A FOUR-NODE QUADRILATERAL ELEMENT FOR FINITE ELEMENT ANALYSIS VIA AN EFFICIENT FORCE METHOD

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ABSTRACT

In this paper an efficient method is developed for the formation of null bases of finite element models comprised of four-node quadrilateral plane stress and plane strain elements corresponding to highly sparse and banded flexibility matrices. This is achieved by associating special graphs to the finite element models and using an element with new equilibrium tractions and stress field for the formation of localized self-equilibrating stress systems. The efficiency of the present method is illustrated through some examples.

Keyword: finite element; graph theory; self-equilibrating stress systems (SESs); null basis; sparse; flexibility matrix; force method

1. INTRODUCTION

The force method of structural analysis in which the member forces are used as unknowns is appealing to engineers since the properties of members of a structure most often depend on the member forces rather than joint displacements. This method was used extensively until 1960. After this the advent of the digital computer and the amenability of the displacement method for computation attracted most researchers. As a result the force method and some of the advantages it offers in non-linear analysis and optimization has been neglected.

Three different approaches are adopted for the force method of structural analysis classified as:

1. Topological force methods
2. Algebraic force methods
3. Mixed algebraic-combinatorial force methods

Topological methods have been developed by Henderson [1] Maunder [2] and Henderson and Maunder [3] for rigid-jointed skeletal structures using the cycle bases of their graph models. Methods suitable for computer programming are due to Kaveh [4-5]. These

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topological methods are generalized to cover different types of skeletal structures such as rigid-jointed frames pin-jointed planar trusses and ball-jointed space trusses [67].


In this paper an efficient method is developed for the formation of null bases for finite element models comprised of four-node quadrilateral plane stress and plane strain elements corresponding to highly sparse and banded flexibility matrices This is achieved by associating a special graph to the finite element model and selecting subgraphs (known as γ-cycles [6]) for the formation of localized self-equilibrating stress systems (null vectors). The efficiency of the present method is illustrated through simple examples.

2. ALGEBRAIC FORCE METHODS

Consider a discrete or discretized structure \( S \) which is assumed to be statically indeterminate. Let \( \mathbf{r} \) denote the \( m \)-dimensional vector of generalized independent element (member) forces and \( \mathbf{p} \) the \( n \)-vector of nodal loads. The equilibrium conditions of the structure can then be expressed as

\[
\mathbf{A}\mathbf{r} = \mathbf{p}. \tag{1}
\]

where \( \mathbf{A} \) is an \( n \times m \) *equilibrium matrix*. The structure is assumed to be rigid and therefore \( \mathbf{A} \) has a full rank i.e. \( t = m - n > 0 \) and \( \text{rank} \mathbf{A} = n \).

The member forces can be written as

\[
\mathbf{r} = \mathbf{B}_0\mathbf{p} + \mathbf{B}_1\mathbf{q} \tag{2}
\]

where \( \mathbf{B}_0 \) is an \( m \times n \) matrix such that \( \mathbf{A}\mathbf{B}_0 \) is an \( n \times n \) identity matrix and \( \mathbf{B}_1 \) is an \( m \times t \) matrix such that \( \mathbf{A}\mathbf{B}_1 \) is an \( n \times t \) zero matrix. \( \mathbf{B}_1 \) and \( \mathbf{B}_0 \) always exist for a structure and in fact many of them can be found for a structure. Each column of \( \mathbf{B}_1 \) is known as a *self-equilibrating stress system* (SES) or a *null vector*. A maximal set of SESs (null vectors) is known as a *statical basis* (null basis). \( \mathbf{B}_1 \) is called a *self-equilibrating stress matrix* (null basis matrix).

