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STUDYING THE CORONA PRODUCT OF GRAPHS UNDER SOME GRAPH INVARIANTS

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Abstract. The corona product $G \circ H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leq i \leq |V(G)|$. In this paper, exact formulas for the eccentric distance sum and the edge revised Szeged indices of the corona product of graphs are presented. We also study the conditions under which the corona product of graphs produces a median graph.

1. Introduction

Suppose G is a graph with vertex and edge sets of $V(G)$ and $E(G)$, respectively. If $x, y \in V(G)$ then the **distance** $d_G(x, y)$ (or d(x,y) for short) between x and y is defined as the length of a minimum path connecting x and y. The **Wiener index** is defined as the summation of distances between all pairs of vertices in the graph under consideration. In other words, The Wiener index index of a graph G can be defined as $\mathrm{W}(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v)$ [1].

The interval $I(u, v)$ between two vertices u and v of a graph G is the set of vertices on shortest paths between u and v. Note that $I(u, v)$ contains u and v. A f it vertex z is called a median for triple $\{u, v, w\}$ if it belongs to shortest paths between any two of u, v, and w. A median graph is an undirected graph in which any triple has a unique median. In other words, Alternatively, the medians of u, v, w can be defined as the vertices in $I(u, v) \cap I(u, w) \cap I(v, w)$. A graph is called **median** if every triple of its vertices has a unique median, namely if $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$, for every triple $u, v, w \in V(G)$ [22].

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Let G and H be two graphs. The **corona product** GoH, Figure 1, is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leq i \leq |V(G)|$ [2].

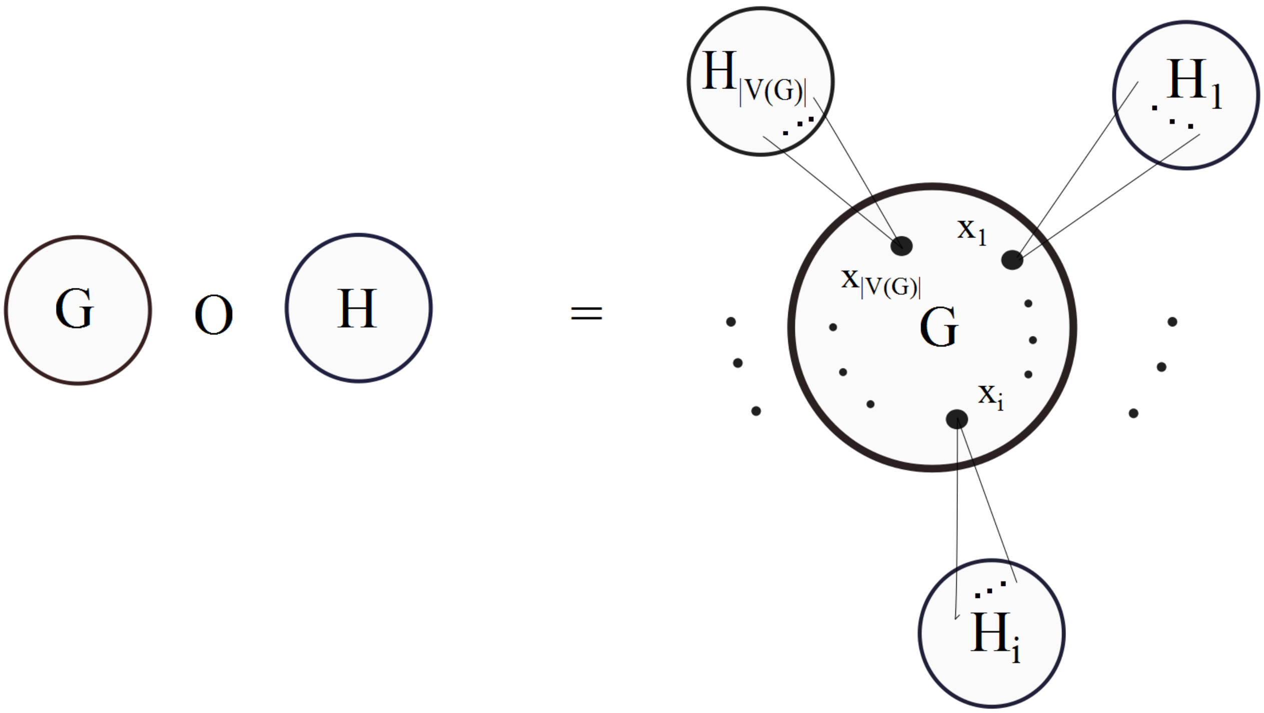

FIGURE 1. The Corona Product $G \circ H$.

