USING FEED-BACK NEURAL NETWORK METHOD FOR SOLVING LINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. In this paper a new method for finding the solution of Fredholm integral equation based on hybrid neural networks have presented. The proposed neural net can get a real input vector and calculates its corresponding output vector. Next a learning algorithm based on the gradient descent method has been defined for adjusting the connection weights. Eventually, we have showed this method in comparison with some numerical methods provides solutions with good generalization and high accuracy. The method is illustrated by several examples with computer simulations.

Key Words: Fredholm integral equations, Hybrid neural nets, Feed-back neural networks (FNN).

2010 Mathematics Subject Classification: Primary: 13A15; Secondary: 13F30, 13G05.

1. INTRODUCTION

Since many mathematical formulations of physical phenomena contain integral equations and these equations are very useful for solving many problems in several applied fields like mathematical physics and engineering, therefore various approaches for solving these problems have been proposed. First time, Taylor expansion approach was presented for solution of integral equations by Kanwal and Liu in [6] and then has
been extended in [5, 13, 16, 18, 17]. Also variational iteration method [7] and Adomian decomposition method [1] are effective and convenient for solving integral equations. The homotopy analysis method (HAM) was proposed by Liao [8, 9, 10, 11] and then has been applied in [1, 3, 4]. In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. Also some theoretical aspects in existence of solutions for linear and nonlinear integral equations have been studied in [12, 13, 14]. Moreover, some different valid methods for solving integral equation have been developed in the last years. For example, artificial neural network approach has been applied for finding the solution of this kind of problems. Effati and Buzhabadi [2] used multilayer perceptron neural networks for solving Fredholm integral equations of the second kind.

In this paper we want to propose a new numerical approach to approximate solution of a Fredholm integral equation. For this aim, we apply an architecture of hybrid neural nets. At first we substitute $N$-th truncation of the Taylor expansion for unknown function $F(t)$ instead of unknown function in the given integral equation. Then we use a two-layer feed-back neural network with $(N + 1)$ input units and $(N + 1)$ output units, where it can get an input vector and calculates its corresponding output vector. The outputs from this neural network are numerically compared with target outputs. Next a cost function is defined that measures the difference between the target output and corresponding actual output. Then the suggested neural net by using a learning algorithm which is based on the gradient descent method adjusts the real connection weights to any desired degree of accuracy.

Here is an outline of the paper. In section 2, the basic notations and definitions of integral equation and Taylor polynomial method are briefly presented. Section 3 describes how to find an approximate solution of the given Fredholm integral equation by using neural network (NN). Finally, the reliability of the Feed-back neural networks (FNN) method versus trapezoidal quadrature rule and the proposed iterative method in [13], is checked to the integral equations in Section 4.

2. Preliminaries

In this section we give a detailed study of integral equations and Taylor expansion which are used in the next sections.
2.1. **Integral equation.** The basic definition of integral equation is given in [2].

**Definition 1.** The Fredholm integral equation of the second kind is

\[
F(t) = f(t) + \lambda (ku)(t),
\]

where

\[
(ku)(t) = \int_a^b K(s, t) F(s) ds.
\]

In Eq. (2.1), \( K(s, t) \) is an arbitrary kernel function over the square \( a \leq s, t \leq b \) and \( f(t) \) is a function of \( t \) where \( a \leq t \leq b \). If the kernel function satisfies \( K(s, t) = 0 \), \( s > t \), we obtain the Volterra integral equation

\[
F(t) = f(t) + \lambda \int_a^t K(s, t) F(s) ds.
\]

