The Polynomials of a Graph

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Abstract. In this paper, we are presented a formula for the polynomial of a graph. Our main result is the following formula:

$$\sum_{u \in V(G)} d_u^k \sum_{j=1}^{k} a_{kj} S_{G}^{(j)}(1).$$

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1. Introduction

The graphs in this paper are connected and simple. Denote the vertex and edge sets of graph $G$ by $V(G)$ and $E(G)$, respectively. For a simple graph $G(p, q)$, we
define the degree sequence of $G$ as

$$S : d_1, d_2, \cdots, d_p$$

where $d_i = \text{deg} v_i$, $1 \leq i \leq p$, and $v_i$'s are vertices of $G$. Suppose $a_0$ is number of vertices of degree 0, $a_1$ the number of vertices of degree 1, ..., and $a_{\Delta(G)}$ is number of number vertices of degree $\Delta(G)$, where $\Delta(G) = \max\{d_i\}$. The polynomial of $G$ is defined as:

**Definition 1.1.** If $S : d_1, d_2, \cdots, d_p$ is a degree sequence of graph $G$. Then the polynomial of graph $G$ is

$$S_G(x) = \sum_{j=0}^{\Delta(G)} a_j x^j$$

Also a polynomial $p(x)$ is said to be graphical if there exists a graph $G$ such that $p(x) = S_G(x)$.

**Example 1.2.** Suppose $G$ is defined by the following diagram:

```
  o---o
  |   |
  o---o
```

Then the degree sequence of $G$ is $S : 0, 1, 1, 2, 3, 3$ and $\Delta(G) = 3$. Thus the polynomial of $G$ is

$$S_G(x) = \sum_{j=0}^{3} a_j x^j$$

where $a_0 = 1$, $a_1 = 2$, $a_2 = 1$ and $a_3 = 2$. Hence we have

$$S_G(x) = 1x^0 + 2x + 1x^2 + 2x^3 = 1 + 2x + x^2 + 2x^3.$$  

**Remark 1.3.** It is easy to see that

$$S_G(x) = \sum_{j=0}^{\Delta(G)} a_j x^j = \sum_{u \in V(G)} x^{d_u}$$

where $d_u$ is the degree of $u$.

**Corollary 1.4.** If $G(p, q)$ is a graph with $p$ vertices and $q$ edges, then we have:
The Polynomials of a graph

\[ S_G(1) = p \]

\[ \sum_{j=0}^{\Delta(G)} ja_j = 2q \]

\[ S'_G(1) = 2q = \sum_{u \in V(G)} d_u \]

Suppose \( P_n, C_n, K_n \) denoted the path, cycle and complete graphs with exactly \( n \) vertices, respectively. Also a general \( k \)-regular graph is denoted by \( G_k \). Then,

\[ S_{P_n}(x) = 2x + (n - 2)x^2 \]

\[ S_{C_n}(x) = nx^2 \]

\[ S_{K_n}(x) = nx^{n-1} \]

\[ S_{G_k}(x) = px^k \]

Definition 1.5. Let \( G_1 \) and \( G_2 \) be two graphs. If \( V(G_1) \cap V(G_2) = \phi \). Then

1. \( G_1 \cup G_2 \) is a graph that \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \)

2. \( G_1 \times G_2 \) is a graph that \( V(G_1 \times G_2) = V(G_1) \times V(G_2) \) and \( \{(u, v), (u', v')\} \in E(G_1 \times G_2) \) if and only if \( u = u' \) and \( \{v, v'\} \in E(G_2) \) or \( v = v' \) and \( \{u, u'\} \in E(G_1) \)

3. \( G_1 + G_2 \) is a graph that \( V(G_1 + G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(u, v) \mid u \in V(G_1), v \in V(G_2)\} \)

Example 1.6. Suppose \( G_1 \) and \( G_2 \) are two graphs such that their diagrams are as follows:

Then the diagram graph \( G_1 \times G_2 \) and \( G_1 + G_2 \) as follows:
Theorem 1.7. If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two graphs, then the polynomial of graphs $G_1 \cup G_2$, $G_1 \times G_2$ and $G_1 + G_2$ are given by

1. $S_{G_1 \cup G_2}(x) = S_{G_1}(x) + S_{G_2}(x)$
2. $S_{G_1 \times G_2}(x) = S_{G_1}(x) \cdot S_{G_2}(x)$
3. $S_{G_1 + G_2}(x) = x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x)$

Proof.