The eccentricity $\varepsilon_G(u)$ is defined as the largest distance between u and other vertices of G. We will omit the subscript G when the graph is clear from the context. The eccentric connectivity index of a graph G is defined as $\xi(G) = \sum_{u \in V(G)} deg_G(u)\varepsilon_G(u)$ [3]. We encourage the reader to consult papers [4, 5] for some applications and papers [6, 7, 8, 9] for the mathematical properties of this topological index. For a given vertex $u \in V(G)$ we define its **distance sum** $D_G(u)$ as $D_G(u) = \sum_{v \in V(G)} d_G(u,v)$. The **eccentric distance sum** of G is summation of all quantity $D_G(u)\varepsilon_G(u)$ over all vertices of G [10]. In other words, $\xi^{SD}(G) = \sum_{u \in V(G)} D_G(u)\varepsilon_G(u)$. The concept of eccentricity also gives rise to a number of other topological invariants. For example, the **total eccentricity** $\zeta(G)$ of a graph G is defined as $\zeta(G) = \sum_{u \in V(G)} \varepsilon_G(u)$.

For an edge $e = ab$ of G, $n_a^G(e)$ denotes the number of vertices closer to a than to b, i.e. $n_a^G(e) = |\{u \in V(G)| d(u,a) < d(u,b)\}|$. In addition, assume that $n_0^G(e)$ is the number of vertices with equal distances to a and b. This means that $n_0^G(ab) = |\{u \in V(G)| d(u,a) = d(u,b)\}|$. The number of edges of G whose distance to the vertex a is smaller than the distance to the vertex b is denoted by $m_a^G(e)$, and the number of edges with equal distances to a and b is denoted by $m_0^G(e)$. The Szeged, edge Szeged, revised Szeged, edge revised Szeged, vertex-edge Szeged and modified vertex-edge Szeged indices of the graph G is defined as follows:

$$\begin{aligned}
\mathrm{Sz}_v(G) &= \sum_{e=uv\in E(G)} n_u^G(e)n_v^G(e),\\
\mathrm{Sz}_e(G) &= \sum_{e=uv\in E(G)} m_u^G(e)m_v^G(e),\\
\mathrm{Sz}_v^*(G) &= \sum_{e=uv\in E(G)} (n_u^G(e)+\frac{n_0^G(e)}{2})(n_v^G(e)+\frac{n_0^G(e)}{2})\\
\mathrm{Sz}_e^*(G) &= \sum_{e=uv\in E(G)} (m_u^G(e)+\frac{m_0^G(e)}{2})(m_v^G(e)+\frac{m_0^G(e)}{2})\\
\mathrm{Sz}_{ev}(G) &= \frac{1}{2}\sum_{e=uv\in E(G)} (m_u^G(e)n_v^G(e)+m_v^G(e)n_u^G(e))\\
\mathrm{Sz}_{ev}^*(G) &= \frac{1}{2}\sum_{e=uv\in E(G)} (m_u^G(e)n_u^G(e)+m_v^G(e)n_v^G(e)).
\end{aligned}$$

see [11, 12, 13, 14, 15] for details.

A graph G is called nontrivial if $|V(G)| > 1$. Throughout this paper we consider only "simple connected nontrivial graphs". The complete and cycle graphs of order n are denoted by K_n and C_n, respectively. A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k. A graph G is said to be (vertex) **distance-balanced**, if $n_a^G(e) = n_b^G(e)$, for each edge $e = ab \in E(G)$, see [16, 17, 18] for details. These graphs first studied by Handa [19] who considered distance-balanced partial cubes. In [20], Jerebič et al. studied distance-balanced graphs in the framework of various kinds of graph products. Similarly, a graph G is said to be **edge distance-balanced**, if $m_a^G(e) = m_b^G(e)$, for each edge $e = ab \in E(G)$ [21].

2. Main Results

In this section, we first present conditions under which the corona product of graphs produces a median graph. Then formulas are given for the eccentric distance sum and edge revised Szeged indices of the corona product of graphs.

Proposition 2.1 (See [22]). *Graphs in which every triple has a median are bipartite.*

Suppose G and H are graphs with disjoint vertex sets. Following Doslić [23], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y; z)$, is defined by identifying the vertices y and z in the union of G and H. Similarly, a link of G and H by vertices y and z is defined as the graph $(G \cdot H)(y; z)$ obtained by joining y and z by an edge in the union of these graphs.