2.2. **Taylor series.** Let us first recall the basic principles of the Taylor polynomial method for solving Fredholm integral equations [6]. Since these results are the key for our problems therefore we explain them. Let us first write the Eq. (2.1) in this form

\[
F(t) = f(t) + \lambda \int_a^b K(s, t) F(s) ds,
\]

where the function \( f(t) \) and the kernel \( K(s, t) \) have been given and \( F(t) \) has to be evaluated. To obtain the solution of the given problem in the form of

\[
F_N(t) = \sum_{i=0}^N \frac{1}{i!} F^{(i)}(c)(t-c)^i, \quad a \leq t, c \leq b,
\]

which is the Taylor polynomial of degree \( N \) at \( t = c \), we first differentiate equation (2.1) \( N \) times with respect to \( t \) and get

\[
F^{(i)}(t) = f^{(i)}(t) + \lambda \int_a^b \frac{\partial^{(i)} K(s, t)}{\partial t^i} F(s) ds, \quad i = 0, ..., N.
\]

The aim of this paper is determining of the coefficients \( F^{(i)}(c) \), \((for i = 0, ..., N)\) in Eq. (2.4). For this aim, we expanded \( F(s) \) in Taylor series at \( s = 0 \) and substituted it’s \( N \)-th truncation in (2.5). Without loss generality we assume that in Eq. (2.3), \( a = 0 \). Because in the otherwise
with doing change variable \( x = s - a \) the lower bound of the integral equation (2.3) transformed to 0. Now we can write:

\[
F^{(i)}(0) = f^{(i)}(0) + \sum_{j=0}^{N} T_{ij} F^{(j)}(0), \quad i = 0, \ldots, N,
\]

where

\[
T_{ij} = \frac{\lambda}{j!} \int_{a}^{b} \frac{\partial^{(i)} K(s, t)}{\partial t^i} \big|_{t=0} s^j ds, \quad j = 0, \ldots, N.
\]

The applied approach for determining the coefficient \( F^{(i)}(0) \) in Eq. (2.6) will be described in section 3.

3. Learning of neural network

This section, first gives a short review on hybrid feed-back neural networks and then will suggest a learning algorithm to obtain an approximate solution of the integral equation.

**Definition 2.** A hybrid neural net is a neural network with real input signals, connection weights and activation function \( f \). However,

i) we can combine input signals and connection weights using a \( t \)-norm, \( t \)-conorm or some other continuous operation.

ii) \( f \) can be any continuous function from input to output.

We know that hybrid neural nets are universal approximators. In other words, they can approximate any continuous function on a compact set to arbitrary accuracy. Since the unknown function \( F(s) \) in Eq. (2.1) must be integrable on interval \([a, b]\), therefore we can approximate it with this neural network.

As applications of hybrid neural nets in this paper, we employ multiplication and addition to combine input signals and connection weights.

3.1. Input-output relations of each unit. Now consider a two-layer hybrid FNN with \((N + 1)\) input units and \((N + 1)\) output units such that all input-output signals and connection weights be real numbers. Suppose an input vector \( A = (A_0, A_1, \ldots, A_N) \) is presented to our FNN, then the input-output relation of each unit can be written as follows:

\[
o_i = A_i, \quad i = 0, 2, \ldots, N.
\]
Output units:

\[ y_j = f(\text{net}_j), \]

\[ \text{net}_j = \sum_{i=0}^{N} w_{ji} o_i, \quad j = 0, 2, ..., N. \]

where \( A_i \) is a real number. In above equation \( w_{ji} \) denotes the connection weight from the \( i \)-th input unit to the \( j \)-th output unit and \( f(x) \) is identical activation function corresponding to output nodes (see Fig. 1).

![Fig. 1. The proposed FNN](image)

3.2. Cost function. Let us define the parameters \( A_i, w_{ji} \) and target vector \( B = (B_0, ..., B_N) \) as following:

\[ A_i = F(i)(0), \quad w_{ji} = \begin{cases} T_{i,j} & j \neq i, \\ T_{i,j} - 1 & j = i, \end{cases} \quad B_j = -f^{(j)}(0), \quad j, i = 0, ..., N. \]

