HAMILTON-CONNECTED PROPERTIES IN CARTESIAN PRODUCT

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Abstract. In this paper, we investigate a problem of finding natural condition to assure the product of two graphs to be hamilton-connected. We present some sufficient and necessary conditions for \( G \square H \) being hamilton-connected when \( G \) is a hamilton-connected graph and \( H \) is a tree or \( G \) is a hamiltonian graph and \( H \) is \( K_2 \).

1. Introduction

In this paper, we consider finite simple graphs, and refer to [1] for terms and notations not defined here. Let \( G = (V, E) \) be a graph. For any vertex \( v \in V \), let \( d_G(v) \) denote the degree of \( v \) in \( G \), and \( \Delta(G) \) denote the maximum degree of \( G \). Let \( c(G) \) be the number of components in \( G \). Denote \( P_m, C_n \) and \( K_{1,j} \) to be a path with \( m \) vertices \((m \geq 2)\), a cycle with \( n \) vertices \((n \geq 3)\) and a star with \( j \) vertices \((j \geq 1)\), respectively.

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. The Cartesian product of \( G_1 \) and \( G_2 \), denoted by \( G_1 \square G_2 \), is the graph with vertex set \( V_1 \times V_2 \) such that the vertices \((x_1, y_1)\) and \((x_2, y_2)\) are adjacent if and only if either \( x_1 = x_2 \in V_1 \) with \( y_1 y_2 \in E_2 \), or \( y_1 = y_2 \in V_2 \) with \( x_1 x_2 \in E_1 \). It follows the definition that for any \((x, y) \in V(G)\),

\[
d_{G_1 \square G_2}(x, y) = d_{G_1}(x) + d_{G_1}(y).
\]

For any \( y \in V_2 \), define \( G_{1y} \) to be the graph with vertex set \( V_{1y} = \{(x, y) \mid x \in V_1\} \) and edge set \( E_{1y} = \{(x_1, y)(x_2, y) \mid x_1 x_2 \in E_1\} \). Similarly, For any \( x \in V_1 \), define \( G_{2x} \) to be the graph with vertex set \( V_{2x} = \{(x, y) \mid x \in V_2\} \) and edge set \( E_{2x} = \{(x, y_1)(x, y_2) \mid y_1 y_2 \in E_2\} \).

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Note that $G_{1y}$ is isomorphic to graph $G_1$, for any $y \in V_2$; and that $G_{2x}$ is isomorphic to graph $G_2$ for any $x \in V_1$. It is clear that

$$V_{1y} \cap V_{1y'} = \emptyset, \ E_{1y} \cap E_{1y'} = \emptyset \text{ for } y \neq y';$$

$$V_{2x} \cap V_{2x'} = \emptyset, \ E_{2x} \cap E_{2x'} = \emptyset \text{ for } x \neq x';$$

$$V_{1y} \cap V_{2x} = \{(x, y)\} \text{ for } x \in V_1; \ y \in V_2;$$

$$E(G_1 \square G_2) = (\cup_{y \in V_2} E_{1y}) \cup (\cup_{x \in V_1} E_{2x});$$

$$V(G_1 \square G_2) = (\cup_{y \in V_2} V_{1y}) = (\cup_{x \in V_1} V_{2x}).$$

A spanning path(cycle) is called a Hamilton path(cycle). A graph $G$ is traceable if it contains a Hamilton path, and hamiltonian if it contains a Hamilton cycle. A graph $G$ is hamilton - connected if there exists a Hamilton path joining any two different vertices of $G$.

Let $F$ be a subgraph of a graph $G$. An ear of $F$ in $G$ is a nontrivial path in $G$ whose ends lie in $F$ but whose internal vertices do not.

A graph $G$ is called a cactus if it has at least 3 vertices, all cycles of $G$ are vertex-disjoint, maximum degree of $G$ is 3 and all vertices of degree 3 are on a cycle of $G$.

We denote by $\mathcal{E}$ the class of graphs with following properties:

(i) any graph $H \in \mathcal{E}$ can be edge-covered by two subgraphs $H_C$ and $H_F$, such that $H = H_C \cup H_F$, $H_C$ and $H_F$ are edge-disjoint, $H_C$ is an edge disjoint union of cycles $C_1, \cdots, C_p$, and $H_F$ is a forest.

