UNITS IN $\mathbb{Z}_2(C_2 \times D_\infty)$

R. K. SHARMA*, P. YADAV AND K. JOSHI

Communicated by Victor Bovdi

Abstract. In this paper we consider the group algebra $R(C_2 \times D_\infty)$. It is shown that $R(C_2 \times D_\infty)$ can be represented by a $4 \times 4$ block circulant matrix. It is also shown that $U(\mathbb{Z}_2(C_2 \times D_\infty))$ is infinitely generated.

1. Introduction

Let $RG$ denote the group ring of a group $G$ over a ring $R$, and $U(RG)$ denote the unit group of $RG$. A lot is known about the unit group of group rings of finite groups. For details see [9], [1] and [2]. In this paper we deal with units in the group algebra $R(C_2 \times D_\infty)$ over a commutative integral domain $R$, by representing it by $4 \times 4$ block circulant matrix. The idea that the group ring $RD_{2n}$ can be written as a block matrix was introduced by Hurley in [6]. Additionally this method was also used by Gildea in [4], [5] to establish the structure of certain unit groups of group algebras. Maciez Mirowicz in [7] studied the group of units $U(RD_\infty)$ of the group ring of the infinite dihedral group $D_\infty$ over a commutative integral domain $R$. He obtained the structure of $U(\mathbb{Z}_2D_\infty)$. In this paper, we extend his results and obtain some subgroups of the unit group $U(\mathbb{Z}_2(C_2 \times D_\infty))$. We have shown that $U(\mathbb{Z}_2(C_2 \times D_\infty))$ is not finitely generated.

Let $R$ be a commutative domain with unity. The infinite dihedral group is a two generator group with a known presentation as:

$$D_\infty = \langle t, x \mid x^2 = 1, xt = t^{-1}x \rangle.$$
C_2 is the cyclic group of order 2 generated by y, that is, C_2 = \langle y \rangle. Since the canonical form of elements of D_\infty is t^i x^j for some \( i \in \mathbb{Z} \) and \( 0 \leq k \leq 1 \) and y commutes with t and x we can write any element \( \alpha \in R(C_2 \times D_\infty) \) in the form: \( \alpha = (a + bx) + (c + dx)y \), where \( a, b, c, d \in RC_\infty \), where \( C_\infty \) denotes an infinite cyclic group.

Let \( C_\infty = \langle t \rangle \) be an infinite cyclic group generated by \( t \) and let \( * : RC_\infty \rightarrow RC_\infty \) be the involution map of the group ring \( RC_\infty \) which comes from the non-trivial automorphism of the group \( C_\infty \), that is, \( \left( \sum_{i \in \mathbb{Z}} a_i t^i \right)^* := \sum_{i \in \mathbb{Z}} a_i t^{-i} \). We can easily get that for any \( a \in RC_\infty \) the relation \( xa = a^* x \) holds.

\[ \begin{align*}
2. \text{Units in } \mathbb{Z}_2(C_2 \times D_\infty) \\
\end{align*} \]

In this section we obtain some infinitely generated subgroups of the unit group of \( \mathbb{Z}_2(C_2 \times D_\infty) \). First, we prove some lemmas.

**Lemma 2.1.** Let \( \theta : R(C_2 \times D_\infty) \rightarrow M_4(\mathbb{R}C_\infty) \) defined by

\[ \theta((a + bx) + (c + dx)y) = \begin{pmatrix} a & b & c & d \\ b^* & a^* & d^* & c^* \\ c & d & a & b \\ d^* & c^* & b^* & a^* \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \]

where \( A = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \) and \( B = \begin{pmatrix} c & d \\ d^* & c^* \end{pmatrix} \). Then \( \theta \) is a monomorphism.

**Proof.** Let \( \alpha = (a_1 + b_1 x) + (c_1 + d_1 x)y \) and \( \beta = (a_2 + b_2 x) + (c_2 + d_2 x)y \). Then \( \alpha \beta = (p + q x) + (r + s x)y \), where \( p = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2 \), \( q = a_1 b_2 + a_2 b_1 + c_1 d_2 + c_2 d_1 \), \( r = a_1 c_2 + a_2 c_1 + b_1 d_2 + b_2 d_1 \) and \( s = a_1 d_2 + a_2 d_1 + b_1 c_2 + b_2 c_1 \).

