On C3-Like Finsler Metrics

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Abstract. In this paper, we study the class of C3-like Finsler metrics which contains the class of semi-C-reducible Finsler metric. We find a condition on C3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent.

Keywords: Finsler metric, C3-like metric, semi-C-reducible metric.


1. Introduction

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians [3][5][14][16]. The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg, and etc [6][7]. In [5], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hojo proves that the converse is true too [2]. A Randers metric \( F = \alpha + \beta \) is just a Riemannian metric \( \alpha \) perturbated by a one form \( \beta \), which has important applications both in mathematics and physics [15].

Let us remark some important curvatures in Finsler geometry. Let \((M, F)\) be a Finsler manifold. The second derivatives of \( \frac{1}{2} F^2_x \) at \( y \in T_xM_0 \) is an inner product \( g_y \) on \( T_yM \). The third order derivatives of \( \frac{1}{2} F^2_x \) at \( y \in T_xM_0 \) is a symmetric trilinear form \( C_y \) on \( T_yM \). We call \( g_y \) and \( C_y \) the fundamental...
form and the Cartan torsion, respectively. The rate of change of $C_y$ along geodesics is the Landsberg curvature $L_y$ on $T_xM$ for any $y \in T_xM_0$. $F$ is said to be Landsbergian if $L = 0$.

In [11], Prasad-Singh introduced a new class of Finsler spaces named by C3-like spaces which contains the class of semi-C-reducible spaces, as special case (see [8], [9], [10]). A Finsler metric $F$ is called C3-like if its Cartan tensor is given by

\[ C_{i j k} = \{ a_i h_{j k} + a_j h_{k i} + a_k h_{i j} \} + \{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \}, \]

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are homogeneous scalar functions on $TM$ of degree -1 and 1, respectively. We have some special cases as follows: (i) if $a_i = 0$, then we have $C_{i j k} = \{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \}$, contracting it with $g^{ij}$ implies that $b_i = 1/(3C^2) I_i$. Then $F$ is a C2-like metric; (ii) if $b_i = 0$, then we have $C_{i j k} = \{ a_i h_{j k} + a_j h_{k i} + a_k h_{i j} \}$, contracting it with $g^{ij}$ implies that $a_i = 1/(n+1) I_i$, and $b_i = q/(3C^2) I_i$, where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on $TM$, then $F$ is a semi-C-reducible metric. It is remarkable that in [3] Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ is semi-C-reducible. Therefore the study of the class of C3-like Finsler spaces will enhance our understanding of the geometric meaning of $(\alpha, \beta)$-metrics.

In this paper, we study C3-like metrics and find a condition on C3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent. More precisely, we prove the following.

**Theorem 1.1.** Let $(M, F)$ be a C3-like Finsler manifold. Suppose that $b_i = b_i(x, y)$ is constant along Finslerian geodesics. Then $F$ is a weakly Landsberg metric if and only if it is a Landsberg metric.

There are many connections in Finsler geometry [12][13]. In this paper, we use the Berwald connection and the $h$- and $v$- covariant derivatives of a Finsler tensor field are denoted by “$\big|_{\text{h}}$” and “$\big|_{\text{v}}$” respectively.

2. Preliminaries

Let $M$ be a $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of $M$.

A Finsler metric on $M$ is a function $F: TM \to [0, \infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0 := TM \setminus \{0\}$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$,
(iii) for each $y \in T_xM$, the following quadratic form $g_y$ on $T_xM$ is positive definite,

\[ g_y(u, v) := \frac{1}{2} \left[ F^2(y + su + tv) \right] |_{s, t = 0}, \quad u, v \in T_xM. \]
Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y+tw}(u, v) \right]_{t=0}, \quad u, v, w \in T_x M.$$ 

The family $C = \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathcal{C}=0$ if and only if $F$ is Riemannian. For $y \in T_x M_0$, define mean Cartan torsion $I_y$ by $I_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, $F$ is Riemannian if and only if $I_y = 0$.

For $y \in T_x M_0$, define the Matsumoto torsion $M_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by $M_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \},$$

and $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{2} g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric $F$ is said to be C-reducible if $M_y = 0$. This quantity is introduced by Matsumoto [5]. Matsumoto proves that every Randers metric satisfies that $M_y = 0$. A Randers metric $F = \alpha + \beta$ on a manifold $M$ is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbed by a one form $\beta = b_i(x)y^i$ on $M$ such that $||\beta||_\alpha < 1$. Later on, Matsumoto-Hōjo proves that the converse is true too.