(ii) there is no vertex in $H_C$ common to more than two cycles among the cycles $C_1, \cdots, C_p$.

(iii) $H \in \mathcal{E}$ has at least two vertices.

We call the pair $(H_C, H_F)$ a cycle - tree covering of $H$.

A graph $H \in \mathcal{E}$ satisfying the following:

(i) for every vertex on exactly one cycle of $H_C$ in the cycle-tree covering of $H$ all its neighbors are either all pendent (vertices of degree one) or all nonpendant, i.e. for such vertex $u$ we have either $d_G^p(u) \geq 0$ and $d_G^{np}(u) = 2$ or $d_G^p(u) = 0$ and $d_G^{np}(u) \geq 2$,

(ii) a vertex common to exactly two cycles in the cycle-tree covering of $H$ has neighbors on these cycles only;

is called generalized cactus. In particular, such cactus is even if all its cycles are of even length.

The generalized b- cactus is a generalized cactus with every branch vertex (vertices of degree more than 3) on a cycle.

We denote by $\mathcal{F}$ the class of graphs with following properties:

(i) any graph $G \in \mathcal{F}$ is union of cycles $C_1, \cdots, C_p$,
(ii) for two cycles for $C_i, C_{i+1}, |V(C_i \cap C_{i+1})| = l_i, 1 \leq i \leq p - 1, 1 \leq l_i \leq \max\left\{\frac{|V(C_i)|}{2}, \frac{|V(C_{i+1})|}{2}\right\}$

(iii) at least one of these cycles is odd cycle.

When $p = 2, l_i = l$, graph $G$ is denoted by $\Theta(l, m, n)$, where $|V(C_1)| = n, |V(C_2)| = m$.

Here we mention some related results. Gould in [4] raised a research problem to find natural conditions to assure the product of two graphs to be hamiltonian. Paulraja in [11] gave the sufficient and necessary conditions for the prism over graphs to be hamiltonian. And Lu et.al in [9] present some sufficient and necessary conditions for $G \Box H$ being hamiltonian when $G$ is a hamiltonian graph and $H$ is tree.

The followings are some results related with our main Theorem.

Theorem 1.1. [1] Let $S$ be a set of vertices of a hamiltonian graph $G$. Then $c(G - S) \leq |S|$.

Theorem 1.2. [11] Let $G$ be a graph. The Cartesian product $G \Box K_2$ is hamiltonian if and only if $G$ has an even generalized b-cactus as a subgraph.

Lemma 1.3. [9] Let $C_n$ be a cycle ($n \geq 3$). For any tree $T$, if $T$ contains a subdivision of $K^{(n)}_{1,3}$ as a subgraph. Then $G = C_n \Box T$ is not traceable, where $K^{(n)}_{1,3}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$.

Lemma 1.4. [7] Suppose that $m$ is an odd integer. Then $C_m \Box K_2$ is hamilton-connected.

Note that when $m$ is an even integer, $C_m \Box K_2$ is not hamilton-connected. In this paper, we shall investigate the sufficient and necessary conditions for $G \Box H$ being hamilton-connected when $G$ is a hamilton-connected graph and $H$ is a tree or $G$ is hamiltonian graph and $H$ is $K_2$. Our main theorems are as follows:

Theorem 1.5. Let $G$ be a hamilton-connected graph, and let $T$ be a tree with maximum degree $\triangle$. Then graph $G \Box T$ is hamilton-connected if and only if $\triangle(T) \leq |V(G)| - 1$ and $T$ contains no subdivision of $K^{(n)}_{1,3}$ as a subgraph, where $K^{(n)}_{1,3}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$.

Theorem 1.6. Let $G$ be a hamiltonian graph. The Cartesian product $G \Box K_2$ is hamilton-connected if and only if:

(i) $G$ is of odd order, or

(ii) $G$ is of even order and $G$ contains $\Theta(1, 2k + 1, 2l + 1)$ as a spanning subgraph, where $\Theta(1, 2k + 1, 2l + 1)$ is the graph union of two odd cycles with a common edge.