Suppose that \( Y = (Y_0, ..., Y_N) \) be the output vector corresponding to the input vector \( A = (A_0, ..., A_N) \). Now we want to introduce how to deduce the learning algorithm from the training set \( \{A; B\} \). To do this, we defined a cost function between the \( j \)th output unit \( Y_j \) and corresponding target output \( B_j \) as follows:

\[ e_j = \frac{(B_j - Y_j)^2}{2}, \quad j = 0, ..., N. \]
In general, the cost function for the input-output pair \( \{A; B\} \) is obtained as:

\[
e = \sum_{j=0}^{N} e_j.
\]

### 3.3. Learning algorithm of the FNN.

The present algorithm intends to acquire the minimum of the cost function in weight space using the method of gradient descent. The combination of the connection weights which minimizes the error function is considered to be a solution of the learning problem. Let real quantities \( A_i \) (for \( i = 0, ..., N \)) are initialized at random values for input signals. We want to update the crisp parameter \( A_i \) such that \( A_i = F^{(i)}(0) \). At first we calculate the parameters \( w_{ji} \), \( T_{ji} \) and \( Y_j \) (for \( j, i = 0, ..., N \)) by using of relations that were described at the beginning of this section. For crisp parameter \( A_i \) adjustment rule can be written as follows:

\[
A_i(r + 1) = A_i(r) + \Delta A_i(r),
\]

\[
\Delta A_i(r) = -\eta \frac{\partial e}{\partial A_i} + \alpha \Delta A_i(r - 1), \quad i = 0, ..., N,
\]

where \( r \) is the number of adjustments, \( \eta \) is the learning rate and \( \alpha \) is the momentum term constant. Thus our problem is to calculate the derivative \( \frac{\partial e}{\partial A_i} \) in (3.5). The derivative \( \frac{\partial e}{\partial A_i} \) can be calculated from the cost function \( e \) in (3.3). We calculated \( \frac{\partial e}{\partial A_i} \) as follows:

\[
\frac{\partial e}{\partial A_i} = \frac{\partial e_1}{\partial A_i} + \cdots + \frac{\partial e_N}{\partial A_i}, \quad i = 0, ..., N.
\]

On the other hand,

\[
\frac{\partial e_j}{\partial A_i} = \frac{\partial e_j}{\partial Y_j} \cdot \frac{\partial Y_j}{\partial \text{net}_j} \cdot \frac{\partial \text{net}_j}{\partial A_i} = (f^{(j)}(0) + Y_j) \cdot \frac{\partial \text{net}_j}{\partial A_i}, \quad j = 0, ..., N,
\]

where

\[
\frac{\partial \text{net}_j}{\partial A_i} = \begin{cases} T_{ji} & ; j \neq i, \\ T_{ij} - 1 & ; j = i. \end{cases}
\]

Let us assume input-output pair \( \{A; B\} \) that is given as training data, is applied for learning of the FNN. Then the learning algorithm can be summarized as follows:

**Learning algorithm**

*Step 1:* \( \eta > 0, \alpha > 0 \) and \( E_{max} > 0 \) are chosen. Then crisp quantities
$A_i$ (for $i = 0, ..., N$) are initialized at random values.

**Step 2:** Let $r := 0$ where $r$ is the number of iterations of the learning algorithm. Then the running error $E$ is set to 0.

**Step 3:** Let $r := r + 1$. Then,

i): Forward calculation: Calculate the output vector $Y$ by presenting the input vector $A$.

ii): Back-propagation: Adjust crisp parameter $A_i$ using the cost function (3.3).

**Step 4:** Cumulative cycle error is computed by adding the present error to $E$.

**Step 5:** The training cycle is completed. For $E < E_{max}$ terminate the training session. If $E > E_{max}$ then $E$ is set to 0 and we initiate a new training cycle by going back to Step 3.