**Lemma 2.1.** ([2]) A Finsler metric $F$ on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $M_y = 0$, $\forall y \in TM_0$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n} \{ h_{ij}I_k + h_{ik}I_j + h_{jk}I_i \} + \frac{q}{C^2} I_i I_j I_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on $TM$ and $C^2 = I^i I_i$. Multiplying the definition of semi-C-reducibility with $g^{ik}$ shows that $p$ and $q$ must satisfy $p + q = 1$. If $p = 0$, then $F$ is called C2-like metric. In [3], Matsumoto and Shibata proved that every $(\alpha, \beta)$-metric is semi-C-reducible. Let us remark that an $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha_0(s)$, where $s = \beta/\alpha$, $\alpha = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha$ is a Riemannian metric and $\beta$ is a 1-form on $M$ [4].

**Theorem 2.2.** ([3][4]) Let $F = \phi (\frac{\beta}{\alpha})$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Then $F$ is semi-C-reducible.

The horizontal covariant derivatives of $C$ along geodesics give rise to the Landsberg curvature $L_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ defined by

$$L_y(u, v, w) := L_{ijk}(y)u^iv^jw^k,$$

where $L_{ijk} := C_{ijk}y^s$, $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The family $L := \{L_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $L = 0$. 

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3. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1.

**Proof of Theorem 1.1:** $F$ is $C3$-like metric

(2) \[ C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\}, \]

where \( a_i = a_i(x, y) \) and \( b_i = b_i(x, y) \) are scalar functions on \( TM \). Multiplying (2) with \( g^{ij} \) implies that

(3) \[ a_i = \frac{1}{n + 1} \{(1 - 2I^m b_m) I_i - C^2 b_i\}, \]

where \( C^2 = I^m I_m \). By plugging (3) in (2), we get

\[
C_{ijk} = \frac{1}{n + 1} \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\} - \frac{2I^m b_m}{n + 1} \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}
\]

(4) \[
- \frac{C^2}{n + 1} \{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\} + \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\},
\]

or equivalently

\[
M_{ijk} = - \frac{2I^m b_m}{n + 1} \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\} - \frac{C^2}{n + 1} \{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\}
\]

(5) \[
+ \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\}.
\]

By taking a horizontal derivation of (5), we have

\[
\tilde{M}_{ijk} = - \frac{2}{n + 1} \{(J^m b_m + I^m b_m') \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}
\]

\[
- \frac{2I^m b_m}{n + 1} \{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} - \frac{C^2}{n + 1} \{b_i' h_{jk} + b_j' h_{ki} + b_k' h_{ij}\}
\]

(6) \[
- \frac{1}{n + 1} \{(J^m I_m + I^m J_m) \{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\}
\]

\[
+ \{b_i J_j I_k + b_i I_j J_k + b_j J_i I_k + b_j I_i J_k + b_k J_i J_j + b_k I_i J_j\}
\]

\[
+ \{b_i' I_j I_k + b_j' I_i I_k + b_k' I_i I_j\},
\]

where $b_i' = b_{ij} g^i$ and

\[
\tilde{M}_{ijk} = L_{ijk} - \frac{1}{n + 1} \{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\}.
\]

Let $F$ be a weakly Landsberg metric. Since $b_i$ is constant along geodesics, i.e., $b_i' = 0$, then (6) reduces to following

(7) \[ L_{ijk} = \frac{1}{n + 1} \{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} = 0. \]

This means that $F$ is a Landsberg metric. \[\square\]

**Corollary 3.1.** Let $(M, F)$ be a weakly Landsberg $C3$-like Finsler manifold. Suppose that $q = q(x, y)$ is constant along Finslerian geodesics. Then $F$ is a Landsberg metric.
Proof. Since $F$ is weakly Landsberg, then (6) reduces to following
\begin{equation}
L_{ijk} = -\frac{C^2}{n+1} \left\{ b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij} \right\} + \left\{ b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j \right\}.
\end{equation}
It is obvious that if $q = q(x,y)$ is constant along Finslerian geodesics, i.e., $q' = 0$ then $F$ is a Landsberg metric. \hfill \Box

**Corollary 3.2.** Let $(M,F)$ be a semi-C-reducible Finsler manifold. Suppose that $q = q(x,y)$ is constant along Finslerian geodesics. Then $F$ is a weakly Landsberg metric if and only if it is a Landsberg metric.

**Proof.** According to Theorem 1.1, a weakly Landsberg semi-C-reducible metric is a Landsberg metric if and only if the following holds
\begin{equation}
0 = b'_i = \frac{q'}{3C^2} I_i + \frac{q}{3C^2} I_i - \frac{q}{3C^4} (I^m J_m + J^m I_m) I_i
\end{equation}
Thus $b'_i = 0$ if and only if $q' = 0$. \hfill \Box

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**References**


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