Now, \( \theta(\alpha \beta) = \begin{pmatrix} p & q & r & s \\ q^* & p^* & s^* & r^* \\ r & s & p & q \\ s^* & r^* & q^* & p^* \end{pmatrix} = \theta(\alpha) \theta(\beta) \). Hence \( \theta \) is a homomorphism.

As \( \theta((a + bx) + (c + dx)y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow a = b = c = d = 0 \), thus \( \theta \) is one-one. Hence the lemma.

**Lemma 2.2.** If \( R \) is a commutative domain then \( \alpha \in \mathcal{U}(RG) \Leftrightarrow \det(\alpha) \in \mathcal{U}(R) \), where \( G \) is \( (C_2 \times D_\infty) \).
Proof. Let $\alpha = \begin{pmatrix} a & b & c & d \\ b^* & a^* & d^* & c^* \\ c & d & a & b \\ d^* & c^* & b^* & a^* \end{pmatrix}$.

Then, $\det(\alpha) = (aa^* - bb^*)^2 + (cc^* - dd^*)^2 + 2(abc^*d^* + a^*b^*cd + ab^*c^*d + a^*bcd^*) - 2(aa^*dd^* + bb^*cc^*) - (a^2c^2 + a^2e^2 + b^2d^2 + b^2e^2)$. Hence we define $\deg \alpha := 0$.

Thus we get, $(\det(\alpha))^* = \det(\alpha)$.

Suppose $\alpha \in \mathcal{U}(RG)$ then there exists $\beta \in RG$ such that $\alpha \beta = 1 \Rightarrow \det(\alpha)\det(\beta) = \det(1) = 1$. Thus $\det(\alpha) \in \mathcal{U}(RC_\infty)$. But $\mathcal{U}(RC_\infty) = \{rt^i \mid i \in \mathbb{Z}, r \in R\}$. Thus we have $\det(\alpha) = rt^i$ for some $i$. But as $(\det(\alpha))^* = \det(\alpha)$, we get $(rt^i)^* = rt^i$. This gives $rt^{-i} = rt^i \Rightarrow i = 0$. Hence $\det(\alpha) = r \in R$.

For $0 \neq a = \sum_{i \in \mathbb{Z}} \alpha_i t^i \in RC_\infty$ we fix:

$$\begin{align*}
\max a & := \max \{ i \mid \alpha_i \neq 0 \} \\
\min a & := \min \{ i \mid \alpha_i \neq 0 \} \\
\deg a & := \max a - \min a = \max aa^*
\end{align*}$$

If $\alpha = a + bxy \in R(C_2 \times D_\infty) \cong \begin{pmatrix} a & 0 & 0 & b \\ 0 & a^* & b^* & 0 \\ 0 & b & a & 0 \\ b^* & 0 & 0 & a^* \end{pmatrix}$ is a non-trivial unit then $a \neq 0$, $b \neq 0$.

Thus $\det(\alpha) = (aa^* - bb^*)^2 \in \mathcal{U}(R)$ from Lemma 2.2. Now $(aa^* - bb^*) \in RC_\infty$. But $\mathcal{U}(RC_\infty) = \{rt^i \mid i \in \mathbb{Z}, r \in R\}$. Thus we have $aa^* - bb^* = rt^i$ for some $i$. But as $(aa^* - bb^*)^* = (aa^* - bb^*)$, we get $(rt^i)^* = rt^i$. This gives $rt^{-i} = rt^i \Rightarrow i = 0$. Therefore, $aa^* - bb^* \in \mathcal{U}(R)$. Hence we define $\deg \alpha = \max aa^* = \max bb^* = \deg b > 0$. For trivial units, we extend this definition by setting $\deg \alpha := 0$.

