For a unital foundation topological \( \ast \)-semigroup \( S \) whose representations separate points of \( S \), we show that the spectrum of the Fourier-Stieltjes algebra \( B(S) \) is a compact semitopological semigroup.

We also calculate \( B(S) \) for several examples of \( S \).

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1. **Introduction**

In [3] Lau studied the subalgebra \( F(S) \) of \( \text{WAP}(S) \) of a topological semigroup \( S \) with involution. If \( G \) is an abelian topological group, then \( F(G) \cong M(\hat{G}) \) where \( \hat{G} \) is the dual group of \( G \). If \( S \) is a topological \( \ast \)-semigroup with an identity, then \( F(S) \) is the linear span of positive definite functions on \( S \). The authors introduced and studied Fourier and Fourier-Stieltjes algebras \( A(S) \) and \( B(S) \) of a foundation topological \( \ast \)-semigroup \( S \) in [1]. When \( S \) is unital, \( B(S) = F(S) \).

Let \( S \) be a locally compact topological semigroup and \( M(S) \) be the Banach algebra of all bounded regular Borel measures on \( S \). We consider the mappings
Let $L_{\mu}$ and $R_{\mu}$ of $S$ into $M(S)$ defined by
\[ L_{\mu}(x) = \mu \ast \delta_x, \quad R_{\mu}(x) = \delta_x \ast \mu \quad (x \in S, \mu \in M(S)), \]
where $\delta_x$ is the point mass at $x$. Then the semigroup algebra $L(S)$ consists of those $\mu \in M(S)$ for which $L_{|\mu|}$ and $R_{|\mu|}$ are continuous with respect to the weak topology of $M(S)$, and $L(S)$ is a Banach subalgebra of $M(S)$. The semigroup $S$ is called foundation if $\bigcup \{ \text{supp} \mu : \mu \in L(S) \}$ is dense in $S$ [6].

A representation of $S$ is a pair $\{ \pi, H_\pi \}$ of a Hilbert space $H_\pi$ and a semigroup homomorphism $\pi : S \to B(H_\pi)$ such that $\pi$ is (weakly) continuous, i.e., the mappings $x \mapsto \langle \pi(x)\xi, \eta \rangle$ are continuous on $S$, for all $\xi, \eta \in H_\pi$, and that $\pi$ is bounded if $\|\pi\| = \sup_{x \in S} \|\pi(x)\| < \infty$. Also $\pi$ is called a $*$-representation if moreover $\pi(x^\ast) = \pi(x)^\ast (x \in S)$, where the right hand side is the adjoint operator. A $*$-representation $\{ \sigma, H \}$ of $L(S)$ is called non-vanishing if for every $0 \neq \xi \in H$, there exists $\mu \in L(S)$ with $\sigma(\mu)\xi \neq 0$. Let $\Sigma(L(S))$ be the family of all $*$-representations of $L(S)$ on a Hilbert space which are non-vanishing, and $\Sigma(S)$ be the family of all continuous $*$-representations $\pi$ of $S$ with $\|\pi\| \leq 1$, then one has a bijective correspondence between $\Sigma(S)$ and $\Sigma(L(S))$ via
\[ \langle \tilde{\pi}(\mu)\xi, \eta \rangle = \int_S \langle \pi(x)\xi, \eta \rangle d\mu(x) \quad (\mu \in L(S), \xi, \eta \in H_\pi = H_\pi). \]

Given $\rho \subseteq \Sigma = \Sigma(S)$ and $\mu \in L(S)$, define $\|\mu\|_\rho = \sup\{ |\tilde{\pi}(\mu)| : \pi \in \rho \}$ and $I_\rho = \{ \mu \in L(S) : \|\mu\|_\rho = 0 \}$. Then $I_\rho$ is clearly a closed two-sided ideal of $L(S)$ and $\|\mu + I_\rho\| = \|\mu\|_\rho$ defines a $C^*$-norm on $L(S)/I_\rho$. The completion of this quotient space in this norm is a $C^*$-algebra which is denoted by $C^*_\rho(S)$. When $\rho = \Sigma$, then the $C^*$-algebra $C^*(S) = C^*_\Sigma(S)$ is called the (full) semigroup $C^*$-algebra of $S$. If $S$ is foundation and $\Sigma$ separates the points of $S$, then $L(S)$ is $*$-semisimple and so $I_\Sigma = \{0\}$. In this case $L(S)$ is a norm dense subalgebra of $C^*(S)$ (see [1] for more details).

