem 3.9. Suppose that $\mathcal{R}1 \subseteq \mathcal{U} \times \mathcal{U}$. The relation $K^\mathcal{L}_R$ (resp. $K^\mathcal{R}_R$) on $R$ defined by:

$$x1K^\mathcal{L}_R(y) \iff x \in K^\mathcal{L}_R(y) (x \in K^\mathcal{R}_R(y))$$

where $K^\mathcal{L}_R(y) = K^\mathcal{L}_R(\{y\})$ (resp. $K^\mathcal{R}_R(y) = K^\mathcal{R}_R(\{y\})$) is a preorder. Furthermore if $\mathcal{R}$ is symmetric, then $K^\mathcal{L}_R$ (resp. $K^\mathcal{R}_R$) is an equivalence relation.
Proof. It is easy to see that $K^C_n$ is reflexive. Now suppose that $x1K^C_ny$ and $y1K^C_nz$. So $x \in K^C_n(y)$ and $y \in K^C_n(z)$. By Theorem 3.7(i) we have $K^C_n(z)$ is $\mathcal{L}R_n$-part thus $K^C_n(y) \subseteq K^C_n(z)$ and hence $x \in K^C_n(z)$. Therefore $K^C_n$ is preorder. Now let $\mathcal{R}$ be symmetric. We prove that $K^C_n$ is symmetric as well. To this end the following is premised:

(1) for all $n \geq 2$ and $x \in R$, $K^C_n(x) = K^C_{n+1,n}(x)$;

(2) $x \in K^C_n(y)$ if and only if $y \in K^C_n(x)$.

We prove (1) by induction on $n$. Suppose that $z \in K^C_n(x)$ so there exists $(u, v) \in \mathcal{R}$ such that $z \in u$ and $v \cap K^C_n(x) \neq \emptyset$. Thus $z \in K^C_{n+1,n}(x)$. Let $K^C_n(x) = K^C_{n+1,n}(x)$ so we have:

$$z \in K^C_{n+1,n}(x) \iff \exists (u, v) \in \mathcal{R}, z \in u, v \cap K^C_n(x) \neq \emptyset$$

$$\iff \exists (u, v) \in \mathcal{R}, z \in u, v \cap K^C_{n+1,n}(x) \neq \emptyset$$

$$\iff z \in K^C_{n+1,n}(x).$$

We also prove (2) by induction on $n$. It is clear that $x \in K^C_n(y)$ if and only if $y \in K^C_n(x)$. Suppose $x \in K^C_n(y)$ if and only if $y \in K^C_n(\mathcal{R})$. Let $x \in K^C_{n+1,n}(y)$ be given, so there exist $(u, v) \in \mathcal{R}$ such that $x \in u$ and $v \cap K^C_n(y) \neq \emptyset$. Therefore there exists $b \in v \cap K^C_n(y)$ and hence $y \in K^C_{n+1,n}(b)$. Since $\mathcal{R}$ is symmetric and $(u, v) \in \mathcal{R}, b \in v$ and $x \in u \cap K^C_{n+1,n}(x) implies that b \in K^C_n(x)$ and hence $y \in K^C_n(x) = K^C_{n+1,n}(x)$. Similarly we can show if $y \in K^C_{n+1,n}(x)$, then $x \in K^C_{n+1,n}(x)$.

\[
\square
\]

Proposition 3.10. Let $\mathcal{R}$ be a relation on $U$ and $A$ be a non-empty subset of the hyperring $R$. The following conditions are equivalent:

(i) $A$ is a $(\mathcal{R}_n^\mathcal{L}R_n^-)$-part $\mathcal{L}R_n^-$-part of $R$;

(ii) $x \in A, (x1K^C_nz)z1K^C_nx \Rightarrow z \in A$.

Proof. (i) $\Rightarrow$ (ii) If $x \in A$ and $z \in R$ such that $z1K^C_nx$, then there exists $(u, v) \in \mathcal{R}$ such that $z \in u$ and $v \cap K^C_n(A) \neq \emptyset$ for some $n \in N$. Since $A$ is a $\mathcal{L}R_n^-$-part by Theorem 3.7, $K^C_n(A)1 \subseteq 1A$ and so $v \cap A \neq \emptyset$. Therefore $u1 \subseteq 1A$ and hence $z \in A$.