4. Numerical examples

This section contains three examples of linear Fredholm integral equations of second kind. In these examples, we illustrate the use of FNN technique to approximate the solutions of the given integral equations. For each example, the computed values of the approximate solution are calculated over a number of iterations and the cost function is plotted. Moreover the present method is compared with trapezoidal quadrature rule and a type of artificial neural networks that has been introduced in [2]. In the following simulations, we use the specifications as follows:

1. Learning constant $\eta = 0.5$.
2. Momentum constant $\alpha = 0.5$.
3. Stoping conditions $E_{max} < 0.0001$.

**Example 1.** Consider the following Fredholm integral equation

$$F(t) = \sin(t) - \frac{1}{4}t + \frac{1}{4} \int_0^{\pi} tsF(s)ds,$$

with the exact solution $F(t) = \sin t$. For this case, we used a polynomial of degree 7. We trained FNN with 8 input units and 8 output units. We choose an initial approximation polynomial of degree 7 for unknown function $F(t)$ as follows:

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\[ F(t) = \frac{1}{2} + \frac{1}{2} t - \frac{1}{4} t^2 - \frac{1}{12} t^3 + \frac{1}{48} t^4 + \frac{1}{240} t^5 - \frac{1}{1440} t^6 - \frac{1}{10080} t^7. \]

From above polynomial, initial input vector can be written as

\[ A = (0.5, 0.5, -0.5, -0.5, 0.5, 0.5, -0.5, -0.5). \]

After 14 iterations the approximate function \( F(t) \) transformed to following form:

\[ F(t) = -1.9841 \times 10^{-4} t^7 + 1.8701 \times 10^{-6} t^6 + 0.0083 t^5 + 0.0953 \times 10^{-6} t^4 - 0.1666 t^3 + 7.9622 \times 10^{-4} t^2 + 0.9986 t + 0.0026. \]

Table 1 shows the approximated solution over a number of iterations and Figs. 2 and 3 show the convergence behaviors for computed values of the parameter \( F_i(r)(0) \) for different numbers of iterations.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( F_i(r)(0) ), ( i = 0, \ldots, 7 )</th>
<th>( e_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.18, 0.65, -0.29, -0.76, 0.24, 0.74, -0.25, -0.75)</td>
<td>0.97460</td>
</tr>
<tr>
<td>2</td>
<td>(-0.10, 0.81, -0.06, -1.02, -0.00, 0.99, -0.00, -1.00)</td>
<td>0.78816</td>
</tr>
<tr>
<td>3</td>
<td>(-0.21, 0.92, 0.07, -1.14, -0.13, 1.12, 0.12, -1.12)</td>
<td>0.62220</td>
</tr>
<tr>
<td>4</td>
<td>(-0.15, 0.97, 0.10, -1.13, -0.12, 1.12, 0.12, -1.12)</td>
<td>0.49024</td>
</tr>
<tr>
<td>5</td>
<td>(-0.04, 0.98, 0.07, -1.05, -0.06, 1.06, 0.06, -1.06)</td>
<td>0.38664</td>
</tr>
<tr>
<td>6</td>
<td>(0.03, 0.99, 0.01, -0.99, 0.00, 1.00, 0.00, -0.99)</td>
<td>0.30538</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>9</td>
<td>(0.00, 1.01, -0.01, -0.90, 0.01, 0.98, -0.01, -0.98)</td>
<td>0.0015990</td>
</tr>
<tr>
<td>10</td>
<td>(-0.00, 1.01, -0.00, -1.00, -0.00, 0.99, -0.00, -1.00)</td>
<td>0.0014420</td>
</tr>
<tr>
<td>11</td>
<td>(-0.00, 1.01, 0.00, -1.00, -0.00, 1.00, 0.00, -1.00)</td>
<td>0.0013027</td>
</tr>
<tr>
<td>12</td>
<td>(-0.00, 1.00, 0.01, -1.00, -0.00, 1.00, 0.00, -1.00)</td>
<td>0.0011794</td>
</tr>
<tr>
<td>13</td>
<td>(0.00, 1.00, 0.00, -1.00, -0.00, 1.00, 0.00, -1.00)</td>
<td>0.0010697</td>
</tr>
<tr>
<td>14</td>
<td>(0.00, 0.99, 0.00, -0.99, 0.00, 1.00, 0.00, -1.00)</td>
<td>0.0009717</td>
</tr>
</tbody>
</table>