2. Proof of Theorem $1.5$

In this section we will give a proof of Theorem $1.5$.

Lemma 2.1. Let $T = K_{1,m}$ be a star, and let $G$ be a hamilton-connected graph with $n$ vertices ($n \geq 3$). If $m \leq n - 1$, then the graph $H = G \Box T$ is hamilton-connected.
Proof. Let $V(K_{1,m}) = \{y_0, y_1, \ldots, y_m\}$, where $d(y_0) = m$ and $d(y_i) = 1$ for $1 \leq i \leq m$. Then $G_{1y_i} \cong G$ for $i = 0, 1, \ldots, m$. Particularly, $V(G_{1y_i}) = \{(v_1, y_i), (v_2, y_i), \ldots, (v_n, y_i)\}$ for $i = 0, 1,\ldots, m$. We shall determine a Hamilton path between any two given vertices in $G \Box T$. We distinguish the following cases.

Case 1. Any $(v_l, y_0), (v_f, y_0) \in G_{1y_0}$, $1 \leq l, f \leq n$.

Because graph $G$ is hamilton-connected, there exists a Hamilton path joining any two distinct vertices of the graph $G$. Let $P_0$ be a Hamilton path of $G_{1y_0}$ between $(v_l, y_0)$ and $(v_f, y_0)$. Let $x_1 = v_l$, $x_n = v_f$ and $P_0 = (x_1, y_0), (x_2, y_0), \ldots, (x_n, y_0))$. Let $P_1$ be a Hamilton path of $G_{1y_i}$ between $(x_i, y_i)$ and $(x_{i+1}, y_i)$. Then

$$P = (P_0 - \{(x_1, y_0), (x_2, y_0), \ldots, (x_m, y_0)\}) \cup P_1 \cup P_2 \cup \cdots \cup P_m \cup \{(x_1, y_0), (x_2, y_0), \ldots, (x_m, y_0)\}$$

is a Hamilton path between $(v_l, y_0)$ and $(v_f, y_0)$ in $H$.

Case 2. Any $(v_l, y_i), (v_f, y_i) \in G_{1y_i}$, $1 \leq l, f \leq n$, $1 \leq i \leq m$.

Let $P_1$ be a Hamilton path between $(v_{l+1}, y_i)$ and $(v_{f+1}, y_i)$ in $G_{1y_i}$. From Case 1, we can find a Hamilton path $P_2$ between $(v_l, y_0)$ and $(v_{l+1}, y_0)$ in $G \Box (T - y_i)$. Then

$$P = P_1 \cup P_2 \cup \{(v_{l+1}, y_0)(v_l, y_0)\}$$

is a Hamilton path between $(v_l, y_0)$ and $(v_f, y_0)$ in $H$.

Case 3. Any $(v_l, y_i), (v_l, y_j) \in G_{1y_j}$, $1 \leq l, f \leq n$, $1 \leq i, j \leq m$, $i \neq j$.

Let $P_1$ be a Hamilton path between $(v_l, y_i)$ and $(v_r, y_i)$ in $G_{1y_i}$ for $1 \leq p \leq m$. Let $P_2$ be a Hamilton path between $(v_f, y_j)$ and $(v_r, y_j)$ in $G_{1y_j}$ for $1 \leq q \leq m$. From Case 1, we can find a Hamilton path $P_3$ between $(v_r, y_0)$ and $(v_r, y_0)$ in $G \Box (T - \{y_i, y_j\})$. Then

$$P = P_1 \cup P_2 \cup \{(v_r, y_0)(v_r, y_0), (v_q, y_0)(v_q, y_0)\}$$

is a Hamilton path between $(v_l, y_i)$ and $(v_f, y_j)$ in $H$.

Case 4. Any $(v_l, y_i), (v_f, y_i) \in G_{1y_i}$, $1 \leq l, f \leq n$, $1 \leq i \leq m$.