(ii) $\Rightarrow$ (i). Let $u \cap A \neq \emptyset$ and $v1\mathcal{R}1u$. So there exists $x \in A \cap u$ and $x \in u, u \cap K^C_n(x) \neq \emptyset$. Now suppose that $z \in v$ is given. So

$$v1\mathcal{R}1u \Rightarrow z \in K^C_n(x), 4 \text{ because } x \in u$$

$$\Rightarrow z1K^C_nx$$

$$\Rightarrow z \in A, 15 \text{ because } x \in A.$$
4. Rings derived from hyperrings

In this section we give the notion of (strongly) normal relation on $\mathcal{U}$ and then we construct a ring from a hyperring.

**Definition 4.1.** Suppose that $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$.

(i) for all $(x, y) \in \mathbb{R}^2$ define the relation $\mathcal{R}_{\mathcal{L} \times \mathcal{R}}$ on $\mathbb{R}$ by:

$$x \mathcal{R}_{\mathcal{L} \times \mathcal{R}} y \iff [x = y_1 \text{ or } \exists (u, v) \in \mathbb{R}^2 \text{ such that } 1x \in u_1 \text{ and } 1y \in v]$$

and $\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^*$ is the transitive closure of $\mathcal{R}_{\mathcal{L} \times \mathcal{R}}$;

(ii) for all $(x, y) \in \mathbb{R}^2$ define the relation $\mathcal{R}_{\mathcal{R} \times \mathcal{R}}$ on $\mathbb{R}$ by:

$$x \mathcal{R}_{\mathcal{R} \times \mathcal{R}} y \iff [x = y_1 \text{ or } \exists (v, u) \in \mathbb{R}^2 \text{ such that } 1x \in u_1 \text{ and } 1y \in v]$$

and $\mathcal{R}_{\mathcal{R} \times \mathcal{R}}^*$ is the transitive closure of $\mathcal{R}_{\mathcal{R} \times \mathcal{R}}$;

(iii) for all $(x, y) \in \mathbb{R}^2$ define the relation $\mathcal{R}_\mathcal{R}$ on $\mathbb{R}$ by:

$$x \mathcal{R}_\mathcal{R} y \iff [x = y_1 \text{ or } \exists (u, v) \in \mathbb{R} \cup \mathbb{R}^{-1} \mathbb{R}^2 \text{ such that } 1x \in u_1 \text{ and } 1y \in v]$$

and $\mathcal{R}_\mathcal{R}^*$ is the transitive closure of $\mathcal{R}_\mathcal{R}$.

**Theorem 4.2.** Suppose that $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$. For all $(x, y) \in \mathbb{R}^2$ we have:

(i) $x1\mathcal{R}_\mathcal{R}^* 1y$ if and only if $x1\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1y$;

(ii) $x1\mathcal{R}_\mathcal{R}^* 1y$ if and only if $x1\mathcal{R}_{\mathcal{R} \times \mathcal{R}}^* 1y$.

**Proof.** (i) It is easy to see that $\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1 \subseteq 1 \mathcal{R}_\mathcal{R}^* 1$. Conversely suppose that $x1\mathcal{R}_\mathcal{R}^* 1y$ so by Theorem 3.9 we have $x \in \mathcal{K}_{n-1, \mathcal{R}}(y)$ for some $n \in \mathbb{N}$. So there exists $(u_1, v_1) \in \mathbb{R}$ such that $x \in u_1$ and $v_1 \mathcal{R}_\mathcal{R}(y) \neq \emptyset$ thus there exists $x_1 \in v_1 \cap \mathcal{K}_{n-1, \mathcal{R}}(y)$ and hence $x1\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1x_1$. Since $x_1 \in \mathcal{K}_{n-1, \mathcal{R}}(y)$, there exists $(u_2, v_2) \in \mathbb{R}$ such that $x_1 \in u_2$ and $v_2 \mathcal{R}_\mathcal{R}(y) \neq \emptyset$. Therefore $x_11\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1x_2$, where $x_2 \in u_2 \cap \mathcal{K}_{n-1, \mathcal{R}}(y)$. As a consequence we conclude that $x_2 \in v_2 \cap \mathcal{K}_{n-1, \mathcal{R}}(y)$ exists such that $x_21\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1x_3$. Thus we have,