Table 1. The approximated solutions with error analysis for Example 1.
The cost function

![Fig. 2. The cost function for Example 1.](image1)

The convergence of the calculated solutions

![Fig. 3. Convergence of the calculated solutions for Example 1.](image2)

**Trapezoidal rule:** Moreover, this example is going to show difference between the FNN approach and the trapezoidal quadrature rule. Consider again Example 1, the trapezoidal quadrature procedure is as follows:

Let the region of integration is subdivided into 4 equal intervals of width $h = \frac{\pi}{8}$, 5 integration nodes $t_i = \frac{(i-1)\pi}{8}$ and $f_i = f(t_i)$ (for $i = 1, \ldots, 5$). After using the TQR for this integral equation, following relations are
derived:

\[ F(t_i) = \sin t_i - \frac{1}{4} t_i \]

\[ + \frac{\pi^2}{64} t_i \left( \frac{1}{4} F\left(\frac{\pi}{8}\right) + \frac{1}{2} F\left(\frac{\pi}{4}\right) + \frac{3}{4} F\left(\frac{3\pi}{8}\right) + \frac{1}{2} F\left(\frac{\pi}{2}\right) \right), \quad i = 1, \ldots, 5. \]

These relations are transformed to linear system

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -0.9849 & 0.0303 & 0.0454 & 0.0303 \\
0 & 0.0303 & -0.9394 & 0.0908 & 0.0606 \\
0 & 0.0454 & 0.0908 & -0.8637 & 0.0908 \\
0 & 0.0606 & 0.1211 & 0.1817 & -0.8789
\end{bmatrix}
\begin{bmatrix}
F(0) \\
F\left(\frac{\pi}{8}\right) \\
F\left(\frac{\pi}{4}\right) \\
F\left(\frac{3\pi}{8}\right) \\
F\left(\frac{\pi}{2}\right)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
-0.2845 \\
-0.5108 \\
-0.6294 \\
-0.6073
\end{bmatrix}.
\]

The solution of above system yields following data points,

\[ \{(0, 0), \left(\frac{\pi}{8}, 0.3846\right), \left(\frac{\pi}{4}, 0.7109\right), \left(\frac{3\pi}{8}, 0.9296\right), \left(\frac{\pi}{2}, 1.0076\right)\}. \]

For the data in above set, the Lagrange interpolation formula is derived as

\[ L_p = 0.0287 t^4 - 0.2036 t^3 + 0.0200 t^2 + 1.0012 t. \]

The approximate to the exact solution of Eq. \((2.3)\) with 7-th truncation limits of Taylor series and trapezoidal quadrature are compared in Fig. 4 with the exact solution.

Fig. 4. Comparison the exact and approximated solutions for Example 1.
Also Table 2 illustrates the absolute values of the errors obtained here and the absolute errors of [2] for this example. As showing, difference between the exact solution and the computed solution is dispensable.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Presented method</th>
<th>50 iterations</th>
<th>100 iterations</th>
<th>Neural network approach</th>
<th>50 iterations</th>
<th>100 iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.08 x 10^{-9}</td>
<td>6.35 x 10^{-9}</td>
<td>2.11 x 10^{-8}</td>
<td>6.35 x 10^{-8}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>4.16 x 10^{-5}</td>
<td>5.80 x 10^{-9}</td>
<td>2.03 x 10^{-5}</td>
<td>5.66 x 10^{-8}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>4.23 x 10^{-5}</td>
<td>5.08 x 10^{-9}</td>
<td>1.89 x 10^{-5}</td>
<td>6.10 x 10^{-8}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>4.41 x 10^{-5}</td>
<td>7.15 x 10^{-9}</td>
<td>2.08 x 10^{-5}</td>
<td>7.03 x 10^{-9}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{16}$</td>
<td>4.68 x 10^{-5}</td>
<td>6.24 x 10^{-9}</td>
<td>2.05 x 10^{-5}</td>
<td>5.21 x 10^{-9}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Absolute errors with $N = 8$ for Example 1.