A complex valued function $u : S \to \mathbb{C}$ is called positive definite if for all positive integers $n$ and all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, and $x_1, x_2, \ldots, x_n \in S$, we have
\[ \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j u(x_i, x_j^\ast) \geq 0. \]

Let $P(S)$ denotes the set of all continuous positive definite functions on $S$. We denote the linear span of $P(S)$ by $B(S)$ and call it the Fourier-Stieltjes algebra of $S$. Let $S$ be a topological $*$-semigroup and $C_c(S)$ be the algebra of all continuous functions on $S$ with compact support. Then the closed subalgebra $(B(S) \cap C_c(S)) \subseteq B(S)$ is denoted by $A(S)$ and is called the Fourier algebra of $S$.

2. **Fourier-Stieltjes algebra**

It is well known that for an abelian topological group $G$, the Fourier and Fourier-Stieltjes algebras $A(G)$ and $B(G)$ are isometrically isomorphic to the group and measure algebras $L^1(G)$ and $M(G)$ of the dual group $\hat{G}$. For a class of commutative foundation topological $*$-semigroup with identity we show that
$B(S)$ is isometrically isomorphic to $M(\hat{S})$. Here $\hat{S}$ is the set of continuous semi-characters on $\hat{S}$ which is a locally compact topological semigroup [3].

**Theorem 2.1.** Let $S$ be a commutative foundation topological $*$-semigroup with identity. For $\lambda \in L(\hat{S})$, define $\lambda : S \to \mathbb{C}$ by

$$\hat{\lambda}(x) = \int_{\hat{S}} \chi(x)d\lambda(\chi) \quad (x \in S).$$

Then the map $\lambda \mapsto \hat{\lambda}$ is a continuous monomorphism from $L(\hat{S})$ into $B(S)$.

**Proof.** $\hat{S}$ is a locally compact topological semigroup [3]. Also for each $\lambda \in L(\hat{S})$ there is a probability measure $\gamma$ on $\hat{S}$ and $\phi \in L^1(\hat{S}, \gamma)$ such that $d\lambda = \phi d\gamma$. We can decompose $\phi$ as

$$\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4),$$

where $\phi_i \geq 0$, for $i = 1, \ldots, 4$. Put $d\lambda_i = \phi_i d\gamma$. Then for each $n \geq 1$, $c_1, \ldots, c_n \in \widehat{\mathbb{C}}$, and $x_1, \ldots, x_n \in S$,

$$\sum_{i,j=1}^n c_i c_j \hat{\chi}_k(x_i x_j^*) = \int_{\hat{S}} \sum_{i,j=1}^n c_i c_j \chi(x_i x_j^*)d\lambda_k(\chi) \geq 0,$$

for $k = 1, \ldots, 4$. Next we show that $\hat{\lambda}_k$ is also continuous. Given $\varepsilon > 0$, there is a measurable subset $K \subseteq \hat{S}$ such that

$$\int_{\hat{S}\setminus K} \phi_k(\chi)d\gamma(\chi) < \varepsilon.$$

By Ascoli’s Theorem, $K$ is equicontinuous. Now given $x_0 \in S$, there is a neighborhood $U$ of $x_0$ in $S$ such that

$$|\chi(x) - \chi(x_0)| < \varepsilon \quad (\chi \in K, x \in U).$$

For each $x \in U$,

$$|\hat{\lambda}_k(x) - \hat{\lambda}_k(x_0)| \leq \int_{\hat{S}} |\chi(x) - \chi(x_0)|d\lambda_k(\chi)$$

$$\leq \int_K |\chi(x) - \chi(x_0)|d\lambda_k(\chi) + \int_{\hat{S}\setminus K} |\chi(x) - \chi(x_0)|d\lambda_k(\chi)$$

$$\leq \varepsilon \lambda_k(K) + 2\varepsilon \leq (2 + \lambda_k(\hat{S}))\varepsilon.$$