Let $\text{sgn}(i)$ denotes the sign of $i$. We consider special non-trivial nilpotent elements in the group ring $R(C_2 \times D_\infty)$:

$$\eta_{ij} = \begin{cases} 1 + \text{sgn}(i)t^jxy & t^j \neq t^jy \\ 1 - \text{sgn}(i)t^jxy & t^j = t^jy \end{cases} = \begin{cases} (-t^{-i} + t^i) & \text{sgn}(i)t^j(t^{-i} - t^i)xy \end{cases} \text{ for } i(\neq 0), j \in \mathbb{Z}.$$ 

Also $\eta_{ij}^2 = 0$ as

$$\begin{align*}
(\eta_{ij})^2 &= (1 \pm t^jxy)t^{ij}(1 \mp t^jy)(1 \mp t^jxy)t^{ij}(1 \mp t^jy) \\
&= (1 \pm t^jxy)t^{ij}(1 - (t^jy)^2)t^{ij}(1 \mp t^jy) \\
&= 0 \text{ because } (t^jy)^2 = 1.
\end{align*}$$

For any $r \in R$, $i$, $j \in \mathbb{Z}$, the element $1 + r\eta_{ij}$ is a unit in $R(C_2 \times D_\infty)$. Also inverse of $1 + r\eta_{ij}$ is $1 - r\eta_{ij}$ because

$$(1 + r\eta_{ij})(1 - r\eta_{ij}) = 1 - r^2(\eta_{ij})^2 = 1.$$
All the units of the above form generate a subgroup of the unit group of $R(C_2 \times D_\infty)$, so let

$$U = \langle 1 + r\eta_{ij} \rangle_{i,j \in \mathbb{Z}, r \in R}$$

For all $k > 0$, $j \in \mathbb{Z}$:

$$V_j^k = \langle 1 + r\eta_{ij} \rangle_{0 \leq i \leq k, r \in R}$$

Obviously, the groups $\{V_j^k\}$ form an ascending system. We set:

$$V_j = \lim_{k \to \infty} V_j^k.$$ 

Natural inclusions induce homomorphisms from the free products:

$$\phi_k : \ast V_j^k \to U \text{ for } k > 0 \text{ and } \phi = \lim_{k \to \infty} \phi_k : \ast V_j \to U.$$ 

Now we describe the groups $V_j^k$. Without loss of generality, we can take $\text{sgn}(i)$ and $\text{sgn}(l)$ to be $+ve$. Thus

$$\eta_{ij} \cdot \eta_{lj} = (1 + t^i xy)^l(1 - t^j xy)(1 + t^i xy)^l(1 - t^j xy)$$

$$= (1 + t^j xy)^l \cdot t^j(1 - t^j xy) = 0,$$

therefore the function $\sigma : R^k \to V_j^k$ given by:

$$\sigma(r_1, \ldots, r_k) = 1 + r_1\eta_{ij} + \ldots + r_k\eta_{kj}$$

$$= \left(1 - \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i \right) + t^j \left(\sum_{i=0}^{k} r_i t^{-i} - \sum_{i=0}^{k} r_i t^i \right) xy$$

$\sigma$ is an isomorphism from the additive group of $R^k$ onto the multiplicative group $V_j^k$. Therefore we obtain isomorphisms $V_j^k \to R^k$ and $V_j \cong \oplus_{i \geq 0} R$

**Lemma 2.3.** Let $k > 0$ and let $w \in \ast \mathbb{Z} V_j^k$ be a non-empty reduced word with the last letter $g$ (i.e., $l(wg^{-1}) < l(w)$), where $l$ denotes the length of the word. If $\phi_k(w) = a + bxy \in U \subseteq U(\mathbb{Z}_2(C_2 \times D_\infty))$, then:

(i) $\deg \phi_k w > 0$ (in particular $\phi_k$ is a monomorphism)

(ii) $g \in V_j^k \iff \max (t^{-j}b + a) < \max \{\max a, \max t^{-j}b\}$ or $\min (t^{-j}b + a) > \min \{\min a, \min t^{-j}b\}$.

**Proof.** We will prove the result by induction on the length of the word $w$. Let $l(w) = 1$. So, $w \in V_j^k$ for some $j$ and so we can write $w$ as:

$$w = \left(1 + \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i \right) + t^j \left(\sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i \right) xy.$$ 

Now, $\phi_k(w) = w \neq 1$ as $w$ is non-empty, hence we have $r_i \neq 0$ for some $1 \leq i \leq k$. Thus, $\deg \phi_k(w) \geq 2i > 0$.