Let $P_1$ be a Hamilton path between $(v_l, y_i)$ and $(v_f, y_i)$ in $G_{1y_i}$. From Case 1, we can find a Hamilton path $P_2$ between $(v_r, y_0)$ and $(v_s, y_0)$ in $G \Box (T - y_i)$ for $\{(v_s, y_i)(v_r, y_i)\} \in P_1$ for $1 \leq r, s \leq n$. Then

$$P = (P_1 - \{(v_r, y_i)(v_s, y_i)\}) \cup \{(v_r, y_i)(v_r, y_0), (v_s, y_i)(v_s, y_0)\} \cup P_2$$

is a Hamilton path between $(v_l, y_i)$ and $(v_f, y_i)$ in $H$.

$\square$
Remark 2.2. By the argument used in Case 1 in the proof of Lemma 2.1, \( |E_{1y_0} \cap E(P)| = \emptyset \) if \( m = n - 1 \). If \( n - 1 > m \), then \( (x_i, y_0)(x_{i+1}, y_0) \in E(P) \) for \( m \leq i \leq (n - 1) \), that is, \( |E_{1y_0} \cap E(P)| = n - m - 1 \). If \( n - 1 < m \), there exists no Hamilton path between \((v_1, y_0), (v_f, y_0) \in G_{1y_0}\).

Corollary 2.3. Let \( G \) be a hamilton-connected graph with \( n \) vertices \((n \geq 3)\). Then the graph \( H = G\Box K_{1,n} \) is not hamilton-connected.

Recall that \( K_{1,3}^{(n)} \) is the graph obtained by identifying every degree 1 vertex of a \( K_{1,3} \) with the center of a \( K_{1,n} \). Note that \( \Delta(K_{1,3}^{(n)}) = n + 1 \). Since a hamilton-connected graph is also hamiltonian, by Lemma 1.3 in \[9\], we have the following corollary.

Corollary 2.4. Let \( G \) be a hamilton-connected graph with \( n \) vertices \((n \geq 3)\). If \( T \) contains a subdivision of \( K_{1,3}^{(n)} \) as a subgraph, then the graph \( H = G\Box T \) is not hamilton-connected.

Proof of Theorem 1.5

Let \( H = G\Box T \) be a hamilton-connected graph and \( \Delta(T) \geq n + 1 \), where \( |V(G)| = n \). If there exists \( y \in V(T) \) such that \( d_T(y) \geq n + 1 \), then \( c(H - G_{1y}) = c(T - y) = d_T(y) \geq n + 1 \). By Theorem 1.1, \( H = G\Box T \) is not hamiltonian and hence is not hamilton-connected, a contradiction. If \( \Delta(T) = V(G) = n \), by Corollary 2.3, \( H \) is not hamilton-connected. Therefore \( \Delta(T) \leq n - 1 \). By Corollary 2.4, \( T \) contains no subdivision of \( K_{1,3}^{(n)} \) as a subgraph.

So it suffices to show that if \( \Delta(T) \leq n - 1 \) and \( T \) contains no subdivision of \( K_{1,3}^{(n)} \) as a subgraph, then \( H = G\Box T \) is a hamilton-connected graph. If \( T \) is a star, then it follows from Lemma 2.1. Therefore we may assume that \( T \) is not a star. By way of contradiction, let \( T \) be a tree with minimal number of vertices such that \( \Delta(T) \leq n - 1 \), \( T \) contains no subdivision of \( K_{1,3}^{(n)} \) as a subgraph and \( H = G\Box T \) is not hamilton-connected.

Note that \( T \) can be viewed as a graph obtained from finite stars \( T_1, T_2, \ldots, T_k \) by connecting their centers with edges and there exists such a star \( T_i \) that is connected to the other stars with only one edge. Without lose of generality, we may assume that \( T_1 \) is only connected to \( T_2 \). Let \( y_i \) the center of \( T_i \) \((i = 1, 2)\). Since \( T - T_1 \) is also a tree and \( |V(T - T_1)| \leq |V(T)| \), \( G\Box(T - T_1) \) is hamilton-connected. Since \( \Delta(T_1) \leq n - 1 \), by Lemma 2.1, \( G\Box T_1 \) is hamilton-connected.

Now we shall construct a Hamiltonian path between any two distinct vertices of \( H = G\Box T \), and then obtain a contradiction.

Case 1. Any \((v_l, y_i), (v_f, y_j) \in G\Box T_1, 1 \leq l, f \leq n\).