1. 
\[ S_{G_1 \cup G_2}(x) = \sum_{u \in V(G_1 \cup G_2)} x^{d_u} = \sum_{u \in V(G_1)} x^{d_u} + \sum_{u \in V(G_2)} x^{d_u} \]

\[ = S_{G_1}(x) + S_{G_2}(x) \]

2. 
\[ S_{G_1 \times G_2}(x) = \sum_{u \in V(G_1 \times G_2)} x^{d_u} = \sum_{u=(u_1, u_2) \in V(G_1 \times G_2)} x^{d_u} \]

\[ = \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} x^{d_{u_1} + d_{u_2}} = \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} x^{d_{u_1}} x^{d_{u_2}} \]

\[ = S_{G_1}(x) \cdot S_{G_2}(x) \]

3. 
\[ S_{G_1 + G_2}(x) = \sum_{u \in V(G_1 + G_2)} x^{d_u} = \sum_{u \in V(G_1)} x^{d_u + p_2} + \sum_{u \in V(G_2)} x^{d_u + p_1} \]

\[ = x^{p_2} \sum_{u \in V(G_1)} x^{d_u} + x^{p_1} \sum_{u \in V(G_2)} x^{d_u} \]

\[ = x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x) \]

Corollary 1.8. If $S_{G_1}(x)$ and $S_{G_2}(x)$ are graphical then

1. $S_{G_1}(x) \cdot S_{G_2}(x)$ is graphical and conversely.
2. $x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x)$ is graphical and conversely.

Example 1.9. The polynomial $S_G(x) = 4x^2 + 4x^3 + x^4$ is graphical, because

\[ S_G(x) = 4x^2 + 4x^3 + x^4 = (2x + x^2)^2 \]
On the other hand, we have the following graph for the polynomial $S_{G_1}(x) = 2x + x^2$.

Hence the polynomial $S_G(x)$ is graphical, because $S_G(x) = S_{G_1}(x) \times S_{G_1}(x)$. Also its graph is as follows:

Example 1.10. The polynomial $S_G(x) = 3x^4 + 2x^3$ is graphical, because

$$S_G(x) = 3x^4 + 2x^3 = 2x^4 + x^4 + 2x^3 = x^3(2x) + x^2(x^2 + 2x)$$

On the other hand, we have the following graphs for the polynomials $S_{G_1}(x) = 2x$ and $S_{G_2}(x) = x^2 + 2x$, respectively:

Hence the polynomial $S_G(x)$ is graphical, because $S_G(x) = x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x)$. Also its graph is as following:
Definition 1.11. Let $G$ be a graph. The polynomial $H_G(x)$ is defined as follows:

$$H_G(x) = \sum_{\{u,v\} \in E(G)} x^{d_u + d_v}$$

Example 1.12. The polynomial $H_G(x) = x^3 + x^3 = 2x^3$ is the graph polynomial of the following graph:

```
  a
 / \  
 b   c
 \
  d
```

Corollary 1.13. Let $G(p, q)$ is a graph with $p$ vertices and $q$ edges. Then we have:

\[
\begin{align*}
H_G(1) &= q \\
H'_G(1) &= \sum_{\{u,v\} \in E(G)} d_u + d_v = \sum_{u \in V(G)} d_u^2 \\
H_{P_n}(x) &= 2x^3 + (n - 3)x^4 \\
H_{C_n}(x) &= nx^4 \\
H_{K_n}(x) &= \frac{n(n-1)}{2}x^{2n-2} \\
H_{G_k}(x) &= qx^2
\end{align*}
\]

Theorem 1.14. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then

1. $H_{G_1 \cup G_2}(x) = H_{G_1}(x) + H_{G_2}(x)$
2. $H_{G_1 \times G_2}(x) = H_{G_1}(x) \cdot S_{G_2}(x^2) + H_{G_2}(x) \cdot S_{G_1}(x^2)$
3. $H_{G_1 + G_2}(x) = x^{2p_2}H_{G_1}(x) + x^{2p_1}H_{G_2}(x) + x^{p_1 + p_2}S_{G_1}(x) \cdot S_{G_2}(x)$

