On C3-Like Finsler Metrics

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Abstract. In this paper, we study the class of of C3-like Finsler metrics which contains the class of semi-C-reducible Finsler metric. We find a condition on C3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent.

Keywords: Finsler metric, C3-like metric, semi-C-reducible metric.


1. Introduction

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians [3][5][14][16]. The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg, and etc [6][7]. In [5], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hojo proves that the converse is true too [2]. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric $\alpha$ perturbated by a one form $\beta$, which has important applications both in mathematics and physics [15].

Let us remark some important curvatures in Finsler geometry. Let $(M, F)$ be a Finsler manifold. The second derivatives of $\frac{1}{2} F^2_x$ at $y \in T_x M_0$ is an inner product $g_y$ on $T_x M$. The third order derivatives of $\frac{1}{3} F^2_x$ at $y \in T_x M_0$ is a symmetric trilinear forms $C_y$ on $T_x M$. We call $g_y$ and $C_y$ the fundamental

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form and the Cartan torsion, respectively. The rate of change of $C_y$ along geodesics is the Landsberg curvature $L_y$ on $T_xM$ for any $y \in T_xM_0$. $F$ is said to be Landsbergian if $L = 0$.

In [11], Prasad-Singh introduced a new class of Finsler spaces named by $C^3$-like spaces which contains the class of semi-C-reducible spaces, as special case (see [8], [9], [10]). A Finsler metric $F$ is called $C^3$-like if its Cartan tensor is given by

$$C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\},$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are homogeneous scalar functions on $TM$ of degree -1 and 1, respectively. We have some special cases as follows: (i) if $a_i = 0$, then we have $C_{ijk} = \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\}$, contracting it with $g^{ij}$ implies that $b_i = 1/(3C^2)I_i$. Then $F$ is a $C^2$-like metric; (ii) if $b_i = 0$, then we have $C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\}$, contracting it with $g^{ij}$ implies that $a_i = 1/(n + 1)I_i$. Then $F$ is a $C$-reducible metric; (iii) if $a_i = p/(n + 1)I_i$ and $b_i = q/(3C^2)I_i$, where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on $TM$, then $F$ is a semi-C-reducible metric. It is remarkable that in [3] Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ is semi-C-reducible. Therefore the study of the class of $C^3$-like Finsler spaces will enhance our understanding of the geometric meaning of $(\alpha, \beta)$-metrics.

In this paper, we study $C^3$-like metrics and find a condition on $C^3$-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent. More precisely, we prove the following.

**Theorem 1.1.** Let $(M,F)$ be a $C^3$-like Finsler manifold. Suppose that $b_i = b_i(x, y)$ is constant along Finslerian geodesics. Then $F$ is a weakly Landsberg metric if and only if it is a Landsberg metric.

There are many connections in Finsler geometry [12][13]. In this paper, we use the Berwald connection and the $h$- and $v$-covariant derivatives of a Finsler tensor field are denoted by “$\nabla$” and “$\nabla$” respectively.

2. Preliminaries

Let $M$ be a $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of $M$.

A Finsler metric on $M$ is a function $F: TM \to [0, \infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0 := TM \setminus \{0\}$;

(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$,

(iii) for each $y \in T_xM$, the following quadratic form $g_y$ on $T_xM$ is positive definite,

$$g_y(u, v) := \frac{1}{2} \left[ F^2(y + su + tv) \right]|_{s,t=0}, \quad u, v \in T_xM.$$

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Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [g_{y+tw}(u, v)] |_{t=0}, \quad u, v, w \in T_x M.$$ 

The family $C := \{C_y\}_{y \in TM}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if $F$ is Riemannian. For $y \in T_x M_0$, define mean Cartan torsion $I_y$ by $I_y(u) := I_{ik}(y)u^k$, where $I_{ik} := g^{jk}C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, $F$ is Riemannian if and only if $I_y = 0$.

For $y \in T_x M_0$, define the Matsumoto torsion $M_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\},$$

and $h_{ij} := FF_{y^j} = g_{ij} - \frac{1}{n+1} g_{ip} g^{pq} g_{jq} y^p$ is the angular metric. A Finsler metric $F$ is said to be C-reducible if $M_y = 0$. This quantity is introduced by Matsumoto [5]. Matsumoto proves that every Randers metric satisfies that $M_y = 0$. A Randers metric $F = \alpha + \beta$ on a manifold $M$ is just a Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ perturbed by a one form $\beta = b_1(x)y^1$ on $M$ such that $||\beta||_\alpha < 1$. Later on, Matsumoto-Hōjō proves that the converse is true too.