Let \( P_1 \) be a Hamilton path between \((v_l, y_i)\) and \((v_f, y_j) \) in \( G\Box T_1 \). Since \( d_{T_1}(y_1) \leq n - 2 \), at least one edge of \( G_{1y_1} \) lies in \( P_1 \). By Remark 2.2, we may assume \((v_1, y_1)(v_2, y_1) \in E_{1y_1} \cap E(P_1) \). Then there exists a Hamilton path \( P_2 \) between \((v_1, y_1), (v_2, y_2) \) in \( G\Box(T - T_1) \). Hence

\[
P = P_2 \cup (P_1 - \{(v_1, y_1)(v_2, y_1)\}) \cup \{(v_1, y_1)(v_2, y_2), (v_2, y_1)(v_2, y_2)\}
\]

is a Hamilton path between \((v_l, y_i)\) and \((v_f, y_j) \) in \( G\Box T \).
Case 2. Any \((v_l, y_l), (v_f, y_f) \in G \square (T - T_1), \ 1 \leq l, f \leq n.\)

The proof of this case is similar to the proof of Case 1. So it is omitted.

Case 3. Any \((v_l, y_l) \in G \square T_1, \ (v_f, y_f) \in G \square (T - T_1), \ 1 \leq l, f \leq n.\)

Let \(P_l\) be a Hamilton path between \((v_l, y_l)\) and \((v_l, y_1)\) in \(G \square T_1\). Let \(P_f\) be a Hamilton path between \((v_f, y_2)\) and \((v_f, y_j)\) in \(G \square (T - T_1)\). Then

\[
P = P_l \cup P_f \cup \{(v_l, y_l)(v_l, y_2)\}
\]

is a Hamilton path between \((v_l, y_l)\) and \((v_f, y_f)\) in \(G \square T\).

3. Proof of Theorem 3.1

In this section we will give the sufficient and necessary condition for \(G \square K_2\) being hamilton-connected. We know that prism over odd cycle is hamilton-connected\(^7\), but the prism over even cycle is not hamilton-connected. We now can consider the case when \(G\) contains \(\Theta(1, 2k + 1, 2l + 1)\) or \(\Theta(1, 2k + 1, 2l + 1)\) as a spanning subgraph. Note that \(G\) a is hamiltonian graph with even order.

Lemma 3.1. Let \(V(P_m) = \{x_1, x_2, \ldots, x_m\}\), and let \(X, Y\) be the bipartite partition of bipartite graph \(G = P_m \square K_2\). Then there exists Hamilton path joining any two vertices \((x_e, y_i) \in X\) and \((x_f, y_j) \in Y\) for \(1 \leq e, f \leq m, 1 \leq i, j \leq 2\), but no Hamilton path joining \((x_1, y_1) \in X\) and \((x_1, y_2) \in Y\) for \(1 < l < m\).

Proof. Let \(V(P_m) = \{x_1, x_2, \ldots, x_m\}\), \(V(K_2) = \{y_1, y_2\}\), and \(V(P_m \square K_2) = \{(x_1, y_1), (x_2, y_1), \ldots, (x_m, y_1), (x_1, y_2), (x_2, y_2), \ldots, (x_m, y_2)\}\). We can see that \(P_m \square K_2\) is a bipartite graph. Let \(X\) and \(Y\) be the bipartite partition of the \(G\). We will show that for any \((x_e, y_i) \in X\) and \((x_f, y_j) \in Y\) for \(1 \leq e, f \leq m, 1 \leq i, j \leq 2\) there exists a Hamilton path joining them, but no Hamilton path joining \((x_1, y_1) \in X\) and \((x_1, y_2) \in Y\) for \(1 < l < m\).

Obviously, there exists Hamilton path between \((x_1, y_1)\) and \((x_1, y_2)\) or \((x_m, y_1)\) and \((x_m, y_2)\) in \(G\). But no Hamilton path between \((x_l, y_1) \in X\) and \((x_l, y_2) \in Y\) for \(1 < l < m\).

Now consider any \((x_e, y_i) \in X\) and \((x_f, y_j) \in Y\), where \(e \neq f\).