$$x1\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1x_11\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1x_21\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1x_3 \ldots 1x_n1\mathcal{R}_{\mathcal{L} \times \mathcal{R}}^* 1y.$$ 

From this follows $\mathcal{K}_{n-1, \mathcal{R}}(y) \mathcal{L} \mathcal{R}_\mathcal{R}^* 1y$ and the proof is complete.

Similarly we have (ii). \[\square\]

**Proposition 4.3.** Suppose that $\mathcal{R}$ is a permutation of finite order in $S_\mathcal{U}$, then $\mathcal{R}_\mathcal{R}^* = \mathcal{R}_\mathcal{R}^*$. \[\square\]

**Definition 4.4.** If $(\mathcal{R}, +, \circ)$ is a hyperring and $\rho \subseteq 1 \mathcal{R} \times \mathcal{R}$ is an equivalence, then we set:

$$A \mathcal{R} B \iff a1\rho b, 5 \forall a \in A, \forall b \in B,$$
for all pairs \((A,B)\) of non-empty subsets of \(R\). The relation \(\rho\) is said to be strongly regular to the left (resp. to the right) if (i) \(x1\rho y \Rightarrow a + x \equiv x + a + y\) and (ii) \(x1\rho y \Rightarrow a \circ x \equiv x \circ a \circ y\) (resp. (i) \(x1\rho y \Rightarrow x + a \equiv a + x + a\) and (ii) \(x1\rho y \Rightarrow a \circ x \equiv a \circ x \circ a \circ y\)), for all \((x,y,a)\) \(\in R^3\). \(\rho\) is called strongly regular if it is (i) strongly regular to the right and to the left and moreover (ii) there exists \(e\) in \(R\) such that: \(\rho(x) = \rho(t)\), for all \(t \in x \circ e \cap e \circ x\).

**Definition 4.5.** Let \(R\) be a hyperring, then

(i) a relation \(\mathcal{R}\) on \(U\) is called normal if for all \(x \in R\), one has \(K^\mathcal{L}_x(x) = K^\mathcal{R}_x(x)\).

(ii) a normal relation \(\mathcal{R}\) on \(U\) is called strongly normal to the left (resp. to the right), if \(\rho^*_\mathcal{L}_x\) (resp. \(\rho^*_\mathcal{R}_x\)) is strongly regular to the left (resp. to the right),

(iii) a normal relation \(\mathcal{R}\) on \(U\) is called strongly normal if \(\rho^*_\mathcal{R}_x\) is strongly regular.

Suppose that \(R1 \subseteq U \times U\). For every element \(x\) of a hyperring \(R\), set:

\[P^n_{\mathcal{L}_x}(x) = \bigcup \{v1 \mid v1 \mathcal{R} u_n, \ u_n = \sum_{i=1}^n k_i \prod_{j=1}^i x_{ij}, \ x \in u_n\};\]

\[P_{\mathcal{L}_x}(x) = \bigcup_{n \geq 1} P^n_{\mathcal{L}_x}(x) \bigcup \{x\};\]

\[\rho^*_\mathcal{L}_x(x) = \{y \in R1 \mid 1y1P^*_{\mathcal{L}_x}1x\}.

**Theorem 4.6.** Let \(R\) be a hyperring and \(\mathcal{R}\) be a relation on \(U\). The following conditions are equivalent:

(i) \(\rho^*_\mathcal{R}_x\) is transitive;

(ii) for every \(x \in R\), \(\rho^*_\mathcal{L}_x(x) = P_{\mathcal{L}_x}(x)\);

(iii) for every \(x \in R\), \(P_{\mathcal{L}_x}(x)\) is a \(\mathcal{L}\mathcal{R}_{x_1}\)-part of \(R\).