Minimizing the complementary potential energy requires that \( \mathbf{r} \) minimize the quadratic form

\[
\frac{1}{2} \mathbf{r}^\top \mathbf{F}_m \mathbf{r}, \tag{3}
\]

subjected to the constraint as in Eq. (1). \( \mathbf{F}_m \) is an \( m \times m \) block diagonal element flexibility matrix. Using Eq. (2) it can be seen that \( \mathbf{q} \) must satisfy the following equation

\[
(\mathbf{B}_1^\top \mathbf{F}_m \mathbf{B}_1)\mathbf{q} = -\mathbf{B}_1^\top \mathbf{F}_m \mathbf{B}_0\mathbf{p} \tag{4}
\]

where \( \mathbf{B}_1^\top \mathbf{F}_m \mathbf{B}_1 = \mathbf{G} \) is the *overall flexibility matrix* of the structure. Computing the
redundant forces $q$ from Eq. (4) $r$ can be found using Eq. (2). The structure of $G$ is again important and its sparsity bandwidth and conditioning govern the efficiency of the force method. For the sparsity of $G$ one can search for a sparse $B_i$ matrix which is often referred to as the sparse null basis problem.

3. NATURAL ASSOCIATE GRAPH AND DOUBLE ASSOCIATE GRAPH OF A FINITE ELEMENT MODEL

Basic definitions from graph theory: A graph $S$ consists of a set of elements $N(S)$ called nodes and a set of elements $M(S)$ called members together with a relation of incidence which associates two distinct nodes with each member known as its ends. Two nodes of a graph are called adjacent if these nodes are the end nodes of a member. A member is considered incident with a node if it is an end node of the member. The degree of a node is the number of members incident with that node. A subgraph $S_i$ of a graph $S$ is a graph for which $N(S_i) \subseteq N(S)$ and $M(S_i) \subseteq M(S)$ and each member of $S_i$ has the same ends as in $S$. A path of $S$ is a finite sequence $P_i = \{n_0, m_1, n_1 ... m_p, n_p\}$, whose terms are alternately distinct nodes $n_i$ and distinct members $m_i$ of $S$ for $1 \leq i \leq p$ and $n_{i-1}$ and $n_i$ are the two ends of $m_i$. Two nodes $n_i$ and $n_j$ are said to be connected in $S$ if there exists a path between these nodes. A cycle is a path $(n_0, m_1, n_1 ... m_p, n_p)$ for which $n_0 = n_p$ and $p \geq 3$; i.e. a cycle is a closed path. The cycles of a graph form a vector space known as the cycle space. The dimension of this space for connected graph $S$ is known as the first Betti number $b_1(S) = M(S) - N(S) + 1$ of the graph where $M(S)$ and $N(S)$ are the number of members and nodes of $S$ respectively.

In order to transfer the topological property of a finite element model to the connectivity of a graph ten different graphs are introduced in [19]. Here the natural associate graph is used and an additional new graph is defined.

Natural associate graph: The Natural Associate Graph (NAG) of a finite element model denoted by NAG (FEM) is constructed as follows:

1. Associate one node of the graph NAG with an element of the finite element model.
2. Two nodes of NAG are connected by a member if the corresponding elements in the finite element model have a common edge.

A finite element model and the corresponding NAG are shown in Figure 1.

![Figure 1. A finite element mesh and the corresponding NAG](www.SID.ir)
**Double Associate Graph:** The double associate graph (DAG) of a finite element model is constructed using the following rules:

1. Associate one node of the graph DAG (FEM) with each element of the finite element model.
2. Two nodes of DAG (FEM) are connected by two members if the corresponding elements in the finite element model have a common edge.

Thus a typical interface of two elements in finite element model is represented by a double member in DAG. The members of the DAG are numbered according to the numbering of the NAG. As an example for a member of NAG numbered as n in the DAG the corresponding two members are numbered as 2n and 2n−1. A finite element model and the corresponding DAG are shown in Figure 2.

![Figure 2. A finite element mesh and the corresponding DAG](image)

4. **PROPERTIES OF THE NEW ELEMENT**

**Assumptions:** In this article it is assumed that the bodies undergoing the action of external forces are perfectly elastic i.e. these bodies resume their initial form completely after the removal of the forces. It will also be assumed that the considered elastic body is homogeneous and isotropic.