By induction on \(m\) suppose that it is true for \(P_k \square K_2\) with \(k < m\). Let \(P_l = \langle x_1, x_2, \ldots, x_{f-1} \rangle\), \(P_f = \langle x_f, x_{f+1}, \ldots, x_m \rangle\). We may assume \(x_e \in V(P_l)\). By the induction hypothesis, there is a Hamilton path \(P'\) between \((x_e, y_1)\) and \((x_{f-1}, y_{j+1})\) in \(P_l \square K_2\) and there exists Hamilton a path \(P''\) between \((x_f, y_j)\) and \((x_f, y_{j+1})\), where \((x_f, y_j)(x_f, y_{j+1}) \in E(P_l \square K_2)\). Then

\[
P = P' \cup P'' \cup \{(x_f, y_{j+1})(x_{f-1}, y_{j+1})\}
\]

is a Hamilton path between \((x_e, y_1)\) and \((x_f, y_j)\) in \(G\).

Proposition 3.2. Let \(G = \Theta(1, 2k, 2l)\). Then \(H = G \square K_2\) is not hamilton-connected.
Proof. Let $C_1$ and $C_2$ be two even cycles, and let $V(C_1) = \{x_1, x_2, \ldots, x_{2k}\}$, $V(C_2) = \{x_1, x_2, x_3, \ldots, x_{2l}\}$, $E(C_1 \cap C_2) = \{x_1x_2\}$, $V(K_2) = \{y_1, y_2\}$.

We suppose that $G \Box K_2$ is hamilton-connected. There exists a Hamilton path $P$ between $(x_1, y_1)$ and $(x_2, y_2)$. Let $C = G \setminus \{x_1x_2\}$, and note that $C$ is an cycle with even order. Then $C \Box K_2$ is not hamilton-connected, and there exists no Hamilton path between $(x_1, y_1)$ and $(x_2, y_2)$ in $C \Box K_2$. So edges $(x_1, y_1)(x_2, y_1)$ or $(x_1, y_2)(x_2, y_2)$ must be contained in $P$, and one of them must be the first or last edge of $P$.

Now let $P_1 = (x_3, x_4, \ldots, x_{2k})$, and let $P_2 = (x_3', x_4', \ldots, x_{2l})$. Say edge $(x_1, y_1)(x_2, y_1)$ is first edge of $P$. From the argument earlier, there must exist Hamilton path between $(x_3, y_1)$ and $(x_{2k}, y_2)$ in $P_1 \Box K_2$. Because $(x_3, y_1)$ and $(x_{2k}, y_2)$ is in same partite in $P_1 \Box K_2$, by the Lemma 3.1, there exists no Hamilton path joining $(x_3, y_1)$ and $(x_{2k}, y_2)$, contradiction. \hfill\Box

Lemma 3.3. Let $G = \Theta(1, 2k + 1, 2l + 1)$. Then $H = G \Box K_2$ is hamilton-connected.

**Proof.** Let $C_1$ and $C_2$ be two odd cycles, and $V(C_1) = \{x_1, x_2, \ldots, x_{2k+1}\}$, $V(C_2) = \{x_1, x_2, x_3, \ldots, x_{2l+1}\}$, $E(C_1 \cap C_2) = \{x_1x_2\}$, $V(K_2) = \{y_1, y_2\}$.

**Case 1.** Any $(x_e, y_i), (x_f, y_j) \in V(C_1 \Box K_2)$, $1 \leq e, f, \leq 2k+1$, $1 \leq i, j \leq 2$.

Let $P_1 = (x_3, \ldots, x_e)$, $P_2 = (x_{e+1}, \ldots, x_f, \ldots, x_{2k+1})$.

Let $X_1, Y_1$ be bipartite partition of $P_1 \Box K_2$; $X_2, Y_2$ be bipartite partition of of $P_2 \Box K_2$. Without loss of generality, we may assume that $(x_3, y_1) \in X_1$, $(x_3, y_2) \in Y_1$, and $(x_{2k+1}, y_1) \in X_2$, $(x_{2k+1}, y_2) \in Y_2$. If $(x_e, y_i) \in X_1$ (if $(x_e, y_i) \in Y_1$ we can choose the $(x_3, y_1)$), by Lemma 3.1, we can find a Hamilton path $P_3$ between $(x_e, y_i)$ and $(x_3, y_2)$ in $P_1 \Box K_2$. Similarly if $(x_f, y_j) \in X_2$ (if $(x_f, y_j) \in Y_2$ we can choose the $(x_{2k+1}, y_1)$), we can find a Hamilton path $P_4$ between $(x_f, y_j)$ and $(x_{2k+1}, y_2)$ in $P_2 \Box K_2$.

