On C3-Like Finsler Metrics

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Abstract. In this paper, we study the class of C3-like Finsler metrics which contains the class of semi-C-reducible Finsler metric. We find a condition on C3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent.

Keywords: Finsler metric, C3-like metric, semi-C-reducible metric.


1. Introduction

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians [3,5,14,16]. The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg, and etc [6,7]. In [5], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hojo proves that the converse is true too [2]. A Randers metric \( F = \alpha + \beta \) is just a Riemannian metric \( \alpha \) perturbated by a one form \( \beta \), which has important applications both in mathematics and physics [15].

Let us remark some important curvatures in Finsler geometry. Let \((M, F)\) be a Finsler manifold. The second derivatives of \( \frac{1}{2} F^2 \) at \( y \in T_x M_0 \) is an inner product \( g_y \) on \( T_x M \). The third order derivatives of \( \frac{1}{2} F^2 \) at \( y \in T_x M_0 \) is a symmetric trilinear forms \( C_y \) on \( T_x M \). We call \( g_y \) and \( C_y \) the fundamental

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form and the Cartan torsion, respectively. The rate of change of $C_y$ along geodesics is the Landsberg curvature $L_y$ on $T_x M$ for any $y \in T_x M_0$. $F$ is said to be Landsbergian if $L = 0$.

In [11], Prasad-Singh introduced a new class of Finsler spaces named by C3-like spaces which contains the class of semi-C-reducible spaces, as special case (see [8], [9], [10]). A Finsler metric $F$ is called C3-like if its Cartan tensor is given by
\begin{equation}
C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + I_i b_j I_k + I_j b_k\},
\end{equation}
where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are homogeneous scalar functions on $TM$ of degree -1 and 1, respectively. We have some special cases as follows: (i) if $a_i = 0$, then we have $C_{ijk} = \{b_i I_j I_k + I_i b_j I_k + I_j b_k\}$, contracting it with $g^{ij}$ implies that $b_i = 1/(3C^2)I_i$. Then $F$ is a C2-like metric; (ii) if $b_i = 0$, then we have $C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\}$, contracting it with $g^{ij}$ implies that $a_i = 1/(n+1)I_i$. Then $F$ is a C-reducible metric; (iii) if $a_i = p/(n+1)I_i$ and $b_i = q/(3C^2)I_i$, where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on $TM$, then $F$ is a semi-C-reducible metric. It is remarkable that, in [3] Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ is semi-C-reducible. Therefore the study of the class of C3-like Finsler spaces will enhance our understanding of the geometric meaning of $(\alpha, \beta)$-metrics.

In this paper, we study C3-like metrics and find a condition on C3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent. More precisely, we prove the following.

**Theorem 1.1.** Let $(M, F)$ be a C3-like Finsler manifold. Suppose that $b_i = b_i(x, y)$ is constant along Finslerian geodesics. Then $F$ is a weakly Landsberg metric if and only if it is a Landsberg metric.

There are many connections in Finsler geometry [12][13]. In this paper, we use the Berwald connection and the $h$- and $v$-covariant derivatives of a Finsler tensor field are denoted by “$|$” and “$\nabla$” respectively.

2. Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_x M$ the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_x M$ the tangent bundle of $M$.

A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) $F$ is $C^\infty$ on $TM_0 := TM \setminus \{0\}$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$,
(iii) for each $y \in T_x M$, the following quadratic form $g_y$ on $T_x M$ is positive definite,
\[ g_y(u, v) := \frac{1}{2} \left[ F^2(y + su + tv) \right]_{s, t = 0}, \quad u, v \in T_x M. \]
Let \( x \in M \) and \( F_x := F|_{T_xM} \). To measure the non-Euclidean feature of \( F_x \), define \( C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R} \) by

\[
C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y+tw}(u, v) \right]_{t=0}, \quad u, v, w \in T_xM.
\]

The family \( C := \{C_y\}_{y \in TM_0} \) is called the Cartan torsion. It is well known that \( C=0 \) if and only if \( F \) is Riemannian. For \( y \in T_xM_0 \), define mean Cartan torsion \( I_y \) by \( I_y(u) := I_i(y)u^i \), where \( I_i := g^{jk}C_{ijk} \) and \( u = u^i \frac{\partial}{\partial x^i}|_x \). By Diecke Theorem, \( F \) is Riemannian if and only if \( I_y = 0 \).

For \( y \in T_xM_0 \), define the Matsumoto torsion \( M_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R} \) by \( M_y(u, v, w) := M_{ijk}(y)u^i w^j w^k \) where

\[
M_{ijk} := C_{ijk} - \frac{1}{n+1} \{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\},
\]

and \( h_{ij} := FF_{y^i y^j} = g_{ij} - \frac{1}{n} g_{ip} g_{jq} y^p y^q \) is the angular metric. A Finsler metric \( F \) is said to be \( C \)-reducible if \( M_y = 0 \). This quantity is introduced by Matsumoto [5]. Matsumoto proves that every Randers metric satisfies that \( M_y = 0 \). A Randers metric \( F = \alpha + \beta \) on a manifold \( M \) is just a Riemannian metric \( \alpha = \sqrt{a_{ij} y^i y^j} \) perturbed by a one form \( \beta = b_i(x) y^i \) on \( M \) such that \( \| \beta \|_\alpha < 1 \). Later on, Matsumoto-Hôjô proves that the converse is true too.