In this paper for higher efficiency a new definition for equilibrating forces of element is presented. Consider a triangular element as illustrated in Figure 3.

![Figure 3. A FE model with a small element highlighted](image)
This is an element with a smaller part separated from the middle of its side. The stresses corresponding to this part are shown in Figure 4.

![Figure 4. Stresses corresponding to a small part separated from the middle of the element](image)

For a small part of this part on the external face there is a normal stress $\sigma_{nn}$ and a shearing stress $\tau_{nt}$ and a little toward inside of this part there is a normal stresses $\sigma_{tt}$ as shown in Figure 5.

![Figure 5. Stresses at the external face](image)

When two elements are connected together then the equilibrium in direction $y'$ is satisfied by $\tau_{nt}$. In the other hand in a small distance from external face toward inside equilibrium in direction $y'$ is satisfied by two equal normal stresses $\sigma_{nt}$. This equilibrium is similar to the previous equilibrium in $y'$ direction due to $\tau_{nt}$.

This concept is used to define new stresses to satisfy the equilibrium on a common interface of two elements as shown in Figure 6.
Therefore for an element the stresses are illustrated in Figure 7.

For solution the stress resultants and their points of actions are at the center of each faces as shown in Figure 8. Furthermore for the analysis truss simulation is used. Therefore bi-action forces are defined in such a manner that the rigidity does not need to be checked as shown in Figure 7.

5. A FOUR-NODE PLAN QUADRILATERAL ELEMENT

This element reveals more information on the stresses compared to the other existing elements. Furthermore the new element satisfies equilibrium equation not only in each corner node but also on each face and along all the diagonals of the element as shown in Figure 9.
This element can be compared with quadratic nine-point element of Lagrange family as shown in Figure 10.

![Figure 9. Equilibrating forces of an element](image1)

As it will be shown later on the nodes 1 to 4 result in Type-2 and Type-3 SESs and the nodes 5 to 8 result in Type-1 SES and node 9 is equilibrated by diagonal force $F_9$. This will be used for the rigidity of the elements. However unexpectedly the sparsity will not only be decreased but it will even be increased.

### 6. DEGREE OF STATICAL INDETERMINACY OF FINITE ELEMENT MODELS

For a structure with $\gamma(S)$ degree of indeterminacy $\gamma(S)$ SESs are needed. Subsequently finite element model corresponding to double associate graph there will be a relation between the DSI and the number of cycles in DAG. This relation is explained in the three subsequent subsections.

#### 6.1 Two member cycles

These cycles are produced on each interface of two adjacent elements. Therefore for their formation it is sufficient to find the number of interfaces. For a model $S$ consisting of $F$ element and $E$ external surfaces (including surfaces of cut-outs) the number of interfaces $I$ is found as
6.2 The cycle corresponding to internal nodes
The number of these cycles is the same as the regional cycles of NAG and can be written as follows:

\[ b_1(A(S)) = M' - N' + 1 \]  
\[ b_1(A(S)) = F - \frac{1}{2} E + 1 \]

6.3 Cycles Corresponding to cut-outs
For every cycle three SESs should be generated however only one cycle is generated around every cut-out in NAG. Therefore for each cut-out two should be added to the number of SESs (i.e. \( b_1(A(S)) \)). Finally summation of three previous equations results in the degree of indeterminacy of the structure which is equal to the number of SESs i.e.

\[ \gamma(S) = 1 + b_1(A(S)) + 2nc \]

Substituting the corresponding terms results in

\[ \gamma(S) = 3F - E + 2nc + 1 \]

7. SELF-EQUILIBRATING STRESS SYSTEMS
A set of SESs corresponding to sparse \( B_1 \) matrix should be formed for the complementary solution. Each one of the SESs represents a null vector. A maximal set if independent null vectors form a null basis as columns of the \( B_1 \) matrix. SESs are categorized in three types and are generated as described in the following:

7.1 Type-1 SESs
Every double member of the DAG correspond a SES. In the other words a double member consisting of members numbered as i and j (i<j) where i represent biaction shear stresses and j represent normal stresses having two non-zero entries in the null basis matrix in the corresponding rows i and j. These double members are called Type-1 SESs. This type of
SES represents equilibrium on the interface between two connected elements and as mentioned before; it corresponds to the nodes 5 to 8 of the quadratic nine-node element of Lagrange family (Figure 10) and satisfies continuity condition between two elements. Two elements before and after connection are shown in Figure 11(a) and Figure 11(b). The number of these double members equals to the number of members of the NAG (FEM).

Using these double members approximately 80% of the columns of a null basis matrix can easily be generated. For finding these double members one can use the adjacency matrix or the node-member incident matrix. For each null vector the highest numbered member is called the *generator* member. This member is unique for each null vector and satisfies the independency between null vectors. In order to achieve the magnitude of the forces of the members the corresponding interface and two adjacent elements with all forces are considered. Here results are obtained for square elements for simplicity. A typical Type-1 SES and the corresponding equilibrating forces are shown in Figure 11(a) and Figure 11(b).

![Equilibrating forces typical Type-1 SES and two possible cases for connection of two elements shown in (c) and (d)](image)

In Figure 11(c) normal bi-action forces are summed with part of the tangential force such that the normal bi-action lays in the direction of a diagonal connecting the two remaining nodes of the two elements. This set of forces creates a $\gamma$-cycle that is once statically indeterminate. Now the generator member is assigned +1 and other forces are determined by the analysis of the remaining determinant truss. The normal and tangential forces must then be rescued. The $\gamma$-cycle and the corresponding null vector are shown in Figure 11© and Figure 11(d) respectively.
7.2 Type-2 SESs
After selection of the Type-1 SESs in order to fulfill the independency of the remaining SESs generators of Type-1 SESs are eliminated. These members have even numbers and represent normal stresses between two elements. As soon as these stresses are eliminated only shear stresses and diagonal forces F9 will remain. For a node common to several elements equilibrium due to these stresses must be satisfied and thus the Type-2 SESs are defined. In order to find these SESs all the regional cycles in NAG bounding single nodes of the finite element model must be formed. Now the members of NAG have odd numbers since the even numbers are the generators of the Type-1 SESs which are eliminated to provide the independency condition. Every regional cycle with n nodes in the NAG of the finite element model corresponds to n interfaces between n elements is called Type-2 SES. The remaining forces for an element after elimination of the generators of the Type-1 SESs are shown in Figure 12(a). A typical Type-2 SES and the corresponding equilibrating forces are shown in Figure 12(b) and Figure 12(c) respectively.

![Figure 12. NAG SES and the entries of a null vector for a model with square elements](image)

Again the highest numbered member is taken as the generator member. For calculating the magnitude of the forces a γ-cycle is created by considering only tangential forces. Then the generator member is assigned +1 and other forces are obtained by the analysis of the remaining determinant truss. These values are shown in Figure 12(c).

7.3 Type-3 SESs
Each regional cycle bounding a cut-out in the finite element model corresponds to a regional cycle of NAG with 3 degrees of statical indeterminacy corresponding to three SESs. For a finite element model with nc cut-outs apart from the SESs corresponding to double members of the graph DAG \( b_2(NAG) + 2nc \) additional SESs should be generated. This is obvious since for each non-cut-out cycle of NAG one SES and for each cut-out cycle of NAG three independent SESs can be generated. These SESs generated around three independent corner nodes of cut-outs are similar to the Type-2 SESs. The highest numbered member is taken as the generator. The entries in the null basis matrix are determined via a γ-cycle that is created.
by considering all the adjacent elements and the corresponding forces. A cut-out in a finite element model and the corresponding NAG are shown in Figure 13(a). Several other kind of external nodes are available and some of which together with the entries of the corresponding null vectors are illustrated in Figures 13(b) (c).