Also, when $l(w) = 1$ then $g = w \in V_j^k$ for some fix $j$. Then $a = \left(1 + \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i \right)$ and $b = t^j \left(\sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i \right)$. This implies that

$$\max (t^{-j}b + a) = 0 < \max \{\max a, \max t^{-j}b\}.$$
Hence the result is true for length 1.

For \( c + dxy \in V_j^k \) we obtain some observations as follows:

1. \( c = c^* \), because

\[
\begin{align*}
  c &= \sum_{i=1}^{k} r_i t^{-i} + 1 + \sum_{i=1}^{k} r_i t^i \\
  c^* &= \sum_{i=1}^{k} r_i t^i + 1 + \sum_{i=1}^{k} r_i t^{-i},
\end{align*}
\]

thus we have, \( c + c^* = 2 \) which implies \( c = c^* \) as char of \( R = \mathbb{Z}_2 \) is 2.

2. \( d^* = dt^{-2j} \), because

\[
\begin{align*}
  d &= \left( \sum_{i=1}^{k} r_i t^{-i} + \sum_{i=1}^{k} r_i t^i \right) t^j \\
  d^* &= \left( \sum_{i=1}^{k} r_i t^i + \sum_{i=1}^{k} r_i t^{-i} \right) t^{-j},
\end{align*}
\]

thus we have, \( dt^{-j} = d^* t^j \) which implies \( d^* = dt^{-2j} \).

3. \( c + t^{-j}d = 1 \).

Now let us assume that the lemma holds for words of length \( n \geq 1 \). Suppose \( l(w) = n + 1 \) implies \( w = v \cdot g \) where \( l(v) = n \) and \( g \in V_j^k \).

Let \( \phi_k(v) = p + qxy \) and \( g = c + dxy \). So

\[
\phi_k(w) = (p + qxy)(c + dxy) = (pc + qd^*) + (pd + qc^*)xy = a + bxy \text{(say)}.
\]

Last word of \( v \) does not belong to \( V_j^k \) and by induction, we obtain the following inequalities:

\[
\begin{align*}
  \max (t^{-j}q + p) &\geq \max \{\max p, \max t^{-j}q\} \quad \ldots \quad (1) \\
  \min (t^{-j}q + p) &\leq \min \{\min p, \min t^{-j}q\} \quad \ldots \quad (2)
\end{align*}
\]

We get

\[
a = pc + qd^* = pc + t^{-2j}qd = pc - t^{-j}q(1 + c) = c(p + t^{-j}q) + t^{-j}q
\]

From (1) it follows that

\[
\max (c(p + t^{-j}q)) = \max c + \max (p + t^{-j}q) \geq \max c + \max t^{-j}q
\]

> \max t^{-j}q

which implies that

\[
\max a = \max (c(p + t^{-j}q) + t^{-j}q) = \max (c(p + t^{-j}q))
\]

> \max t^{-j}q

\ldots \quad (3)

Similarly, using (1) we get

\[
\max a = \max (c(p + t^{-j}q)) > \max p \quad \ldots \quad (4)
\]

By applying similar calculations and replacing \( \max \) by \( \min \), we can obtain \( \min a < \min p \). Thus \( \deg \phi_k(w) = \deg a = \max a - \min a > \max p - \min p = \deg \phi_k(v) > 0 \). Which completes the induction for (i).

Now, we will prove (ii) part. Let \( g \in V_j^k \), then by using the above mentioned observations we have,

\[
t^{-j}b + a = t^{-j}(pd + qc^*) + (pc + qd^*) = t^{-j}pd + t^{-j}qc + pc + t^{-2j}qd
\]

\[
= (p + t^{-j}q)(c + t^{-j}d) = p + t^{-j}q \text{ since } c + t^{-j}d = 1.
\]

Therefore, \( \max (t^{-j}b + a) = \max (p - t^{-j}q) \leq \max \{\max p, \max t^{-j}q\} \). But \( \max p < \max a \).

Thus, we get \( \max t^{-j}b = \max (p + t^{-j}q + a) > \max t^{-j}q \) by using (3) and (4). So

\[
\max (t^{-j}b + a) \leq \max \{\max p, \max t^{-j}q\} < \max \{\max a, \max t^{-j}b\}.
\]

By replacing \( \max \) by \( \min \) and applying the same type of calculations we can easily get that:
Since in $Z$

Thus ($\phi$ is one-one because for $1 \phi$ isomorphism. The mapping $\phi$

or $\min(t^{-j}b+a) > \min\{\min a, \min t^{-j}b\}$.