By Lemma 1.4, there is a Hamilton path $P_5$ between $(x_1, y_2)$ and $(x_2, y_2)$ in $C_2 \Box K_2$. Then

$$P = P_1 \cup P_2 \cup \{(x_1, y_2)(x_{2k+1}, y_2), (x_2, y_2)(x_3, y_2)\}$$

is a Hamilton path between $(x_e, y_i)$ and $(x_f, y_j)$ in $G \Box K_2$.

**Case 2.** Any $(x_e', y_i'), (x_f', y_j') \in V(C_2 \Box K_2)$, $3 \leq e, f \leq 2l+1$, $1 \leq i, j \leq 2$.

The proof of this case is similar to that of Case 1. So it is omitted.

**Case 3.** Any $(x_e, y_i) \in V(C_1 \Box K_2)$, $(x_f', y_j) \in V(C_2 \Box K_2)$, $1 \leq e \leq 2k+1$, $3 \leq f \leq 2l+1$, $1 \leq i, j \leq 2$.

Let $P_{2l-2} = (x_3, x_4, \ldots, x_{2l+1})$ and let $X, Y$ be bipartite partition of $P_{2l-2} \Box K_2$. Consider the vertices $(x_{2l+1}, y_1), (x_{2l+1}, y_2)$ or $(x_3, y_1), (x_3, y_2)$ that are in different partite sets. Say the former, we can assume $(x_{2l+1}, y_1) \in X, (x_{2l+1}, y_2) \in Y$.

Without loss of generality, we may assume $(x_f', y_j) \in X$, by Lemma 3.1, there is a Hamilton path $P_1$ between $(x_f', y_j)$ and $(x_{2l+1}, y_2)$ in $P_{2l-2} \Box K_2$. Also there is a Hamilton path $P_2$ between $(x_1, y_2)$ and $(x_e, y_i)$ in $C_1 \Box K_2$. Then

$$P = P_1 \cup P_2 \cup \{(x_{2l+1}, y_2)(x_1, y_2)\}$$
is a Hamilton path between \((x_e, y_i)\) and \((x_f, y_j)\) in \(G \Box K_2\).

\[\Box\]

**Proposition 3.4.** Suppose that \(H = C_n \Box K_2\) is a hamilton-connected graph. Then \(n\) is an odd integer.

**Proof.** Let \(V(C_n) = \{x_1, x_2, \ldots, x_n\}\), \(V(K_2) = \{y_1, y_2\}\). Let \(P_1\) be a Hamilton path between \((x_1, y_1)\) and \((x_2, y_2)\), then \(P_1 = \langle (x_1, y_1), (x_1, y_2), (x_n, y_1), (x_n, y_2), (x_{n-1}, y_1), \ldots, (x_3, y_1), (x_2, y_1), (x_2, y_2) \rangle\) and \(P_1\) contains every pillar of the prism. Let \(X = V(G_{1y_1})\), then \(|\partial(X)| = 2t + 1\), where \(\partial(X)\) is set of edges with one end in \(X\).

Since \(H = C_n \Box K_2\) is an odd graph, and by \(|\partial(X)| + 2e(X) = \sum_{v \in X} d(v)\), we have \(|\partial(X)| = |X| \mod 2\), so \(|X| = |V(G_{1y_1})| = n\) is an odd integer.

Conclude all the above, we can obtain the sufficient and necessary condition for prism over hamiltonian graph being hamilton-connected, that is Theorem 1.6. Therefore we can give a sufficient condition for \(G \Box P_m\) being hamilton-connected.