This shows that $\hat{\lambda}_k \in P(S)$, for $k = 1, \ldots, 4$, and so $\hat{\lambda} \in B(S)$. Next we have

$$\|\hat{\lambda}\|_{B(S)} = \sup_{S} |\int_{\hat{S}} \hat{\lambda}(x)d\mu(x)| = \sup_{S} |\int_{\hat{S}} \chi(x)d\lambda(\chi)d\mu(x)|$$

$$\leq \int_{\hat{S}} \int_{S} |\chi(x)d\mu(x)|d\|\lambda\| \leq \|\lambda\|,$$

where the supremum is taken over all $\mu \in L(S)$ with $\|\mu\|_{S} \leq 1$ (see [1]). Also the last inequality follows from the fact that each semi-character $\chi \in \hat{S}$ could be regarded as a representation of $S$. 
When $\lambda$ is positive, we also have
\[
\|\lambda\| = \int_S \chi(e) d\lambda(\chi) = \hat{\lambda}(e) \leq \|\hat{\lambda}\|_{B(S)},
\]
since $\chi(e) = 1$, for each $\chi \in \hat{S}$, where $e$ is the identity of $S$. In general, $\lambda = (\lambda_1 - \lambda_2) + i(\lambda_3 - (\lambda_4)$, with $\lambda_k$’s positive, and we have
\[
\|\lambda\| \leq \sum_{i=1}^{4} \|\lambda_i\| \leq \sum_{i=1}^{4} \|\hat{\lambda}_i\|.
\]
In particular the map $\lambda \mapsto \hat{\lambda}$ is injective.

Finally, for $\lambda, \mu \in L(S)$ and $x \in S$ we have
\[
(\lambda * \mu)(x) = \int_S \chi(x) d(\lambda * \mu)(\chi) = \int_S \int_S \chi(x) \zeta(x) d\lambda(\chi) d\mu(\zeta) = \hat{\lambda}(x) \hat{\mu}(x),
\]
and we are done. \(\square\)

**Remark 2.2.** In the group case, the range of the above map is $A(S)$. We don’t know if this is the case for foundation semigroups.

Following [4] we say that $S$ is of type $U$ if it has a dense subsemigroup $U$ which is a union of groups. Then to each $x \in U$ there corresponds an element $x' \in U$ (the inverse of $x$ in the group to which $x$ belongs) such that $xx'$ and $x'x$ are idempotents and $xx' = x, x'x = x'$.

In [5] a concept of positive definite functions is defined for semigroups of type $U$. We denote the set of positive definite functions on $U$ by $P(U)$. When $U$ is an increasing union or a disjoint union of groups, this element $x'$ is unique for each $x \in U$. When the latter holds and the map $x \mapsto x'$ is continuous we say that $S$ is of type $U$. In this case the map $x \mapsto x'$ on $U$ extends to a continuous map $x \mapsto x'$ on $S$ and $S$ becomes a topological $*$-semigroup. In this case we can talk about positive definite functions on $S$ in the sense of section 1. If $U$ is an increasing union or a disjoint union of groups, each open in $S$, then $S$ is of type $U$. If $S$ is of type $U$, then it is easy to see that for each $f \in C_0(S)$, $f \in P(S)$ if and only if $f|_U \in P(U)$. In particular for a unital commutative semigroup $S$ of type $U$ we have $B(S) = R(S)$ [5, 7.2.5]. Now the following result follows from [5] immediately.

**Proposition 2.3.** If $S$ is a commutative foundation $*$-semigroup of type $U$ with identity, then the map $\lambda \mapsto \hat{\lambda}$ is a linear isometry of $M(\hat{S})$ onto $B(S)$.

Note that if we consider the semigroup of integers $\mathbb{Z}$ with trivial involution $n^* = n$, then we have $B(\mathbb{Z}) \neq R(\mathbb{Z})$ [7].
3. Spectrum of the Fourier Algebra

In this section we show that for a unital foundation topological *-semigroup $S$, the spectrum of $B(S)$ is a compact unital semitopological semigroup. Let $S$ be a unital foundation topological *-semigroup with identity $e$ and $\Omega = \Omega(S)$ be the family of all continuous *-representations $\omega$ of $S$ in a $W^*$-algebra $M_\omega$ with $\|\omega\| \leq 1$. Let $\omega_\Omega$ be the universal representation of $S$ in the $\ell^\infty$ direct sum $M_\Omega = \sum_{\omega \in \Omega} \oplus M_\omega$. Then the predual $(M_\Omega)_*$ is the $\ell^1$ direct sum $\sum_{\omega \in \Omega} \oplus (M_\omega)_*$ and for each $\psi \in (M_\Omega)_*$ we have $u = \psi \circ \omega_\Omega \in B(S)$ and $\|u\| \leq \|\psi\|$ [1, 3.1, 3.4], [7].