Lemma 2.2. *Suppose G and H are connected rooted graphs with respect to the rooted vertices a and b, respectively. Then*

(i) $(G\cdot H)(a;b)$ *is median if and only if* G *and* H *are median.*

(ii) $(G\sim H)(a;b)$ *is median if and only if* G *and* H *are median.*

Proof. Consider a triple x,y,z of vertices of $(G\cdot H)(a;b)$. By definition of splice, it is clear that every triple of vertices of copy of G or H has a unique median in $(G\cdot H)(a;b)$ if and only if it has a unique median in G or H, respectively. Suppose $x,y\in V(G)$ and $z\in V(H)$. Then, a shortest z,x-path is containing a shortest a,x-path and a shortest z,y-path is containing a shortest a,y-path. Therefore, x,y,z has a unique median in $(G\cdot H)(a;b)$ if and only if x,y,a has a unique median in G. A similar argument proves the part (ii). $\square$

In the next result, we study the conditions under which corona product produce a median graph.

Proposition 2.3. *Let* G *and* H *be connected graphs. Then* $G\circ H$ *is median if and only if* G *is median and* $H\cong K_1$.

Proof. Let G and H be connected graphs. By definition of corona product, $G\circ H$ is containing no cycles of odd length if and only if G is containing no cycles of odd length and $H\cong K_1$. Then, by Proposition 2.1, if every triple of $G\circ H$ has a median, then G is containing no cycles of odd length and $H\cong K_1$. Therefore, the result is follows from Lemma 2.2. $\square$

Proposition 2.4. *Let* G *and* H *be graphs. Then*

$$\begin{aligned}\xi^{SD}(G\circ H) &= (|V(H)|+1)^2\xi^{SD}(G)-4|V(G)||E(H)|\\ &+2\Big(|V(G)||V(H)|^2+|V(G)||V(H)|-|V(H)|-|E(H)|\Big)\zeta(G)\\ &+2\Big(2|V(H)|^2+3|V(H)|+1\Big)\mathrm{W}(G)\\ &+2|V(G)||V(H)|\Big(2|V(G)||V(H)|+\frac{3}{2}|V(G)|-2\Big).\end{aligned}$$

Proof. Let G' be the copy of G and H_i be the i-th copy of H in $G\circ H$, $1\le i\le |V(G)|$. Then, $G\circ H$ is obtained by joining each vertex of the i-th copy of H to the i-th vertex (x_i) of G. A vertex of $G\circ H$ corresponding to the vertex u' in H is denoted by u. Also, we denote a vertex of $G\circ H$ corresponding to the vertex v' in G by v. Therefore

$$d_{G\circ H}(u,v) = \begin{cases} d_G(u',v') & \text{if } u,v\in V(G')\\ d_G(u',x_i)+1 & \text{if } u\in V(G'), v\in V(H_i)\\ d_G(x_i,x_j)+2 & \text{if } u\in V(H_i), v\in V(H_j), i\neq j\\ 2 & \text{if } u,v\in V(H_i), uv\notin E(H_i)\end{cases}$$

and so, for every $u\in V(G')\cap V(G\circ H)$, we have:

$$\begin{aligned} D_{G\circ H}(u) &= (|V(H)|+1)D_G(u')+|V(G)||V(H)|,\\ \varepsilon_{G\circ H}(u) &= \varepsilon_G(u')+1.\end{aligned}$$

This implies that,

$$\xi_1^{SD}(G\circ H)=\sum_{u\in V(G')\cap V(G\circ H)} D_{G\circ H}(u)\varepsilon_{G\circ H}(u)=(|V(H)|+1)\xi^{SD}(G)$$
$$+|V(G)||V(H)|\zeta(G)+2(|V(H)|+1)\mathrm{W}(G)+|V(G)|^2|V(H)|.$$

On the other hand, for every $u\in V(H_i)\cap V(G\circ H)$, we have:

$$D_{G\circ H}(u)=(|V(H)|+1)D_G(x_i)-deg_H(u')+2|V(G)||V(H)|+|V(G)|-2,$$
$$\varepsilon_{G\circ H}(u)=\varepsilon_G(x_i)+2,$$

and so,

$$\begin{aligned}\xi_2^{SD}(G\circ H)=&\sum_{i=1}^{|V(G)|}\sum_{u\in V(H_i)\cap V(G\circ H)} D_{G\circ H}(u)\varepsilon_{G\circ H}(u)\\
&=|V(H)|(|V(H)|+1)\xi^{SD}(G)\\
&+(2|V(G)||V(H)|^2+|V(G)||V(H)|-2|V(H)|-2|E(H)|)\zeta(G)\\
&+4(|V(H)|+1)|V(H)|\mathrm{W}(G)-4|V(G)||E(H)|\\
&+2|V(G)||V(H)|(2|V(G)||V(H)|+|V(G)|-2).\end{aligned}$$

By summation of $\xi_1^{SD}(G\circ H)$ and $\xi_2^{SD}(G\circ H)$, the result can be proved. □

The girth of a graph G is the length of a shortest cycle in G; if G has no cycles then we define the girth of G to be infinite.