**Proof.** (i) \(\Rightarrow\) (ii) For every pair \((x,y)\) of elements of \(R\) we have:

\[y \in \rho^*_\mathcal{L}_x(x) \iff y1\rho^*_\mathcal{L}_x1x \iff y1\rho_{\mathcal{L}_x}1x \iff y \in P_{\mathcal{L}_x}(x).\]

(ii) \(\Rightarrow\) (iii) Let \((u,v)\) \(\in \mathcal{R}\) such that \(u \cap P_{\mathcal{L}_x}(x) \neq \emptyset\) be given. So \(u \cap \rho^*_\mathcal{L}_x(x) \neq \emptyset\) and hence there exists \(z \in R\) such that \(z \in u\) and \(z \in \rho^*_\mathcal{L}_x(x)\), thus \(z \in K^\mathcal{L}_x(x)\), by Theorem 4.2. On the other hand, \(z \in K^\mathcal{L}_x(z)\), so \(u \cap K^\mathcal{L}_x(z) \neq \emptyset\) and hence \(v1 \subseteq K^\mathcal{R}_x(z)\), because \(v1 \subseteq K^\mathcal{R}_x(z)\) is a \(\mathcal{L}\mathcal{R}_{y_1}\)-part of \(R\), by Theorem 3.7. Now suppose that \(t \in v\) is an arbitrary element, thus \(t \in K^\mathcal{L}_x(x)\) and hence \(t1\rho^*_\mathcal{L}_x1x\). Therefore \(t \in \rho^*_\mathcal{L}_x(x) = P_{\mathcal{L}_x}(x)\) and so \(v1 \subseteq P_{\mathcal{L}_x}(x)\).

(iii) \(\Rightarrow\) (i) Let \(x, y\) and \(z\) in \(R\) be given such that \(x1\rho_{\mathcal{L}_x}1y\) and \(y1\rho_{\mathcal{L}_x}1z\). Since \(x1\rho_{\mathcal{L}_x}1y\), there exists \((u,v)\) \(\in \mathcal{R}\) such that \(x \in u\) and \(y \in v\). Therefore \(v \cap P_{\mathcal{L}_x}(y) \neq \emptyset\) and since \(P_{\mathcal{L}_x}(y)\) is a \(\mathcal{L}\mathcal{R}_{y_1}\)-part, \(u1 \subseteq P_{\mathcal{L}_x}(y)\) and hence \(x \in P_{\mathcal{L}_x}(y)\). We can see that \(P_{\mathcal{L}_x}(y)1 \subseteq P_{\mathcal{L}_x}(z)\), because \(y1\rho_{\mathcal{L}_x}1z\) and so by above \(y \in P_{\mathcal{L}_x}(z)\). Therefore \(x \in P_{\mathcal{L}_x}(z)\) and hence \(x1\rho_{\mathcal{L}_x}1z\). \(\square\)

**Proposition 4.7.** If \(\mathcal{R}\) is a normal relation on \(U\), then:

(i) \(\mathcal{R}^{-1}\) is a normal relation;
(ii) \( \rho^*_e = \rho^*_n \) and \( \rho^*_e \) is an equivalence relation.

Proof. The proof follows from Proposition 3.4 and Theorem 4.2. \( \square \)

**Theorem 4.8.** Suppose that \((R, +, \circ)\) is a hyperring and \(\mathcal{R}\) is a strongly normal relation on \(\mathcal{U}\). A ring structure turns out to be defined on \(R/\rho^*_n\) with respect to the operations:

\[
\rho^*_n(x) + \rho^*_n(y) = \rho^*_n(z), \text{ where } 1 \leq z \leq x + y.
\]

\[
\rho^*_n(x) \circ \rho^*_n(y) = \rho^*_n(z), \text{ where } 1 \leq z \leq x \circ y.
\]