Proof. (1) is trivial. To prove (2), we have:

\[
H_{G_1 \times G_2}(x) = \sum_{\{u,v\} \in E(G_1 \times G_2)} x^{d_u + d_v}
\]

\[
= \sum_{\{u_1,v_1\}, \{u_2,v_2\} \in E(G_2)} x^{2d_{u_1} + d_{u_2} + d_{v_1}}
\]

\[
+ \sum_{\{u_1,v_1\} \in E(G_1), u_2 = v_2} x^{d_{u_1} + d_{v_1} + 2d_{u_2}}
\]

\[
= \sum_{\{u_2,v_2\} \in E(G_2)} x^{d_{u_2} + d_{v_2}} \sum_{u_1 \in V(G_1)} (x^2)^{d_{u_1}}
\]

\[
+ \sum_{\{u_1,v_1\} \in E(G_1)} x^{d_{u_1} + d_{v_1}} \sum_{u_2 \in V(G_2)} (x^2)^{d_{u_2}}
\]

\[
= H_{G_1}(x)S_{G_2}(x^2) + H_{G_2}(x)S_{G_1}(x^2)
\]
\[ H_{G_1+G_2}(x) = \sum_{\{u,v\} \in E(G_1+G_2)} x^{d_u+d_v} \]
\[ = \sum_{\{u,v\} \in E(G_1)} x^{d_u+d_v+2p_2} + \sum_{\{u,v\} \in E(G_2)} x^{d_u+d_v+2p_1} + \sum_{u \in V(G_1), v \in V(G_2)} x^{d_u+d_v+p_1+p_2} \]
\[ = x^{2p_2} \sum_{\{u,v\} \in E(G_1)} x^{d_u+d_v} + x^{2p_1} \sum_{\{u,v\} \in E(G_2)} x^{d_u+d_v} + x^{p_1+p_2} \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} x^{d_u} \sum_{x^{d_v}} \]
\[ = x^{2p_2} H_{G_1}(x) + x^{2p_1} H_{G_2}(x) + x^{p_1+p_2} S_{G_1}(x)S_{G_2}(x) \]

Example 1.15. Consider the following diagrams for graphs $G_1$ and $G_2$:

![Diagram](attachment:image.png)

then, we have:

\[ H_{G_1}(x) = 2x^3 \quad S_{G_1}(x) = 2x + x^2 \]
\[ H_{G_2}(x) = 3x^4 \quad S_{G_2}(x) = 3x^2 \]

Thus:

\[ H_{G_1+G_2}(x) = x^6(2x^3) + x^6(3x^4) + x^6(2x + x^2)(3x^2) \]
\[ = 2x^9 + 3x^{10} + 6x^9 + 3x^{10} = 8x^9 + 6x^{10} \]

Hence the diagram $G_1 + G_2$ is:
Corollary 1.16.

\[
\sum_{u \in V(G_1 \times G_2)} d_u^2 = p_2 \sum_{u \in V(G_1)} d_u^2 + p_1 \sum_{u \in V(G_2)} d_u^2 + 8q_1 q_2
\]

Proof. We know that

\[
H_{G_1 \times G_2}(x) = H_{G_1}(x)S_{G_2}(x^2) + H_{G_2}(x)S_{G_1}(x^2)
\]

Hence,

\[
\begin{align*}
H'_{G_1 \times G_2}(x) &= H'_{G_1}(x)S_{G_2}(x^2) + 2xH_{G_1}(x)S'_{G_2}(x^2) \\
&\quad + H'_{G_2}(x)S_{G_1}(x^2) + 2xH_{G_2}(x)S'_{G_1}(x^2)
\end{align*}
\]

Therefore

\[
\begin{align*}
H'_{G_1 \times G_2}(1) &= H'_{G_1}(1)S_{G_2}(1) + 2H_{G_1}(1)S'_{G_2}(1) \\
&\quad + H'_{G_2}(1)S_{G_1}(1) + 2H_{G_2}(1)S'_{G_1}(1)
\end{align*}
\]