To show $g \in V^k_j$. Suppose $g \in V^k_j$ then if $\max(t^{-j}b+a) = \max\{\max a, \max t^{-j}b\}$ for $j, l \in Z$ or $\min(t^{-j}b+a) > \min\{\min a, \min t^{-j}b\}$. Also $\max(t^{-j}b+a) < \max\{\max a, \max t^{-j}b\}$ then $\max a = \max t^{-j}b$ and hence $\max(t^{-j}b+a) < \max t^{-j}b = j + \max b$. Therefore if $\max(t^{-j}b+a) < \max\{\max a, \max t^{-j}b\}$ and $\max(t^{-j}b+a) < \max\{\max a, \max t^{-j}b\}$ then

$$\max(t^{-j}b+a + t^{-j}b+a) \leq \max\{\max(a+t^{-j}b), \max(a+t^{-j}b)\} < \max\{\max b-j, \max b-l\}.$$

But $\max(t^{-j}b+a + t^{-j}b+a) = \max(t^{-j}b+t^{-j}b) = \max b + \max t^{-j} + t^{-j}$. So when $t^{-j} - t^{-l} \neq 0$, we have $\max(t^{-j}b+t^{-j}b) = \max\{\max b-j, \max b-l\}$ which gives a contradiction. Therefore $t^{-j} - t^{-l} = 0$, i.e., $j = l$ and hence $g = a + bx \in V^k_j$ for some $j$.

Similarly we can prove the converse if $\min(t^{-j}b+a) > \min\{\min a, \min t^{-j}b\}$ and $\min(t^{-j}b+a) > \min\{\min a, \min t^{-j}b\}$.

Theorem 2.4. Let $G = \langle U, D \rangle$, where $U = \langle 1 + r \eta_j \rangle, j \in Z, r \in R$ and $D \cong (C_2 \times D_\infty) \times U(Z_2)$, i.e., $D$

denote the group of trivial units of $Z_2(C_2 \times D_\infty)$. Then:

(i) $U \cong \ast_{j\in Z} V^k_j \cong \ast_{Z \oplus N} R^+$, where $R^+$ denotes the additive group of the ring $R = Z_2$.

(ii) $G = UD$.

Proof. (i) To prove (i), we have to show that the homomorphism $\phi = \lim_k \phi_k : \ast_j V^k_j \to U$ is an isomorphism. The mapping $\phi$ is onto because each generator $1 + r \eta_j$ lies in the image of $\phi$. Further, $\phi$ is one-one because for $1 \neq w \in \ast_j V^k_j$ there exists $k \in N$ such that $w \in \ast_j V^k_j$ and by above lemma $\phi(w) = \phi_k(w) \neq 1$. This proves part (i).

(ii) In order to prove (ii) it is enough to show that:

1. $U \cap D = \{1\}$
2. $U$ is a normal subgroup of $G$

1. If $1 \neq \alpha \in U$ then by previous lemma $\deg \alpha > 0$ so $\alpha \notin D$. Thus $U \cap D = \{1\}$.

2. $D \cong (C_2 \times D_\infty) \times U(Z_2)$, where $C_2 = \langle y \rangle$, $y, U(Z_2)$ are contained in center of $Z_2(C_2 \times D_\infty)$, therefore it is sufficient to show that $tUt^{-1} \subseteq U$ and $xUx^{-1} \subseteq U$.

Since in $Z_2(C_2 \times D_\infty)$, $\eta_j = (t^{-i} + t^i) + t^j(t^{-i} + t^i)xy$.

Thus $t \eta_j t^{-1} = (t^{-i} + t^i) + t^{j+2}(t^{-i} + t^i)xy$.