Proof. We will prove that the operation \(\oplus\) is well defined. Let \(\rho^*_n(x_0) = \rho^*_n(x_1)\) and \(\rho^*_n(y_0) = \rho^*_n(y_1)\). It is necessary to verify that \(\rho^*_n(x_0) + \rho^*_n(y_0) = \rho^*_n(x_1) + \rho^*_n(y_1)\). By hypothesis \((m, n) \in \mathbb{N}^2\), \((z_0, z_1, ..., z_m) \in R^{m+1}\) and \((t_0, t_1, ..., t_n) \in R^{n+1}\) exist such that

\[
x_0 = z_0 \rho_n 1 z_1 \rho_n 1 z_2 1 ... 1 z_m - 1 \rho_n 1 z_m = x_1
\]

and

\[
y_0 = t_0 \rho_n 1 t_1 \rho_n 1 t_2 1 ... 1 t_n - 1 \rho_n 1 t_n = y_1
\]

Since \(\mathcal{R}\) is normal, for all \(u \in z_s - t_s - 1\) and \(v \in z_s + t_s\), where \(1 \leq s \leq k\) and \(k = \min\{m, n\}\), we have \(u \rho_n v\). Therefore \(\rho^*_n(x_0) + \rho^*_n(y_0) = \rho^*_n(x_1) + \rho^*_n(y_1) = ... = \rho^*_n(z_k) + \rho^*_n(t_k) = \rho^*_n(a_{k+i}) \oplus \rho^*_n(b_{k+i})\), where \(k + 1 \leq k + i \leq \max\{m, n\}\) and:

\[
(a_{k+i}, b_{k+i}) = \begin{cases} 
(x_1, t_{k+i}) & \text{if } k = m; \\
(z_{k+i}, y_1) & \text{if } k = n.
\end{cases}
\]

Hence \(\oplus\) is well defined. Similarly the operation \(\circ\) is well defined and Theorem 31 of [2] shows that \((R, \rho^*_n, \oplus)\) is a group. By strongly normality of \(\mathcal{R}\) we conclude that \((R, \rho^*_n, \circ)\) is a monoid with unit \(\rho^*_n(e)\). The commutativity of \(\oplus\) is related with the existence of the unit in multiplication. Since \(\mathcal{R}\) is strong, there exists \(e \in R\) such that \(\rho(x) = \rho(t)\) for all \(t \in x \circ e \cap e \circ x\) which means \(\rho^*_n(e)\) is the unit of multiplication so we have:

\[
[p^*_n(x) \circ p^*_n(y)] \circ [p^*_n(e) \circ p^*_n(e)] = [p^*_n(x) \circ [p^*_n(e) \circ p^*_n(e)]] \circ [p^*_n(y) \circ [p^*_n(e) \circ p^*_n(e)]] = (p^*_n(x) \circ p^*_n(y)) \circ (p^*_n(e) \circ p^*_n(e)) = (p^*_n(x) \circ p^*_n(y)) \circ (p^*_n(x) \circ p^*_n(y)) = (p^*_n(x) \circ p^*_n(y)) \circ (p^*_n(x) \circ p^*_n(y)).
\]

So \(\rho^*_n(x) \circ p^*_n(y) \circ p^*_n(y) = (p^*_n(x) \circ p^*_n(y)) \circ (p^*_n(x) \circ p^*_n(y))\) gives, \(\rho^*_n(x) \circ p^*_n(y) = p^*_n(y) \circ p^*_n(x)\).

Let \((R, +, \circ)\) and \((R', +', \circ')\) be two hyperrings. We say that \(f : R \to R'\) is a homomorphism if for every \((x, y) \in R^2\) we have \(f(x + y) = f(x) +' f(y)\) and \(f(x \circ y) = f(x) \circ' f(y)\).
Definition 4.9. Let \( R \) be a hyperring and \( \mathcal{R} \) be a strongly normal relation on \( U \). If \( \varphi_R : R \to R/\mathcal{R}_R^* \) be the canonical projection, we set \( \omega_R = \varphi_R^{-1}(1_R/\mathcal{R}_R^*) \), and called the heart of \( \varphi_R \).