**Example 2.** Let us consider the integral equation

$$ F(t) = e^t - \frac{e^{t+1} - 1}{t+1} + \int_0^1 e^t F(s) ds, $$

with the exact solution $F(t) = e^t$. We trained the fuzzy neural network as described in last example. Before starting calculations, we assumed that $F^{(i)}(0) = 0.5, (for \; i = 0, ..., N)$. Now we have:

$$ F(t) = \frac{1}{2} + \frac{1}{2} t + \frac{1}{4} t^2 + \frac{1}{12} t^3 + \frac{1}{48} t^4 + \frac{1}{240} t^5. $$

After 30 iterations the approximate function $F(t)$ transformed to the follow function:

$$ F(t) = 0.0083 \; t^5 + 0.0412 \; t^4 + 0.1647 \; t^3 + 0.4933 \; t^2 + 0.9849 \; t + 0.9736. $$

Numerical results can be found in Table 3. Similarly Figs. 5 and 6 show the accuracy of the solution or the convergence behaviors for computed values of the parameters $F^{(i)}_r(0)$ where $r$ is the number of iterations.
<table>
<thead>
<tr>
<th>$r$</th>
<th>$F^{(i)}(0), \ i = 0, \ldots, 5$</th>
<th>$e_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.47, 0.50, 0.51, 0.52, 0.52, 0.53)</td>
<td>0.17971</td>
</tr>
<tr>
<td>2</td>
<td>(0.45, 0.50, 0.52, 0.54, 0.55, 0.56)</td>
<td>0.15448</td>
</tr>
<tr>
<td>3</td>
<td>(0.44, 0.50, 0.53, 0.56, 0.57, 0.58)</td>
<td>0.13222</td>
</tr>
<tr>
<td>4</td>
<td>(0.42, 0.51, 0.55, 0.59, 0.61)</td>
<td>0.11463</td>
</tr>
<tr>
<td>5</td>
<td>(0.41, 0.51, 0.56, 0.59, 0.61, 0.63)</td>
<td>0.10083</td>
</tr>
<tr>
<td>6</td>
<td>(0.40, 0.52, 0.57, 0.60, 0.63, 0.65)</td>
<td>0.08995</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>24</td>
<td>(0.94, 0.96, 0.97, 0.97, 0.97, 0.96)</td>
<td>0.0010974</td>
</tr>
<tr>
<td>25</td>
<td>(0.95, 0.97, 0.97, 0.98, 0.98, 0.97)</td>
<td>0.0010737</td>
</tr>
<tr>
<td>27</td>
<td>(0.96, 0.97, 0.97, 0.98, 0.98, 0.98)</td>
<td>0.0010505</td>
</tr>
<tr>
<td>28</td>
<td>(0.96, 0.98, 0.98, 0.98, 0.98, 0.98)</td>
<td>0.0010278</td>
</tr>
<tr>
<td>29</td>
<td>(0.97, 0.97, 0.97, 0.98, 0.98, 0.99)</td>
<td>0.0010056</td>
</tr>
<tr>
<td>30</td>
<td>(0.97, 0.98, 0.98, 0.98, 0.98, 0.99)</td>
<td>0.0009838</td>
</tr>
</tbody>
</table>

Table 3. The approximated solutions with error analysis for Example 2.

![Error Analysis](image-url)  
**Fig. 5.** The cost function for Example 2.
Fig. 6. Convergence of the calculated solutions for Example 2.