Similarly, $x \eta_j x^{-1} = (t^{-i} + t^i) + t^{-j}(t^{-i} + t^i)xy$ and hence

$$tUt^{-1} = t(1 + r \eta_j t^{-1} = 1 + r \eta_{i(j+2)} \subseteq U$$

$$xUx^{-1} = x(1 + r \eta_j x^{-1} = 1 + r \eta_{i(-j)} \subseteq U$$

This completes the proof of the theorem.
Remark 2.5. By Lemma 2.3, \( 1 \neq \alpha \in \text{im } \phi_k \) it implies that \( \deg \alpha > 0 \) so \( \text{im } \phi_k \cap D = \{1\} \). Also by the above Theorem 2.4, \( \text{im } \phi_k \) is a normal subgroup of \( \langle \text{im } \phi_k, D \rangle \). Thus \( \langle \text{im } \phi_k, D \rangle = \text{im } \phi_k D \).

Proposition 2.6. \( U \) and \( G \) are infinitely generated subgroups of unit group of \( \mathbb{Z}_2(C_2 \times D_{\infty}) \).

Proof. If \( \alpha_1, \alpha_2, \ldots, \alpha_n \in U \) then there exists \( k \in \mathbb{N} \) such that \( \alpha_1 \alpha_2 \ldots \alpha_n \in \phi(*_j V_j^k) \). But \( 1 + \eta_{(k+1)j} \notin \text{im } \phi_k \) because \( \phi_{k+1} \) is a monomorphism. Therefore, \( \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \neq U \). Similarly, if \( \beta_1, \beta_2, \ldots, \beta_n \in G \) then by above theorem there exists \( k \in \mathbb{N} \) such that \( \beta_1 \beta_2 \cdots \beta_n \in \phi_k D \). But by above remark, \( 1 + \eta_{(k+1)j} \notin \text{im } \phi_k D \) because \( \phi_{k+1} \) is a monomorphism. Therefore, \( \langle \beta_1, \beta_2, \ldots, \beta_n \rangle \neq G \). \( \square \)

Theorem 2.7. \( \text{Any } \alpha = a + bxy \in U(\mathbb{Z}_2(C_2 \times D_{\infty})) \) is in \( G \) (as above defined).

Proof. \( \alpha = a + bxy \). If \( \deg \alpha = 0 \) then \( \alpha \) is a trivial unit. Hence we assume that \( \deg \alpha > 0 \).

Let \( j = \max b - \max a \), and \( k = \min \{\min(a + t^{-j}b) - \min a, \max a - \max(a + t^{-j}b)\} \).

Observe that \( aa^* - bb^* = 1 \) this implies that \( aa^* \neq bb^* \) and hence \( a \neq t^{-j}b \) since if \( a = t^{-j}b \) then \( aa^* = bb^* \). Also \( k \geq 1 \) because \( \max aa^* = \max bb^* \) and \( \deg \alpha = \deg a = \deg b \) this gives \( \max a - \min a = \max b - \min b \) and thus \( j = \max a - \max b = \min a - \min b \). Hence \( \min(t^{-j}b) = \min a - \min b + \min b = \min a \), so \( \min(a + t^{-j}b) > \min a > 0 \). Similarly, \( \max a > \max(a + t^{-j}b) \).

\[
1 + \eta_{kj} = \begin{pmatrix} a + bxy \end{pmatrix} [(1 + t^k + t^{-k}) + t^j(t^k + t^{-k})xy] \\
= [a + t^k(a + t^{-j}b) + t^{-k}(a + t^{-j}b)] \\
+ [b(1 + t^k + t^{-k}) + a(t^j(t^k + t^{-k}))xy].
\]

Let \( h = t^k(a + t^{-j}b) + t^{-k}(a + t^{-j}b) \).

\[
\max h = k + \max(a + t^{-j}b)(\text{as } k \geq 1) \\
\leq \max a - \max(a + t^{-j}b) + \max(a + t^{-j}b) \\
\leq \max a \text{ (by using definition of } k). \\
\]

Similarly, we can show that \( \min h \geq \min a \).

By definition of \( k \) either \( \max h = \max a \) or \( \min h = \min a \).

Case 1 If \( \max h = \max a \). Then \( \max(a + h) < \max a \).

\[
\deg(1 + \eta_{kj}) = \deg(a + h) = \max(a + h) - \min(a + h) \\
< \max a - \min a = \deg \alpha.
\]

Similarly, if \( \min h = \min a \), we can get \( \deg(\alpha(1 + \eta_{kj})) < \min a \).