Theorem 4.10. Let \((R, +, \circ)\) is a hyperfield (i.e, \((R, +, \circ)\) be a hyperring and \((R, \circ)\) is a hypergroup) and \( B \) is a non-empty subset of \( R \), then we have \( \omega_R \circ B = B \circ \omega_R = \varphi_R^{-1}(\varphi_R(B)) \).

Proof. Clearly \( \varphi_R^{-1}(\varphi_R(B)) = \{ x \in R \mid \exists b \in B : \varphi_R(b) = \varphi_R(x) \} \). Let \( y \in \varphi_R^{-1}(\varphi_R(B)) \), thus for some \( b \in B \) \( \varphi_R(b) = \varphi_R(y) \). Since \((R, \circ)\) is a hypergroup, \( u \in R \) exists such that \( y = b \circ u \), so \( \varphi_R(y) = \varphi_R(b) \circ \varphi_R(u) \). Since \((R/\mathcal{R}_R^*, \circ)\) is a group and \( \varphi_R(b) = \varphi_R(y) \), we obtain \( \varphi_R(u) = 1_R/\mathcal{R}_R^* \) and so \( u \in \varphi_R^{-1}(1_R/\mathcal{R}_R^*) = \omega_R \). Therefore, \( \varphi_R^{-1}(\varphi_R(B)) \subseteq B \circ \omega_R \).

\( \square \)

Conversely if \( z \in B \circ \omega_R \), then \( \varphi_R(z) \in \varphi_R(B) \) and so \( z \in \varphi_R^{-1}(\varphi_R(B)) \).

It is proved that \( \omega_R \circ B = \varphi_R^{-1}(\varphi_R(B)) \) by a similar way and we obtain \( \varphi_R^{-1}(\varphi_R(B)) = \omega_R \circ B = B \circ \omega_R \).

\( \square \)

Theorem 4.11. If \((R, +, \circ)\) is a hyperfield and \( B \) is a non-empty subset of \( R \), then we have \( \omega_R \circ B = B \circ \omega_R = \overline{\mathcal{R}_u}(B) \).

Proof. If \( \varphi_R(b) = \varphi_R(x) \) then \( x \in \overline{\mathcal{R}_u}(b) \). Therefore \( \varphi_R^{-1}(\varphi_R(B)) = \bigcup_{b \in B} \overline{\mathcal{R}_u}(b) = \overline{\mathcal{R}_u}(B) \).

\( \square \)

5. \( \mathcal{R} \)-parts and \( A_{\mathcal{R}} \)-hyperrings

We recall that a \( K_{\mathcal{R}} \)-hypergroup is a hypergroup constructed from a hypergroup \((H, \circ)\) and a family \( \{A(x)\}_{x \in H} \) of non-empty subsets that are mutually disjoint. Put \( K_{\mathcal{R}} = \bigcup_{x \in H} A(x) \) and define the hyperoperation \( * \) on \( K_{\mathcal{R}} \) as following,

\[ \forall (a, b) \in K_{\mathcal{R}}^2, 2a \in A(x), b \in A(y), 3a \ast b \overset{\text{def}}{=} \bigcup_{z \in x \circ y} A(z). \]

\((H, \circ)\) is a hypergroup if and only if \((K_{\mathcal{R}}, *)\) is a hypergroup. In this case \( K_{\mathcal{R}} \) is said to be a \( K_{\mathcal{R}} \)-hypergroup generated by \( H \).

Now let \((R, +, \ast)\) be a commutative hyperring, \( S_r, r \in R \) be a family of non-empty sets indexed in \( R \) such that for all \( r_1, r_2 \in R, r_1 \neq r_2, S_{r_1} \cap S_{r_2} = \emptyset \).

We set \( A = \bigcup_{r \in R} S_r \) and we define the hyperoperations \( \uplus \) and \( \circ \) in \( A \) in the following way:

\[ \forall (x, y) \in S_{r_1} \times S_{r_2}, \quad x \uplus y = \bigcup_{t \in r_1 \uplus r_2} S_t \quad \text{and} \quad x \circ y = \bigcup_{u \in r_1 \circ r_2} S_u. \]

It is easy to see that the structure \((A, \uplus, \circ)\) is a hyperring. The hyperring \((A, \uplus, \circ)\) is called a \( A_{\mathcal{R}} \)-hyperring with suport \( A \) or \( A_{\mathcal{R}} \)-hyperring generated by
For all \( P \in P^*(R) \), let \( S(P) = \bigcup_{x \in P} S_x \).