**Lemma 2.1.** ([2]) A Finsler metric $F$ on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $M_y = 0$, $\forall y \in TM_0$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n} \{h_{ij} I_k + h_{ik} I_j + h_{jk} I_i\} + \frac{q}{C^2} I_i I_j I_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on $TM$ and $C^2 = I^i I_i$. Multiplying the definition of semi-C-reducibility with $g^{ik}$ shows that $p$ and $q$ must satisfy $p + q = 1$. If $p = 0$, then $F$ is called $C_2$-like metric. In [3], Matsumoto and Shibata proved that every $(\alpha, \beta)$-metric is semi-C-reducible. Let us remark that an $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s)$, where $s = \beta/\alpha$, $\phi(x)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha$ is a Riemannian metric and $\beta$ is a 1-form on $M$ [4].

**Theorem 2.2.** ([3][4]) Let $F = \phi(s)\alpha$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Then $F$ is semi-C-reducible.

The horizontal covariant derivatives of $C$ along geodesics give rise to the Landsberg curvature $L_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ defined by

$$L_y(u, v, w) := L_{ijk}(y)u^i v^j w^k,$$

where $L_{ijk} := C_{ijk}y^s$, $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The family $L := \{L_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $L = 0$. 

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3. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1.

**Proof of Theorem 1.1:** If $F$ is $C3$-like metric

\[(2) \quad C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + b_j I_i I_k + b_k I_j I_k\},\]

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are scalar functions on $TM$. Multiplying (2) with $g^{ij}$ implies that

\[(3) \quad a_i = \frac{1}{n + 1} \{(1 - 2I^m b_m)I_i - C^2 b_i\},\]

where $C^2 = I^m I_m$. By plugging (3) in (2), we get

\[(4) \quad C_{ijk} = \frac{1}{n + 1} \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\} - \frac{2I^m b_m}{n + 1} \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}\]

\[+ \frac{C^2}{n + 1} \{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\} + \{b_i I_j I_k + b_j I_i I_k + b_k I_j I_k\},\]

or equivalently

\[(5) \quad M_{ijk} = -\frac{2I^m b_m}{n + 1} \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\} - \frac{C^2}{n + 1} \{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\}\]

\[+ \{b_i I_j I_k + b_j I_i I_k + b_k I_j I_k\}.\]

By taking a horizontal derivation of (5), we have

\[(6) \quad \tilde{M}_{ijk} = -\frac{2}{n + 1} (J^m b_m + I^m b'_m) \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}\]

\[+ \frac{2I^m b_m}{n + 1} \{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} - \frac{C^2}{n + 1} \{b_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\}\]

\[+ \{b_i J_j I_k + b_j J_i I_k + b_j J_i J_k + b_j I_j I_k + b_k J_i I_j + b_k I_j J_k + b_k I_i J_j\}\]

\[+ \{b'_i I_j I_k + b'_j I_i I_k + b'_k I_j J_k\},\]

where $b'_i = b_{i|x}^y$ and

\[
\tilde{M}_{ijk} = \tilde{L}_{ijk} = \frac{1}{n + 1} \{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\}.\]

Let $F$ be a weakly Landsberg metric. Since $b_i$ is constant along geodesics, i.e., $b'_i = 0$, then (6) reduces to following

\[(7) \quad L_{ijk} = \frac{1}{n + 1} \{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} = 0.\]

This means that $F$ is a Landsberg metric. \qed

**Corollary 3.1.** Let $(M, F)$ be a weakly Landsberg $C3$-like Finsler manifold. Suppose that $q = q(x, y)$ is constant along Finslerian geodesics. Then $F$ is a Landsberg metric.
Proof. Since $F$ is weakly Landsberg, then (6) reduces to following

$$L_{ijk} = -\frac{C^2}{n+1}\{b'_ih_{jk} + b'_kh_{ji} + b'_{ki}h_{ij}\} + \{b'_IJ_iI_k + b'_JI_iI_k + b'_KI_iI_j\}. \quad (8)$$

It is obvious that if $q = q(x,y)$ is constant along Finslerian geodesics, i.e., $q' = 0$ then $F$ is a Landsberg metric. \hfill \Box

**Corollary 3.2.** Let $(M,F)$ be a semi-C-reducible Finsler manifold. Suppose that $q = q(x,y)$ is constant along Finslerian geodesics. Then $F$ is a weakly Landsberg metric if and only if it is a Landsberg metric.

Proof. According to Theorem 1.1, a weakly Landsberg semi-C-reducible metric is a Landsberg metric if and only if the following holds

$$0 = b'_i = \frac{q'}{3C^2}I_i + \frac{q}{3C^2}J_mJ_i - \frac{q}{3C^4}(I^mJ_m + J^mI_m)I_i \quad (9)$$

Thus $b'_i = 0$ if and only if $q' = 0$. \hfill \Box

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