**Trapezoidal rule:** For this example the trapezoidal rule with 5 integration nodes can be written as follows:

\[
F(t_i) = e^{t_i} - \frac{e^{t_{i+1}} - 1}{t_i + 1}
+ \frac{1}{8} (F(0) + 2e^{\frac{t_i}{4}}F\left(\frac{1}{4}\right) + 2e^{\frac{t_i}{2}}F\left(\frac{1}{2}\right) + 2e^{\frac{3t_i}{4}}F\left(\frac{3}{4}\right) + e^{t_i}F(1)),
\]

where

\[
t_i = \frac{i - 1}{4}, \quad i = 1, \ldots, 5.
\]

The solution of the linear system which is caused from above relations, yields following data points:

\[
\{(0,0.9156), (0.25, 1.1938), (0.5, 1.5538), (0.75, 2.0194), (1, 2.6213)\}.
\]

Similarly, the lagrange interpolation formula is came from above interpolation pairs as

\[
L_p = 0.0733 \ t^4 + 0.1435 \ t^3 + 0.5150 \ t^2 + 0.9739 \ t + 0.9156.
\]
In Fig. 7, the differences between 5-th truncation limit of the Taylor series and the TQR with exact solution are quite noticeable. Similarly Table 4 illustrates the absolute values of the errors obtained here and the absolute errors of [2] for example 2.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Presented method</th>
<th>Neural network approach [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50 iterations</td>
<td>100 iterations</td>
</tr>
<tr>
<td>0</td>
<td>$4.08 \times 10^{-4}$</td>
<td>$3.65 \times 10^{-10}$</td>
</tr>
<tr>
<td>1/8</td>
<td>$8.20 \times 10^{-5}$</td>
<td>$1.15 \times 10^{-10}$</td>
</tr>
<tr>
<td>1/4</td>
<td>$3.17 \times 10^{-4}$</td>
<td>$3.40 \times 10^{-10}$</td>
</tr>
<tr>
<td>1/2</td>
<td>$3.37 \times 10^{-4}$</td>
<td>$3.70 \times 10^{-10}$</td>
</tr>
<tr>
<td>1</td>
<td>$1.09 \times 10^{-3}$</td>
<td>$8.05 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

Table 4. Absolute errors with $N = 6$ for Example 2.

From examples (1) and (2) we can conclude that to get the best approximating solution for unknown function, the truncation limit $N$ must be chosen large enough.

**Example 3.** Consider the following integral equation problem

$$F(t) = 2t + \int_0^1 (t + s)F(s)ds,$$
with the exact solution $F(t) = -12t - 8$. Similarly, we applied proposed scheme for this problem. The learning started by

$$F(t) = t^2 - 10t - 10.$$  

After 313 iterations, the approximate function $F(t)$ is given by

$$F(t) = -4.1380 \times 10^{-4} t^2 - 11.9394 t - 7.9675.$$  

Numerical result can be found in tables 5 and 6. Figs. 8 and 9 show the convergence behaviors for computed values of the parameters $F_r^{(i)}(0)$, where $r$ is number of iterations. Moreover, exact solution and the approximated solution are compared in Fig. 10.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$F_r^{(i)}(0)$, $i = 0, 1, 2$</th>
<th>$e_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-1.1312, -0.9347, 0.624132)</td>
<td>1.38980</td>
</tr>
<tr>
<td>2</td>
<td>(-1.2556, -0.8737, 0.138329)</td>
<td>1.13430</td>
</tr>
<tr>
<td>3</td>
<td>(-1.3606, -0.8232, -0.232404)</td>
<td>0.92432</td>
</tr>
<tr>
<td>4</td>
<td>(-1.4478, -0.7824, -0.352726)</td>
<td>0.77131</td>
</tr>
<tr>
<td>5</td>
<td>(-1.5204, -0.7495, -0.244871)</td>
<td>0.66086</td>
</tr>
<tr>
<td>6</td>
<td>(-1.5808, -0.7232, -0.051282)</td>
<td>0.58075</td>
</tr>
<tr>
<td></td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>308</td>
<td>(-7.9668, -11.9381, -0.000845)</td>
<td>0.00100390</td>
</tr>
<tr>
<td>309</td>
<td>(-7.9669, -11.9384, -0.000841)</td>
<td>0.00100290</td>
</tr>
<tr>
<td>310</td>
<td>(-7.9670, -11.9386, -0.000838)</td>
<td>0.00100200</td>
</tr>
<tr>
<td>311</td>
<td>(-7.9672, -11.9389, -0.000834)</td>
<td>0.00100110</td>
</tr>
<tr>
<td>312</td>
<td>(-7.9673, -11.9392, -0.000831)</td>
<td>0.00100020</td>
</tr>
<tr>
<td>313</td>
<td>(-7.9675, -11.9394, -0.000827)</td>
<td>0.00099922</td>
</tr>
</tbody>
</table>