**Theorem 5.1.** Let \( \mathcal{R} \) be a relation on \( U \). Then \( P \) is \( \mathcal{L}\mathcal{R}_U \)-part of \( R \) if and only if \( S(P) \) is \( \mathcal{L}\mathcal{R}_U \)-part of \( A_U \), where the relation \( \mathcal{R} \) is defined as follows:

\[
\sum_{i=1}^{n} \prod_{j=1}^{l_i} x_{ij} \mathcal{R} \sum_{i=1}^{m} \prod_{j=1}^{k_i} y_{ij} \iff \bigcup_{v \in \prod_{i=1}^{n} x_{ij}} S_v \cup \bigcup_{u \in \prod_{i=1}^{k} y_{ij}} S_u.
\]

**Proof.** Let \( S(P) \) be a \( \mathcal{L}\mathcal{R}_U \)-part of \( A_U \) and \( (\prod_{i=1}^{n} x_i, \prod_{i=1}^{m} y_i) \in \mathcal{R} \) such that \( \prod_{i=1}^{m} y_i \cap P \neq \emptyset \) be given. So \( \bigcup_{v \in \prod_{i=1}^{n} x_i} S_v \cup \bigcup_{u \in \prod_{i=1}^{m} y_i} S_u \) and we have,

\[
\prod_{i=1}^{m} y_i \cap P \neq \emptyset \Rightarrow \exists p \in P, \text{ such that } p \in \prod_{i=1}^{m} y_i
\]

\[
\Rightarrow \exists p \in P, \text{ such that } S_p \subseteq \bigcup_{u \in \prod_{i=1}^{m} y_i} S_u
\]

\[
\Rightarrow \bigcup_{v \in \prod_{i=1}^{n} x_i} S_v \cap S(P) \neq \emptyset
\]

\[
\Rightarrow \bigcup_{v \in \prod_{i=1}^{n} x_i} S_v \subseteq S(P), \text{ because } S(P) \text{ is a } \mathcal{L}\mathcal{R}_U \text{ - part.}
\]

Now suppose that \( t \in \prod_{i=1}^{n} x_i \) is given. Then \( S_t \subseteq S(P) \) and so there exists \( q \in P \) such that \( S_t \cap S_q \neq \emptyset \). Therefore \( t = q \) and hence \( t \in P \), thus \( \prod_{i=1}^{n} x_i \subseteq 1P \). For the proof of the converse implication let \( \sum_{i=1}^{n} \prod_{j=1}^{l_i} z_{ij} \cap S(P) \neq \emptyset \) and \( \sum_{i=1}^{k} \prod_{j=1}^{k_i} z_{ij} \) be given. Therefore there exists \( x_{ij} \in A \) such that for all \( 1 \leq i \leq m', 1 \leq j \leq k_i \), \( z_{ij} \in S_{x_{ij}} \). Suppose that \( u \in \bigcup_{y \in \prod_{i=1}^{k} y_{ij}} S_y \), thus \( u \in S_{y_0} \) for some \( y_0 \in \prod_{i=1}^{n} x_i \). Since \( u \in S(P) \), then there exists \( y_1 \in P \) such that \( u \in S_{y_1} \). Therefore \( S_{y_0} \cap S_{y_1} \neq \emptyset \), which implies \( y_0 = y_1 \in \prod_{i=1}^{n} x_i \). Since
6. Conclusion

In this paper we introduce and analyze a generalization of the notion of a complete part in a hyperring. We call this generalization $\mathcal{R}$-part of a hyperring.

Several properties are investigated, such as the structure of $\mathcal{R}$-closures of a subset. This research can be continued, for instance in the study of some particular classes of hyperrings.

References