(b) A node common to three elements  
(c) A node common to two elements

Figure 13. Elements associated with a cut-out and two typical external nodes
8. TYPE-3 SESS FOR A CUT-OUT

According to the above discussion for each node of the cut-out an equilibrium equation and a corresponding SES could be formed. However each cut-out has just three degrees of indeterminacy. As an example a cut-out confined to several elements is shown in Figure 14.

![Figure 14. A cut-out confined to several elements](image)

If the normal stresses are removed then the remaining structure and the forces will be similar to a planar truss. For changing this cut-out consisting of n nodes to a triangulated region n–3 members are required to be drawn from an arbitrary node K to the other nodes except two adjacent nodes. Every line represents a relation between two nodes and results in a dependency. If \( f(x) \) is a dependent linear function corresponding to each additional assumed line and \( fe(x) \) is the equilibration equation in node X and K is the starting point of a cut-out \( f(x) \) transmits node K to node S. Then one can write

\[
\begin{align*}
  f(x) &= ax + b \\
  \text{for } x = k : fe(x) &= fe(k) \\
  \text{for } x = s : fe(x) &= fe(s) \\
  fe(s) &= fe(fe(k)) \\
  fe(s) &= afe(k+b)
\end{align*}
\]

As a result it can be seen that the equilibrium equation for node S depends on the equilibrium equation of the node K. As a result only one of these n–2 equations is independent. This equation and the two independent equations corresponding to the two
adjacent nodes are employed to form the SESs. These SESs are formed by the approach explained in the previous section. Many choices exist to obtain these SESs. These choices permits to obtain banded null basis matrix.

8.1 Algorithm
Step 1: Use an efficient method for nodes and members numbering of the NAG (FEM) (e.g. see [20] for such an algorithm).
Step 2: Generates the DAG and number the corresponding members according to the numbering of the NAG.
Step 3: Find all the double members and assign 1 to the corresponding positions in the null basis matrix (Type-1 SESs).
Step 4: Find Type-2 SESs using the presented procedures and assign 1 to the corresponding positions in the null basis matrix.
Step 5: Find an efficient starting node for each cut-out and generate Type-3 SESs.
Step 6: Calculate the values of members in all Type-3 SESs considering the boundary conditions and put those into the corresponding position in the null basis matrix.
Step 7: Order the generated null vectors using the number of their generators in an ascending order.

9. PARTICULAR SOLUTION

In the force method of analysis the complementary solution provides several relations among the unknown internal forces and particular solution present a scalar solution. This solution operates on a determinant structure which is obtained by removing $\gamma(S)$ unknown forces. In order to generate this determinate structure all generator members are eliminated. The number of these generators is equal to the degree of indeterminacy $\gamma(S)$ of the model. By removing the generators corresponding stresses are eliminated. These stresses are mostly normal stresses with some being shear stresses. Hence only shear stresses remain. These stresses can be transformed to the bi-action forces as described in Section 4.

Bi-action forces in the direction of each side of elements are similar to the axial forces in a truss. The $B_0$ matrix can then be constructed by applying unit values of external load P. It is evident that external loads must be either concentrated loads or should be converted into concentrated loads.

9.1 Advantages of the present method
In the present method for each element more force components are employed compared to the previous methods. This means apart from the shear stresses normal stresses can also be calculated. On the other hand the method is such that the overall statical indeterminacy of the system does not change. Though the number of components in the problem is increased and DSI has stayed constant however the necessary matrices do not need additional input data and the matrices are sparser and better structured. This increases the importance of the problem. The new element has also better equilibrium since all the forces are in equilibrium and each element is in equilibrium under the hydrostatic normal stresses. On the other hand
the interface between two elements is in equilibrium. These equilibriums are provided as self equilibrating systems of Type-1. Finally each node of the FEM is also in complete equilibrium. These equilibriums are provided by SESs of Type-2 and Type-3.