On the other hand, we know that \(H_G(1) = q\), \(H'_G(1) = \sum_{u \in V(G)} d_u^2\), \(S_G(1) = p\) and \(S'_G(1) = 2q\). Thus

\[
\sum_{u \in V(G) \times G_1} d_u^2 = p_2 \sum_{u \in V(G_1)} d_u^2 + 4q_1 q_2 + p_1 \sum_{u \in V(G_2)} d_u^2 + 8q_1 q_2
\]

\[
= p_2 \sum_{u \in V(G_1)} d_u^2 + p_1 \sum_{u \in V(G_2)} d_u^2 + 8q_1 q_2
\]

\[
\square
\]

Definition 1.17. Let \(G\) be a graph. The polynomial \(F_G(x)\) is defined as follows:

\[
F_G(x) = \sum_{u \in V(G)} d_u x^{d_u}
\]

Example 1.18. The polynomial of the graph \(G\) defined by the following graph is \(H_G(x) = 2x + 2x^2\).
Theorem 1.20. Let 

\[ \text{Corollary 1.19. We have:} \]

\[ F_G(1) = S_G'(1) \quad F'_G(1) = H'_G(1) \]

\[ F_{P_n}(x) = 2x + 2(n - 2)x^2 \quad F_{C_n}(x) = 2nx^2 \]

\[ F_{K_n}(x) = n(n - 1)x^{n-1} \quad F_{G_k}(x) = kp x^k \]

Theorem 1.20. Let \( G_1(p_1, q_1) \) and \( G_2(p_2, q_2) \) be two graphs. Then

1. \( F_{G_1 \cup G_2}(x) = F_{G_1}(x) + F_{G_2}(x) \)
2. \( F_{G_1 \times G_2}(x) = F_{G_1}(x) \cdot S_{G_2}(x) + F_{G_2}(x) \cdot S_{G_1}(x) \)
3. \( F_{G_1 + G_2}(x) = x^{p_2} F_{G_1}(x) + p_2 x^{p_2} S_{G_1}(x) + x^{p_1} F_{G_2}(x) + p_1 x^{p_1} S_{G_2}(x) \)

Proof. (1) is trivial. Prove (2), we have:

\[
F_{G_1 \times G_2}(x) = \sum_{u \in V(G_1 \times G_2)} d_u x^{d_u} \\
= \sum_{(u_1, u_2) \in V(G_1) \times V(G_2)} (d_{u_1} + d_{u_2}) x^{d_{u_1} + d_{u_2}} \\
= \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} (d_{u_1} + d_{u_2}) x^{d_{u_1} + d_{u_2}} \\
= \sum_{u_2 \in V(G_2)} x^{d_{u_2}} \sum_{u_1 \in V(G_1)} d_{u_1} x^{d_{u_1}} + \sum_{u_1 \in V(G_1)} x^{d_{u_1}} \sum_{u_2 \in V(G_2)} d_{u_2} x^{d_{u_2}} \\
= F_{G_1}(x) . S_{G_2}(x) + F_{G_2}(x) . S_{G_1}(x)
\]

\[
F_{G_1 + G_2}(x) = \sum_{u \in V(G_1 + G_2)} d_u x^{d_u} \\
= \sum_{u \in V(G_1)} (d_u + p_2) x^{d_u + p_2} + \sum_{u \in V(G_2)} (d_u + p_1) x^{d_u + p_1} \\
= x^{p_2} \sum_{u \in V(G_1)} d_u x^{d_u} + p_2 x^{p_2} \sum_{u \in V(G_1)} x^{d_u} \\
+ x^{p_1} \sum_{u \in V(G_2)} d_u x^{d_u} + p_1 x^{p_1} \sum_{u \in V(G_2)} x^{d_u} \\
= x^{p_2} F_{G_1}(x) + p_2 x^{p_2} S_{G_1}(x) + x^{p_1} F_{G_2}(x) + p_1 x^{p_1} S_{G_2}(x)
\]

\[ \square \]
Definition 1.21. Let $G$ be a graph. The polynomial $W_G(x)$ is defined as following:

$$W_G(x) = \sum_{\{u,v\} \in E(G)} (d_u + d_v)x^{d_u+d_v}$$

Example 1.22. Consider the following diagram for the graph $G$. Then $W_G(x) = 3x^3 + 3x^3 = 6x^3$.

![Diagram of a graph with labeled vertices a, b, c, and d.]