For $u \in B(S)$ and $x, y \in S$ let $u_x(y) = u(yx)$ then $u_x \in B(S)$ with $\|u_x\| \leq \|u\|$ [1, 3.4]. This means that the right translation operators $\tau_x : B(S) \to B(S)$ defined by

$$\tau_x(u) = u_x \quad (x \in S, u \in B(S)),$$

are bounded with $\|\tau_x\| \leq 1$.

**Definition 3.1.** For $u \in B(S)$ and $f \in B(S)^* = W^*_\Omega(S)$ define $E_f(u) : S \to \mathbb{C}$ by

$$E_f(u)(x) = \langle f, u_x \rangle \quad (x \in S).$$

**Lemma 3.2.** For $f \in W^*_\Omega(S)$, $E_f : B(S) \to B(S)$ is a bounded linear operator which commutes with right translation operators and $\|E_f\| = \|f\|.$

**Proof.** Let $u \in B(S)$ and choose $\psi \in (M_\Omega)_*$ with $u = \psi \circ \omega_\Omega$ and $\|u\| = \|\psi\|$, then $u(x) = \langle \omega_\Omega(x), \psi \rangle$, for $x \in S$. Given $\zeta \in (M_\Omega)_*$ and $m \in M_\Omega$ define $\zeta m \in (M_\Omega)_*$ by $\langle n, \zeta m \rangle = \langle mn, \zeta \rangle$ for $n \in M_\Omega$. Also $m, \zeta$ is defined similarly. For each $x, y \in S,$

$$u_x(y) = u(yx) = \langle \omega_\Omega(yx), \psi \rangle = \langle \omega_\Omega(x), \psi \omega_\Omega(x) \rangle,$$

hence $u_x = (\psi \omega_\Omega(x)) \circ \omega_\Omega$. To each $f \in W^*_\Omega(S)$ there corresponds $f^\circ \in M_\Omega$ defined by $(f^\circ, \zeta) = (f, \zeta \circ \omega_\Omega)$, for $\zeta \in (M_\Omega)_*$. Then

$$E_f(u)(x) = \langle f, u_x \rangle = \langle f, (\psi \omega_\Omega(x)) \circ \omega_\Omega \rangle = \langle f^\circ, \psi \omega_\Omega(x) \rangle = \langle \omega_\Omega(x), f^\circ \psi \rangle,$$

so $E_f(u) = (f^\circ \psi) \circ \omega_\Omega \in B(S)$ with $\|E_f(u)\| \leq \|f^\circ \psi\| \leq \|u\| \|f\|$, that is $\|E_f\| \leq \|f\|$. On the other hand $\|f\| = \|E_f(u)(e)\| \leq \|E_f(u)\| \leq \|E_f\| \|u\|$, hence $\|E_f\| = \|f\|$. Finally, for $x, y \in S,$

$$(E_f(u))(y) = E_f(u)(yx) = \langle f, u_{yx} \rangle = \langle f, (u_x)y \rangle = E_f(u)(y),$$

and so $E_f$ commutes with right translation operators. \qed

Let $L(B(S))$ be the space of all bounded linear operators on $B(S)$ and $L_0(B(S))$ be the closed subspace of $L(B(S))$ consisting of those operators which commute with all right translation operators $\tau_x$ on $B(S)$.