Table 5. The approximated solutions with error analysis for Example 3.
Fig. 8. The cost function for Example 3.

Fig. 9. Convergence of the calculated solutions for Example 3.

**Trapezoidal rule**: Similarly, the trapezoidal rule with 5 integration nodes can be written as follows:

\[ F(t_i) = 2t_i + \]
\[ \frac{1}{8}(t_i F(0) + 2(t_i + 0.25)F(\frac{1}{4}) + 2(t_i + 0.5)F(\frac{1}{2}) + 2(t_i + 0.75)F(\frac{3}{4}) + (t_i + 1)F(1)), \]

where

\[ t_i = \frac{i - 1}{4}, \quad i = 1, \ldots, 5. \]
The solution of the linear system which is caused from above relations, yields following data points:

\{(0, -7.3333), (0.25, -10), (0.5, -12.6667), (0.75, -15.3333), (1, -18)\}.

Similarly, by using the procedure which has been used for the previous examples, the Lagrange interpolation formula is derived as

\[ L_p = -\frac{32}{3} t - \frac{22}{3}. \]

![Graph](image.png)

Fig. 10. Comparison the exact and approximated solutions for Example 3.

<table>
<thead>
<tr>
<th>(t)</th>
<th>Presented method (N = 4)</th>
<th>Neural network approach (N = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50 iterations</td>
<td>100 iterations</td>
</tr>
<tr>
<td>0</td>
<td>(8.11 \times 10^{-3})</td>
<td>(7.06 \times 10^{-5})</td>
</tr>
<tr>
<td>0.1</td>
<td>(2.01 \times 10^{-2})</td>
<td>(8.69 \times 10^{-5})</td>
</tr>
<tr>
<td>0.2</td>
<td>(2.86 \times 10^{-2})</td>
<td>(1.90 \times 10^{-4})</td>
</tr>
<tr>
<td>0.5</td>
<td>(3.37 \times 10^{-2})</td>
<td>(3.04 \times 10^{-4})</td>
</tr>
<tr>
<td>1</td>
<td>(3.70 \times 10^{-2})</td>
<td>(3.73 \times 10^{-4})</td>
</tr>
</tbody>
</table>

Table 6. Absolute errors with \(N = 4\) for Example 3.

It is clear that using of this method we can find the analytical solution for these kinds of equations, if the exact solution of given problem be a polynomial.
5. Conclusions

In this paper, an architecture of feed-back neural networks has been proposed to approximate solution of a linear Fredholm integral equation of the second kind. Presented hybrid FNN in this study was a method for computing the coefficients in the Taylor expansion of unknown function. It is clear that to get the best approximate solution of the given equation, the truncation limit \( N \) must be chosen large enough. An interesting feature of this method is finding the analytical solution for the integral equation, if the exact solution of the problem be a polynomial of degree \( N \) or less than \( N \). Also, we have compared the purposed method with trapezoidal quadrature rule and, a structure of artificial neural networks which have been widely used to solve these kinds of equations. The analyzed examples illustrate the ability and reliability of the present method. The obtained solutions, in comparison with the exact solutions admit a remarkable accuracy. With the availability of this methodology, now it will be possible to investigate the approximate solution of other kinds of integral equations.

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References

Neural network method for solving linear Fredholm integral equations


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