Corollary 1.23. We have:

- $W_G(1) = \sum_{\{u,v\} \in E(G)} d_u + d_v = \sum_{u \in V(G)} d_u^2$
- $W_{P_n}(x) = 6x^3 + 4(n-3)x^4$
- $W_{K_n}(x) = n(n-1)^2x^{2n-2}$

Theorem 1.24. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then

1. $W_{G_1 \cup G_2}(x) = W_{G_1}(x) + W_{G_2}(x)$
2. $W_{G_1 \times G_2}(x) = 2F_{G_1}(x^2).H_{G_2}(x) + S_{G_1}(x^2).W_{G_2}(x) + 2F_{G_2}(x^2).H_{G_1}(x) + S_{G_2}(x^2).W_{G_1}(x)$
3. $W_{G_1 + G_2}(x) = x^{2p_2}W_{G_1}(x) + 2p_2x^{2p_2}H_{G_1}(x) + x^{2p_1}W_{G_2}(x) + 2p_1x^{2p_1}H_{G_2}(x) + x^{p_1+p_2}F_{G_1 \times G_2}(x) + (p_1 + p_2)x^{p_1+p_2}S_{G_1 \times G_2}(x)$
Proof. (1) is trivial. To prove (2), we consider the following equation:

\[ W_{G_1 \times G_2}(x) = \sum_{\{u,v\} \in E(G_1 \times G_2)} (d_u + d_v)x^{d_u+d_v} \]

\[ = \sum_{\{u_1=v_1, u_2=v_2\} \in E(G_2)} (2d_{u_1} + d_{u_2} + d_{v_2})x^{2d_{u_1}+d_{u_2}+d_{v_2}} \]

\[ + \sum_{u_2=v_2 \in E(G_2)} (2d_{u_2} + d_{v_1} + d_{v_1})x^{2d_{u_2}+d_{v_1}+d_{v_1}} \]

\[ = 2\sum_{u_1 \in V(G_1)} (x^2)^d_{u_1} \sum_{\{u_2,v_2\} \in E(G_2)} x^{d_{u_2}+d_{v_2}} \]

\[ + \sum_{u_1 \in V(G_1)} (x^2)^d_{u_1} \sum_{\{u_2,v_2\} \in E(G_2)} (d_{u_2} + d_{v_1})x^{d_{u_2}+d_{v_1}} \]

\[ + 2\sum_{u_2 \in V(G_2)} (x^2)^d_{u_2} \sum_{\{u_1,v_1\} \in E(G_1)} x^{d_{u_2}+d_{v_1}} \]

\[ + \sum_{u_2 \in V(G_2)} (x^2)^d_{u_2} \sum_{\{u_1,v_1\} \in E(G_1)} (d_{u_1} + d_{v_1})x^{d_{u_1}+d_{v_1}} \]

\[ = 2F_{G_1}(x^2).H_{G_2}(x) + S_{G_1}(x^2).W_{G_1}(x) \]

\[ + 2F_{G_2}(x^2).H_{G_1}(x) + S_{G_2}(x^2).W_{G_1}(x) \]

\[ W_{G_1+G_2}(x) = \sum_{\{u,v\} \in E(G_1+G_2)} (d_u + d_v)x^{d_u+d_v} \]

\[ = \sum_{\{u,v\} \in E(G_1)} (d_u + d_v + 2p_2)x^{d_u+d_v+2p_2} \]

\[ + \sum_{\{u,v\} \in E(G_2)} (d_u + d_v + 2p_1)x^{d_u+d_v+2p_1} \]

\[ + \sum_{u \in V(G_1), v \in V(G_2)} (d_u + d_v + p_1 + p_2)x^{d_u+d_v+p_1+p_2} \]

\[ = x^{2p_2} \sum_{\{u,v\} \in E(G_1)} (d_u + d_v)x^{d_u+d_v} + 2p_2x^{2p_2} \sum_{\{u,v\} \in E(G_1)} x^{d_u+d_v} \]

\[ + x^{2p_1} \sum_{\{u,v\} \in E(G_2)} (d_u + d_v)x^{d_u+d_v} + 2p_1x^{2p_1} \sum_{\{u,v\} \in E(G_2)} x^{d_u+d_v} \]