In relation with SESs of Type-2 one can see a similarity between the present element and Ladeveze-Maunder’s element since in the latter element the equilibrium is also studied under traction stresses. The main difference is that in Ladeveze-Maunder’s element all the forces are transformed into traction stresses and each cycle has two DSI and the analysis is performed by the force method for each node separately. This increases the dependency of the problem to the topological and boundary conditions limiting the generality of the problem. This is also mentioned in Ladeveze-Maunder’s work [21].

10. NUMERICAL RESULTS

Example 1: A finite element model is shown in Figure 15. This model is analyzed using plane quadrilateral elements having the following properties:

Thickmess = 0.1m  \( E = 2e + 8 \text{kN/m}^2 \) \( \nu = 0.3 \)
number of quadrilateral elements = 280
number of external edges = 68
number of members of the NAG = 526 (Type-1 SESs)
the first Betti number of the NAG = 247 (Type-2 SESs)
DSI = 773.

Figure 15. A finite element model with 280 elements

Patterns of the null basis matrices constructed using the LU factorization Kaveh-Koohestani method [23] and the present algorithm are illustrated in Figure 16. It can easily be observed that the null basis matrix formed by the present method is much sparser than the other two approaches.
Figure 16. Patterns and the number of nonzero entries of null bases; (a) Present algorithm; (b) Kaveh-Koohestani method; (c) LU factorization.

Patterns of the flexibility matrices using the present algorithm and Kaveh-Koohestani method are illustrated in Figure 17.

Figure 17. Patterns and the number of nonzero entries of the flexibility matrices; (a) Present algorithm; (b) Kaveh-Koohestani method.
For LU factorization the null basis contains 18587 entries and Kaveh-Koohestani method contains 4016 entries while the present method leads to only 2528 entries. Also for Kaveh-Koohestani method the flexibility matrix contains 11193 entries while the present method leads to only 3997 entries.

**Example 2:** A finite element model is shown in Figure 18. This model is analyzed using the plane quadrilateral elements with the following properties:

- Thickness = 0.1m
- $E = 2 \times 10^8 \text{kN/m}^2$
- $\nu = 0.3$
- number of quadrilateral elements = 272
- number of external edges = 88
- number of members of the NAG = 500 (Type-1 SESs)
- the first Betti number of the NAG = 229 (Type-2 SESs)
- DSI = 735.

Patterns of the null basis matrices constructed using the LU factorization Kaveh-Koohestani method [23] and the present algorithm are illustrated in Figure 19. It can easily be observed that the null basis matrix formed by the present method is much sparser than the other two approaches.
Figure 19. Patterns and the number of nonzero entries of null bases matrices; (a) Present algorithm; (b) Kaveh-Koohestani method; (c) LU factorization.

Patterns of the flexibility matrices using the present algorithm and Kaveh-Koohestani method are illustrated in Figure 20.

Figure 20. Patterns and the number of nonzero entries of flexibility matrices; (a) Present algorithm; (b) Kaveh-Koohestani method.
For LU factorization the null basis contains 17061 entries and Kaveh-Koohestani method contains 3944 entries while the present method leads to only 2384 entries. Also for Kaveh-Koohestani method the flexibility matrix contains 10685 entries while the present method leads to only 4005 entries.

**Example 3:** A finite element model is shown in Figure 21. This model is analyzed using plane quadrilateral elements with the following properties:

- Thickness = 0.1m
- $E = 2e + 8kN/m^2$
- $\nu = 0.3$
- number of quadrilateral elements = 560
- number of external edges = 108
- number of members of the NAG = 1066 (Type-1 SESs)
- the first Betti number of the NAG = 507 (Type-2 SESs)
- $DSI= 1573$.