**Theorem 3.3.** Let $S$ be a unital foundation topological *-semigroup with identity $e$, then $B(S)^*$ is isometrically isomorphic to $L_0(B(S))$ and $B(S)^*$ is homeomorphic to the space $\text{End}(L_0(B(S)))$ consisting of non-zero endomorphisms of $L_0(B(S))$. In particular $B(S)^*$ is a compact unital semitopological semigroup.
Theorem. By above lemma, the map \( f \mapsto E_f \) is an isometric isomorphism from \( B(S)^* \) into \( L_0(B(S)) \). Given \( E \in L_0(B(S)) \) define \( f \in B(S)^* \) by \( \langle f, u \rangle = E(u)(e), \) for \( u \in B(S) \). Then

\[
E_f(u)(x) = (f, u_x) = E(u_x)(e) = E(u)_x(e) = E(u)(x),
\]

for \( x \in S \) and \( u \in B(S) \). Therefore \( E_f = E \). Now it is easy to check that \( f \) is multiplicative if and only if \( E_f \) is an endomorphism. Next \( B(S)^* \) is isomorphic with the \( w^* \)-closed linear span of \( \{\omega_\Omega(x) : x \in S\} \) in \( M_\Omega \) [1, 2.1]. Now for each net \( \{f_\alpha\} \subseteq B(S)^* \), \( E_{f_\alpha} \to E_f \) in WOT if and only if \( E_{f_\alpha}(u) \to E_f(u) \) weakly, for each \( u \in B(S) \), that is \( \langle m, E_{f_\alpha}(u) \rangle \to \langle m, E_f(u) \rangle \), for \( m \in B(S)^* \), which in turn is equivalent to \( \langle f_\alpha, \psi \rangle \mapsto \langle f, \psi \psi \rangle \) for \( \psi \in (M_\Omega)_{\ast} \). But \( B(S) \) is unital and so \( (M_\Omega)_{\ast} \), hence the latter is equivalent to \( \langle f_\alpha, \psi \rangle \to \langle f, \psi \rangle \), for \( \psi \in (M_\Omega)_{\ast} \), that is \( f_\alpha \to f \) in \( w^* \)-topology. \( \square \)

4. Examples

In this section we calculate the algebras \( A(S) \) and \( B(S) \) in various examples. One class of examples are semigroups of type \( U \) \[4\].

The following example shows that the existence of an identity is needed in Proposition 2.3.

Example 4.1. Let \( S = \mathbb{N} \cup \{0\} \) with discrete topology and multiplication \( n.m = \delta_{mn} n \) for \( n, m \in S \). Then each singleton \( \{n\} \) is the trivial group and \( S \) is of type \( U \). In this case \( R(S) = \ell^1(\mathbb{N}) \cup \mathbb{C} \) [5, 3.1.6], whereas \( B(S) = \operatorname{span}\{f \in c_0(S) : f(n) \geq f(0) \geq 0\} \).

Example 4.2. Let \( S \) be the unit ball of \( L^\infty(\Omega, \mu) \) with pointwise multiplication and \( w^* \)-topology. We assume that \( \mu \) is a finite measure on \( \Omega \). Put

\[
U = \{ f \in S : |f| = 1 \text{ or } 0 \}. 
\]

In this case \( f' = f \) if \( f \neq 0 \) and \( f' = 0 \). We claim that the map \( f \mapsto f' \) is continuous on \( U \). Let \( f_\alpha \to f \) in \( w^* \)-topology, i.e.

\[
\int_\Omega g f_\alpha d\mu \to \int_\Omega g f d\mu \quad (g \in L^1(\Omega, \mu)).
\]

Then we have

\[
\int_\Omega g (f_\alpha - f) d\mu = \left( \int_\Omega g \right) (f_\alpha - f) \to 0,
\]

for each \( g \in L^1(\Omega, \mu) \). This shows that \( S \) is of type \( U \). In particular \( B(S) = R(S) \).

Example 4.3. Let \( S = G \cup \{\infty\} \) be a one-point compactification of a locally compact group \( G \). If \( \{g_\alpha\} \) is a net in \( G \) and \( g_\alpha \to \infty \) in \( S \), then \( g_\alpha^{-1} \to \infty \) in \( S \). If \( g_\alpha \to g \) in \( G \) then \( g_\alpha^{-1} \to g^{-1} \) in \( G \). Hence \( S \) is of type \( U \). Also \( S \) is unital with identity \( \infty \). If \( G \) is abelian, then \( B(S) = R(S) = M_0(\hat{G}) \oplus \mathbb{C} \), where \( M_0(\hat{G}) = \{ \mu \in M(\hat{G}) : \mu \in C_0(G) \} \) [5, 5.1.3].
Example 4.4. Let $S = ([0,1], \max)$ with involution $x^* = x$. Then $S$ is a compact abelian unital semigroup and $\hat{S}$ is an idempotent semigroup. Indeed