**Lemma 2.1.** ([2]) A Finsler metric \( F \) on a manifold of dimension \( n \geq 3 \) is a Randers metric if and only if \( M_y = 0 \), \( \forall y \in TM_0 \).

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

\[
C_{ijk} = \frac{p}{1+n} \{ h_{ij} I_k + h_{jkl} f + h_{ikl} f \} + \frac{q}{n+2} I_i J_j I_k,
\]

where \( p = p(x, y) \) and \( q = q(x, y) \) are scalar function on \( TM \) and \( C^2 = I^2 I_i \). Multiplying the definition of semi-C-reducibility with \( g^{ik} \) shows that \( p \) and \( q \) must satisfy \( p + q = 1 \). If \( p = 0 \), then \( F \) is called \( C_2 \)-like metric. In [3], Matsumoto and Shibata proved that every \((\alpha, \beta)\)-metric is semi-C-reducible. Let us remark that an \((\alpha, \beta)\)-metric is a Finsler metric on \( M \) defined by \( F := \alpha \phi(s) \), where \( s = \beta/\alpha \), \( \phi(s) \) is a \( C^\infty \) function on the \((-b_0, b_0)\) with certain regularity, \( \alpha \) is a Riemannian metric and \( \beta \) is a 1-form on \( M \) [4].

**Theorem 2.2.** ([3][4]) Let \( F = \phi(s) \alpha \) be a non-Riemannian \((\alpha, \beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Then \( F \) is semi-C-reducible.

The horizontal covariant derivatives of \( C \) along geodesics give rise to the Landsberg curvature \( L_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R} \) defined by

\[
L_y(u, v, w) := L_{ijk}(y)u^i w^j w^k,
\]

where \( L_{ijk} := C_{ijk|y^s} u = u^i \frac{\partial}{\partial x^s}|_x \), \( v = v^i \frac{\partial}{\partial x^i}|_x \) and \( w = w^i \frac{\partial}{\partial x^i}|_x \). The family \( L := \{L_y\}_{y \in TM_0} \) is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if \( L = 0 \).
3. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1.

**Proof of Theorem 1.1**: $F$ is $C^3$-like metric

(2) 
$$C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + b_j I_i I_k + b_i I_j I_k\},$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are scalar functions on $TM$. Multiplying (2) with $g^{ij}$ implies that

(3) 
$$a_i = \frac{1}{n+1}\{(1 - 2I^m b_m)I_i - C^2 b_i\},$$

where $C^2 = I^m I_m$. By plugging (3) in (2), we get

(4) 
$$C_{ijk} = \frac{1}{n+1}\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\} - \frac{2I^m b_m}{n+1}\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}$$

or equivalently

(5) 
$$M_{ijk} = -\frac{2I^m b_m}{n+1}\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\} - \frac{C^2}{n+1}\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\}$$

By taking a horizontal derivation of (5), we have

(6) 
$$\tilde{M}_{ijk} = -\frac{2}{n+1}\{(J^m b_m + I^m b'_m)\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}$$

$$- \frac{2I^m b_m}{n+1}\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} - \frac{C^2}{n+1}\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\}$$

$$- \frac{1}{n+1}\{J^m I_m + I^m J_m\}\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\}$$

$$+ \{b_i J_j I_k + b_i J_k I_j + b_j J_i I_k + b_j J_i I_k + b_k J_i J_j + b_k J_i J_j\}$$

where $b'_i = b_{i|x}$ and

$$\tilde{M}_{ijk} = L_{ijk} - \frac{1}{n+1}\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\}.$$

Let $F$ be a weakly Landsberg metric. Since $b_i$ is constant along geodesics, i.e.,

$b'_i = 0$, then (6) reduces to following

(7) 
$$L_{ijk} = \frac{1}{n+1}\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} = 0.$$

This means that $F$ is a Landsberg metric. \hfill $\square$

**Corollary 3.1.** Let $(M, F)$ be a weakly Landsberg $C^3$-like Finsler manifold. Suppose that $q = q(x, y)$ is constant along Finslerian geodesics. Then $F$ is a Landsberg metric.
Proof. Since $F$ is weakly Landsberg, then (6) reduces to following
\begin{equation}
L_{ijk} = -\frac{C^2}{n+1}\{b'_ih_{jk} + b'_jh_{ki} + b'_kh_{ij}\} + \{b'_iJ_JI_k + b'_jI_Ik + b'_kI_Ij\}.
\end{equation}
It is obvious that if $q = q(x,y)$ is constant along Finslerian geodesics, i.e., $q' = 0$ then $F$ is a Landsberg metric. □

Corollary 3.2. Let $(M,F)$ be a semi-C-reducible Finsler manifold. Suppose that $q = q(x,y)$ is constant along Finslerian geodesics. Then $F$ is a weakly Landsberg metric if and only if it is a Landsberg metric.

Proof. According to Theorem 1.1, a weakly Landsberg semi-C-reducible metric is a Landsberg metric if and only if the following holds
\begin{equation}
0 = b'_i = \frac{q'}{3C^2}I_I + \frac{q}{3C^2}I_i - \frac{q}{3C^2}(I^mI_m + J^mI_m)I_i
\end{equation}
Thus $b'_i = 0$ if and only if $q' = 0$. □

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