Patterns of the null basis matrices constructed using the LU factorization Kaveh-Koohestani method [23] and the present algorithm are illustrated in Figure 22. It can easily be observed that the null basis matrix formed by the present method is much sparser than the other two approaches.
Figure 22. Patterns and the number of nonzero entries of null bases matrices; (a) Present algorithm; (b) Kaveh-Koohestani method; (c) LU factorization.

Pattern of the flexibility matrices using the present algorithm and Kaveh-Koohestani method are illustrated in Figure 23.

Figure 23. Patterns and the number of nonzero entries of flexibility matrices; (a) Present algorithm; (b) Kaveh-Koohestani method
For LU factorization the null basis contains 58097 entries and Kaveh-Koohestani method contains 8216 entries while the present method leads to only 5144 entries. Also for Kaveh-Koohestani method the flexibility matrix contains 23153 entries while the present method leads to only 8209 entries.

**Example 4:** A finite element model and the corresponding double associate graph are shown in Figure 24(a) and (b) respectively. This model is analyzed using plane quadrilateral elements with the following properties:

Thickness = 0.1m E = 2e + 8kN/m² ν = 0.3 number of quadrilateral elements = 88 number of external edges = 52 number of members of the NAG = 150 (Type-1 SESs) the first Betti number of the NAG = 63 (Type-2 SESs) DSI = 289.

Patterns of the null basis and flexibility matrices obtained by using the present algorithm are illustrated in Figure 25.
Example 5: A butterfly disk model is shown in Figure 26. This model is analyzed using plane quadrilateral elements with the following properties:

- Thickness = 0.1m
- $E = 2 \times 10^6 \text{kN/m}^2$
- $\nu = 0.3$
- number of quadrilateral elements = 360
- number of external edges = 144
- number of members of the NAG = 648 (Type-1 SESs)
- the first Betti number of the NAG = 289 (Type-2 SESs)
- $\Delta SI = 940$.

![Figure 26. A finite element model with 360 quadrilateral elements](image)

Patterns of the null basis and flexibility matrices constructed by the present algorithm are illustrated in Figure 27.

![Figure 27. Patterns of the null basis and flexibility matrices constructed using the present algorithm](image)

Example 6: A finite element model is shown in Figure 28. This model is analyzed using plane quadrilateral elements with the following properties:

- Thickness=0.1m
- $E=2 \times 10^6 \text{kN/m}^2$
- $\nu=0.3$
- number of quadrilateral elements=16
- number of external edges=16
- number of members of the NAG=24 (Type-1 SESs)
- the first Betti number
of the NAG=9 (Type-2 SESs) DSI= 33.

Figure 28. A finite element model with 16 quadrilateral elements

Stresses are obtained at the central point of each interface. The stresses are in natural coordinate and are transported to the corner nodes and subsequently are transformed to Cartesian coordinates. Finally normal stresses in x and y directions and shear stresses are calculated by a special equation developed in Ref. [20]. In the other hand the stresses corresponding to the plane stress elements are obtained by the displacement method. All of these stresses are provided in Table 1.

Table 1. Stresses obtained by the displacement method and the present force method

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<th>Present force method</th>
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11. CONCLUDING REMARKS

Many problems solved by the present method and the comparison of the results with those of the standards FE programs indicate the high accuracy of the method. The flexibility matrices based on the selected SESs are highly sparse and narrowly banded compared to other algebraic force methods. This is due to the use of regional cycles of the NAGs and appropriate ordering of the selected SESs.

The comparative study of the present algorithm and other existing approaches such as the LU factorization, Turn-back method and REDUC and Kaveh-Koohestani method [22] shows the superiority of the present graph-theoretical approach. In the present algorithm due to the reduction of the floating point operations, the results have higher accuracy than the other pure algebraic methods since nearly 80% of the null vectors are selected with algebraic operations and the remaining operations are performed on limited and definite vectors.

In the present method nodal numbering of a finite element model is less important and only a suitable ordering of the members of the NAG is required for reducing the bandwidth of the flexibility matrices.

REFERENCES