\[ + x^{p_1+p_2} \sum_{u \in V(G_1), v \in V(G_2)} (d_u + d_v)x^{d_u+d_v} \]

\[ + (p_1 + p_2)x^{p_1+p_2} \sum_{u \in V(G_1), v \in V(G_2)} x^{d_u+d_v} \]

\[ = x^{2p_2}W_{G_1}(x) + 2p_2x^{2p_2}H_{G_1}(x) + x^{2p_1}W_{G_2}(x) + 2p_1x^{2p_1}H_{G_2}(x) \]

\[ + x^{p_1+p_2}F_{G_1 \times G_2}(x) + (p_1 + p_2)x^{p_1+p_2}S_{G_1 \times G_2}(x) \]
In the end of this paper, we define a new triangle $\mathcal{A}$ as follows:

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 3 & 1 \\
\mathcal{A} = & 1 & 7 & 6 & 1 \\
& 1 & 15 & 25 & 10 & 1 \\
& 1 & 31 & 90 & 65 & 15 & 1 \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

that entry $a_{ij}$ of triangle $\mathcal{A}$ is:

\[
a_{ij} = \begin{cases} 
1 & j = 1 \text{ or } j = i \\
a_{(i-1)(j-1)} + ja_{(i-1)j} & 1 < j < i 
\end{cases}
\]

**Theorem 1.25.** If $G$ is a graph with the polynomial $S_G(x)$, then

\[
\sum_{u \in V(G)} d_u^k = \sum_{j=1}^k a_{kj} S_G^{(j)}(1)
\]

where $k \in \mathbb{N}$ and $a_{kj} \in \mathcal{A}$.

**Example 1.26.** Let $G$ is a graph, such that its diagram is as following:

```
\begin{tikzpicture}
\node (1) at (0,0) {	extbullet};
\node (2) at (-0.5,-1) {	extbullet};
\node (3) at (0.5,-1) {	extbullet};
\draw (1) -- (2);
\draw (1) -- (3);
\end{tikzpicture}
```

Hence the degree sequence and the polynomial $S_G(x)$ are "1, 1, 2" and $2x + x^2$, respectively. Thus for $k = 3$ we have:

\[
\sum_{u \in V(G)} d_u^3 = 1^3 + 1^3 + 2^3 = 10
\]

On the other hand, we have $S'_G(1) = 4$, $S''_G(1) = 2$, $S'^{(3)}_G(1) = 0$, $a_{31} = 1$, $a_{32} = 3$ and $a_{33} = 1$. Therefore

\[
\sum_{j=1}^{3} a_{3j} S_G^j(1) = 1 \times 4 + 3 \times 2 + 1 \times 0 = 10
\]
Proof of Theorem 1.25. According to remark (1.3) \( S_G(x) = \sum_{u \in V(G)} x^{d_u} \).

Hence,

\[(1.1) \quad S'_G(x) = \sum_{u \in V(G)} d_u x^{d_u-1} \]

therefore

\[ S'_G(1) = \sum_{u \in V(G)} d_u \]

On the other hand, according to table of \( A \) for \( k = 1 \), we have:

\[ \sum_{j=1}^{1} a_{1j} S_G^{(j)}(1) = a_{11} S'_G(1) = S'_G(1) \]

From above relations, we obtain that the theorem (1.25) for \( k = 1 \) is true. Now from the relation (1.1), we have \( xS'_G(x) = \sum_{u \in V(G)} d_u x^{d_u} \) then

\[(1.2) \quad S'_G(x) + x S''_G(x) = \sum_{u \in V(G)} d_u^2 x^{d_u-1} \]

therefore

\[ S'_G(1) + S''_G(1) = \sum_{u \in V(G)} d_u^2 \]

On the other hand, according to table of \( A \) for \( k = 2 \), we have:

\[ \sum_{j=1}^{2} a_{2j} S_G^{(j)}(1) = a_{21} S'_G(1) + a_{22} S''_G(1) = S'_G(1) + S''_G(1) \]

From two relations before, we obtain that the theorem (1.25) for \( k = 2 \) is true. Similarly from the relation (1.2), we can prove the theorem (1.25) for \( k = 3 \). Therefore, if we continue the above process, then the proof is completed.

References