$$\hat{S} = \{\chi_{[0,x]} : x \in S\}.$$ 

In particular $\hat{S}$ separates the points of $S$ (and so does $\Sigma(S)$.) Also

$$L^1(S) = \{f : S \to \mathbb{C} : f \text{ measurable and } \int_0^1 |f(x)| dx < \infty\}$$

is a Banach algebra with convolution

$$f \ast g(x) = f(x) \int_0^x g(t) dt + g(x) \int_0^x f(t) dt.$$ 

$L^1(S)$ has a bounded approximate identity. Let $f : S \to \mathbb{C}$ be positive definite, then

$$\sum_{i,j=1}^n c_i \bar{c}_j f(x_i x_j^*) \geq 0,$$

for each $n \geq 1, c_1, \ldots, c_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in S$. Once put $n = 1, c_1 = 1$, and $x_1 = x$, and then put $n = 2, c_1 = c_2 = \sqrt{-1}$, and $x_1 = x, x_2 = y$ to get

$$f(x) \geq 0, \quad f(x) - 2f(xy) + f(y) \geq 0,$$

for each $x, y \in S$. This shows that $f$ is non-negative and non-increasing. Conversely all such functions are positive definite, and so $A(S) = B(S) = BV[0,1]$. In particular $A(S)$ is regular and natural [4, 4.4.35]. Also $B(S)$ is not a dual space [5]. Note that in this case $S$ is not foundation [7] (compare with [1].) The convolution product of two elements in $L^2(S)$ is defined as above. In particular for $g(x) = 1$ and

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases},$$

we have $f, g \in L^2(S)$, but

$$f \ast g(x) = x^2 \sin(\frac{1}{x}) + \int_0^x tsin(\frac{1}{t}) dt,$$

for $x \neq 0$ and $f \ast g(0) = 0$. It is easy to see that $f \ast g \not\in BV[0,1]$. In particular $A(S) \neq L^1(S) \ast L^1(S)$.

Example 4.5. Let $S = (\mathbb{R}^+, +)$ with involution $x^* = x$. Then $S$ is a locally compact commutative unital $*$-semigroup. If $f : S \to \mathbb{C}$ is continuous and positive definite, then in the corresponding inequality, once put $n = 1, c_1 = 1$, and $x_1 = \frac{x}{2}$, and then put $n = 2, c_1 = 1, c_2 = -1$, and $x_1 = \frac{x}{2}, x_2 = \frac{y}{2}$ to get

$$f(x) \geq 0, \quad f(x) - 2f(\frac{x}{2} + \frac{y}{2}) + f(y) \geq 0,$$
for each \( x, y \in S \). This shows that \( f \) is non negative and convex. Conversely we know that \( \mathbb{R}^+ \simeq \mathbb{R}^+ \) [4] and we have the Laplace transform
\[
\hat{\mu}(x) = \int_0^\infty e^{-xt}d\mu(t),
\]
for \( \mu \in M(\mathbb{R}^+) \), and these are exactly the elements of \( B(\mathbb{R}^+) \) [2].

**Example 4.6.** Let \( S = (\mathbb{N} \cup \{0\}, +) \) with involution \( x^* = x \). Then \( S \) is a discrete abelian unital semigroup. If \( f : S \to \mathbb{C} \) is positive definite, then in the corresponding inequality, once put \( n = 1, c_1 = 1, x_1 = n \), and then put \( n = 2, c_1 = c_2 = 1 \) and \( x_1 = 0, x_2 = n \), or \( c_1 = 1, c_2 = -1 \) and \( x_1 = m, x_2 = n \) to get
\[
f(2n) \geq 0, \quad f(0) - 2f(n) + f(2n) \geq 0, \quad f(2m) - 2f(m + n) + f(2n) \geq 0,
\]
for each \( m, n \in S \). It follows from the first and second inequality that \( f \) is real valued. In this case \( S \simeq [-1, 1] \) with multiplication. Hence \( B(S) \simeq M[-1, 1] \).

**References**