Proposition 2.5. *Let H be a k-regular graph of girth >4. Then, for every graph G we have:*

$$\begin{aligned}\mathrm{Sz}_e^*(G\circ H)=&\mathrm{Sz}_e^*(G)+\frac{k+2}{2}|V(H)|(\mathrm{Sz}_{ev}(G)-\mathrm{Sz}_{ev}^*(G))+\frac{(k+2)^2}{4}|V(H)|^2\mathrm{Sz}_v^*(G)\\
&+\frac{k+2}{4}|V(G)||V(H)||E(G)|^2+\frac{k}{8}|V(G)||V(H)|\Big(\frac{k+2}{2}|V(G)||V(H)|+|E(G)|\Big)^2\\
&+\frac{k(k+1)+1}{2}|V(G)||V(H)|\Big(\frac{k+2}{2}|V(G)||V(H)|+|E(G)|-\frac{k(k+1)+1}{2}\Big).\end{aligned}$$

Proof. The summation of $(m_u^{G\circ H}(e)+\frac{m_0^{G\circ H}(e)}{2})(m_v^{G\circ H}(e)+\frac{m_0^{G\circ H}(e)}{2})$ over all edges of a copy of G in $G\circ H$, is equal to:

$$\begin{aligned}Sz_1^*=&\mathrm{Sz}_e^*(G)+(|V(H)|+|E(H)|)(\mathrm{Sz}_{ev}(G)-\mathrm{Sz}_{ev}^*(G))+(|V(H)|+|E(H)|)^2\mathrm{Sz}_v^*(G)\\
&+\frac{1}{2}|V(G)||E(G)|^2(|E(H)|+|V(H)|).\end{aligned}$$

Suppose G' is the copy of G and H_i is the i-th copy of H in $G\circ H$, $i=1,2,\ldots,|V(G)|$. It follows from the edge structure of H that, for every edge $e=uv$ of $G\circ H$ such that $uv\in E(H_i)$, we have:

$$m_u^{G\circ H}(e)=m_v^{G\circ H}(e)=k(k-1)+1,$$
$$m_0^{G\circ H}(e)=|E(G)|+\frac{(k+2)}{2}|V(G)||V(H)|-2(k(k-1)+1).$$

Thus, the summation of $(m_u^{G\circ H}(e)+\frac{m_0^{G\circ H}(e)}{2})(m_v^{G\circ H}(e)+\frac{m_0^{G\circ H}(e)}{2})$ over all edges of copies of H_i, is equal to:

$$Sz_2^* = \frac{k}{8}|V(G)||V(H)|\Big(\frac{k+2}{2}|V(G)||V(H)|+|E(G)|\Big)^2.$$

On the other hand, for every edge $e=uv$ of $G\circ H$ such that $u\in V(G')$ and $v\in V(H_i)$, we have:

$$\begin{aligned} m_u^{G\circ H}(e) &= |V(G)||E(H)|+|E(G)|+|V(G)||V(H)|-k^2-1,\\ m_v^{G\circ H}(e) &= k, \qquad m_0^{G\circ H}(e)=k(k-1)+1 \end{aligned}$$

and hence,

$$\begin{aligned} Sz_3^* &= \sum_{\substack{e=uv\in E(G\circ H),\\ u\in V(G'),v\in V(H_i)}} (m_u^{G\circ H}(e)+\frac{m_0^{G\circ H}(e)}{2})(m_v^{G\circ H}(e)+\frac{m_0^{G\circ H}(e)}{2})\\ &= \frac{k(k+1)+1}{2}|V(G)||V(H)|\Big(\frac{k}{2}|V(G)||V(H)|+|V(G)||V(H)|+|E(G)|-\frac{k(k+1)+1}{2}\Big). \end{aligned}$$

By summation of Sz_1^*, Sz_2^* and Sz_3^*, the result can be proved. □

If G is edge and vertex distance-balanced graph, then it is clear that $\mathrm{Sz}_{ev}(G)=\mathrm{Sz}_{ev}^*(G)$. So, we have:

Corollary 2.6. *If H is a k-regular graph of girth >4 and G is an edge and vertex distance-balanced graph, then*

$$\begin{aligned} \mathrm{Sz}_e^*(G\circ H) &= \mathrm{Sz}_e^*(G)+\frac{(k+2)^2}{4}|V(H)|^2\mathrm{Sz}_v^*(G)+\frac{k+2}{4}|V(G)||V(H)||E(G)|^2\\ &+\frac{k(k+1)+1}{2}|V(G)||V(H)|\Big(\frac{k+2}{2}|V(G)||V(H)|+|E(G)|-\frac{k(k+1)+1}{2}\Big)\\ &+\frac{k}{8}|V(G)||V(H)|\Big(\frac{k+2}{2}|V(G)||V(H)|+|E(G)|\Big)^2. \end{aligned}$$

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