Hence by induction we can show that \( \alpha \in G \).

\( \square \)

Corollary 2.8. Every unit in \( U(\mathbb{Z}_2(C_2 \times D_{\infty})) \) of the form \( ax + by = (a + bxy)x \in G \).

If \( \alpha = a + bx \in R(C_2 \times D_{\infty}) \cong \begin{pmatrix} a & b & 0 & 0 \\
b^* & a^* & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & b^* & a^* \end{pmatrix} \) is a non-trivial unit then \( a \neq 0, b \neq 0 \).

Thus \( \text{det}(\alpha) = (aa^* - bb^*)^2 \in U(R) \) from Lemma 2.2. Now \( (aa^* - bb^*) \in RC_{\infty} \). Therefore, \( aa^* - bb^* \in U(R) \) as we have shown earlier.

Hence \( \deg \alpha = \max aa^* = \max bb^* = \deg b > 0 \).
By considering special non-trivial nilpotent elements in the group ring \( R(C_2 \times D_\infty) \) of the form:
\[
\delta_{ij} = (1 + \text{sgn}(i)t^{|i|}(1 - \text{sgn}(i)t^{|j|})x)
\]
\[
= (-t^{-|i|} + t^{|i|}) + \text{sgn}(i)t^{|j|}(t^{-|i|} - t^{|i|})x \quad \text{for } i(\neq 0), \ j \in \mathbb{Z}.
\]
For any \( r \in R, i, j \in \mathbb{Z} \), the element \( 1 + r\delta_{ij} \) is a unit in \( R(C_2 \times D_\infty) \). Also inverse of \( 1 + r\delta_{ij} \) is \( 1 - r\delta_{ij} \) because
\[
(1 + r\delta_{ij})(1 - r\delta_{ij}) = 1 - r^2(\delta_{ij})^2 = 1.
\]
All the units of the above form generate a subgroup of the unit group of \( R(C_2 \times D_\infty) \). Let
\[
V = \langle 1 + r\delta_{ij} \rangle_{i,j \in \mathbb{Z}, r \in R}
\]
For all \( k > 0, j \in \mathbb{Z} \), define
\[
V_j^k = \langle 1 + r\delta_{ij} \rangle_{i<k, r \in R}
\]
Obviously, the groups \( V_j^1 \subseteq V_j^2 \subseteq \cdots \) is an ascending chain. We set:
\[
V_j = \lim_k V_j^k
\]
Thus, we can get following results, that follows from [7].

**Theorem 2.9** ([7]). Let \( G' = \langle V, D \rangle \), where \( V = \langle 1 + r\delta_{ij} \rangle_{i,j \in \mathbb{Z}, r \in R} \) and \( D \cong (C_2 \times D_\infty) \times U(\mathbb{Z}_2) \), i.e., \( D \) denotes the group of trivial units of \( \mathbb{Z}_2(C_2 \times D_\infty) \). Then:

(i) \( V \cong \ast_{j \in \mathbb{Z}} V_j \cong \ast_{\mathbb{Z} \oplus \mathbb{N}} R^+ \), where \( R^+ \) denotes the additive group of the ring \( R \).

(ii) \( G' = VD \).

**Corollary 2.10.** \( V \) and \( G' \) are infinitely generated subgroups of \( U(\mathbb{Z}_2(C_2 \times D_\infty)) \).

**Theorem 2.11.** Any \( \alpha = a + bx \in U(\mathbb{Z}_2(C_2 \times D_\infty)) \) is in \( G' \) (as above defined).

**Corollary 2.12.** Every unit in \( U(\mathbb{Z}_2(C_2 \times D_\infty)) \) of the form \( ay + bx = (ay + bx)y \in G' \).

**References**


R. K. Sharma  
Department of Mathematics, Indian Institute of Technology Delhi, Delhi, India  
Email: rksharma@maths.iitd.ernet.in  
Email: rksharmaiitd@gmail.com

Pooja Yadav  
Department of Mathematics, Kamla Nehru College, Delhi University, Delhi, India  
Email: iitd.pooja@gmail.com

Kanchan Joshi  
Department of Mathematics, Delhi University, Delhi, India  
Email: kanchan.joshi@gmail.com