**Theorem 3.5.** Let \(m\) be an integer, where \(m \geq 2\). Then \(H = G \Box P_m\) is hamiltonian-connected, if:

(i) \(G\) is a hamiltonian graph with odd order, or

(ii) \(G\) is a hamiltonian graph with even order, and \(G\) contains \(\Theta(1, 2k + 1, 2l + 1)\) as a spanning subgraph, where \(\Theta(1, 2k + 1, 2l + 1)\) is the graph union of two odd cycles with a common edge.

**Proof.** Let \(C_n\) be a Hamilton cycle of \(G\), \(V(C_n) = \{x_1, x_2, \ldots, x_n\}\), \(V(P_m) = \{y_1, y_2, \ldots, y_m\}\).

We prove it by induction on \(m\).

When \(m = 2\), by Theorem 1.6 it is right. Now we suppose that \(G \Box P_{m-1}\) is hamilton-connected when \(G\) satisfies those conditions. We shall show that \(H = G \Box P_m\) is hamilton-connected.

**Case 1.** Any \((x_e, y_i), (x_f, y_j) \in V(C_n \Box P_{m-1})\), \(1 \leq e, f \leq n\), \(1 \leq i, j \leq m - 1\).

By induction hypothesis there is a Hamilton path \(P_1\) between \((x_e, y_i)\) and \((x_f, y_j)\) in \(G \Box P_{m-1}\). Without loss of generality, assume \((x_i, y_{m-1})(x_{i+1}, y_{m-1}) \in E(C_n \Box P_{m-1} \cap P_1)\). Then

\[
P = (P_1 \setminus \{(x_i, y_{m-1})(x_{i+1}, y_{m-1})\}) \cup \{(x_i, y_{m-1})(x, y_m), (x_{i+1}, y_{m-1})(x_{i+1}, y_m)\} \cup \{(x_i, y_m)(x_{i-1}, y_m), (x_{i-2}, y_m), \ldots, (x_{i+2}, y_m)(x_{i+1}, y_m)\}
\]

is a Hamilton path between \((x_e, y_i)\) and \((x_f, y_j)\) in \(H\).

**Case 2.** Any \((x_e, y_i) \in V(C_n \Box P_{m-1})\), \((x_f, y_m) \in V(G_{1y_m})\), \(1 \leq e, f \leq n\), \(1 \leq i \leq m - 1\).

By induction hypothesis there is a Hamilton path \(P_1\) between \((x_e, y_i)\) and \((x_{f+1}, y_{m-1})\) in \(G \Box P_{m-1}\). Then

\[
P = P_1 \cup \{(x_{f+1}, y_{m-1})(x_{f+1}, y_m)\} \cup \{(x_{f+1}, y_m)(x_{f+2}, y_m), (x_{f+2}, y_m)(x_{f+3}, y_m), \ldots, (x_{f-1}, y_m)(x_f, y_m)\}
\]

is a Hamilton path between \((x_e, y_i)\) and \((x_f, y_m)\) in \(H\).
Case 3. Any \((x_e, y_m), (x_f, y_m) \in V(G_{1y_m}), 1 \leq e, f \leq n.\)

Let \(P_1 = \langle (x_{e+1}, y_m), (x_{e+2}, y_m), \ldots, (x_f, y_m) \rangle, P_2 = \langle (x_{f+1}, y_m), (x_{f+2}, y_m), \ldots, (x_e, y_m) \rangle.\) By induction hypothesis there is a Hamilton path \(P_3\) between \((x_{e+1}, y_{m-1})\) and \((x_{f+1}, y_{m-1})\) in \(G \square P_{m-1}.\) Then

\[
P = P_3 \cup \{ (x_{e+1}, y_{m-1}) (x_{e+1}, y_m), (x_{f+1}, y_{m-1}) (x_{f+1}, y_m) \} \cup P_1 \cup P_2
\]
is a Hamilton path between \((x_e, y_m)\) and \((x_f, y_m)\) in \(H.\)

By the proof of the Lemma 3.3 we can obtain the following corollary.

**Corollary 3.6.** Let \(G\) be a graph such that \(G \square K_2\) is a hamilton-connected graph. If \(H = G \cup P\) with \(|E(G \cap P)| = 1\), where \(P\) is an ear of \(G\), then \(H \square K_2\) is hamilton-connected.

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### References


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