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# FINITE SIMPLE GROUPS WITH NUMBER OF ZEROS SLIGHTLY GREATER THAN THE NUMBER OF NONLINEAR IRREDUCIBLE CHARACTERS

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Abstract. The aim of this paper is to classify the finite simple groups with the number of zeros at most seven greater than the number of nonlinear irreducible characters in the character tables. We find that they are exactly $A_5$, $L_2(7)$ and $A_6$.

## 1. Introduction

Let $G$ be a finite group, and $\mathrm{Irr}_1(G)$ the set of nonlinear irreducible characters of $G$. For $\chi \in \mathrm{Irr}_1(G)$, we say that an element $g \in G$ is a vanishing element of $\chi$ if $\chi(g) = 0$. Let $\upsilon(\chi)$ denote the set of vanishing elements of $\chi$. Clearly, $\upsilon(\chi)$ is a union of some conjugacy classes of $G$. A well-known theorem of Burnside asserts that $\upsilon(\chi)$ is not empty for any $\chi \in \mathrm{Irr}_1(G)$. A number of papers have been devoted to the study of the zeros of the characters of a finite group (see [12] and [18]). In [1], Y. Berkovich and L. Kazarin posed the following question: Let $G$ be a nonabelian group. For $\chi \in \mathrm{Irr}_1(G)$, let $z(\chi)$ be the number of conjugacy classes of $G$ contained in $\upsilon(\chi)$ minus one, and set $z(G) = \sum_{\chi \in \mathrm{Irr}_1(G)} z(\chi)$. Classify the finite simple groups $G$ with small $z(G)$.

In this article, we study all finite simple groups whose number of character zeros exceeds the number of nonlinear irreducible character by at most seven. The main result of this paper is as follows.

**Theorem.** Let $G$ be a nonabelian simple group. Then $z(G) \leq 7$ if and only if $G$ is isomorphic to $A_5$, $L_2(7)$ or $A_6$.

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Notation is standard and taken from [9]. In particular, let $\pi(G)$ denote the set of all prime divisors of $|G|$, $\pi_e(G)$ the set of all element orders of $G$, $k(G)$ the number of conjugacy classes of $G$, and $k_G(N)$ the number of conjugacy classes of $G$ contained in $N$, where $N$ is a normal subset of $G$.

## 2. Proof of Theorem

We will use frequently the following lemma (see [4, Theorem 3.2]).

**Lemma 2.1.** *Let $G$ be a nonabelian simple group. Then $k(G) - |\pi_e(G)| = 1$ if and only if $G$ is isomorphic to $A_5$ or $L_2(7)$.*

**Lemma 2.2.** *Let $G$ be a finite group and let $r \in \pi(G)$ where $r$ is an odd prime. Suppose that $G$ does not contain elements of orders 8, $2p$, or $rq$, where $p$ and $q$ are odd primes and $r \neq q$. If $G$ has just one conjugacy class of $r$-elements, then $r = 5$ or 3.*

*Proof.* Let $R$ be a Sylow $r$-subgroup of $G$. Write $N_G(R) = RB$, where $B$ is a $r$-complement of $N_G(R)$. Since all $r$-elements of $G$ compose exactly one conjugacy class of $G$, we may conclude that $B \neq \{1\}$ and all elements in $Z(R) - \{1\}$ are $B$-conjugate (see [9, Lemma 12.30]). It follows by the hypothesis that $B$ acts fixed point freely on $Z(R)$, and thus $Z(R)B$ is a Frobenius group with kernel $Z(R)$ and a complement $B$. Furthermore, since all elements in $Z(R) - \{1\}$ are $B$-conjugate, we may conclude that $Z(R)B$ is a 2-transitive Frobenius group. Hence $|B| = |Z(R)| - 1$ is even, and it follows by [6, Hauptsatz V.8.7]) that a Sylow 2-subgroup $P$ of $B$ is a quaternion group of order 8 or a cyclic group whose order is at most 4. Note that $B$ has a central element of order 2 (see [6, Hauptsatz V.8.18]); thus it follows by the hypothesis that $r^n - 1 = |Z(R)| - 1 = |B| = 8$, 4 or 2. Then $r = 5$ or 3. The proof is complete. □

**Proposition 2.3.** *Let $G \cong A_n$ or $L_2(q)$, where $n \geq 5$ and $q \geq 4$. If $z(G) \leq 7$, then $G \cong A_5$, $L_2(7)$ or $L_2(9)$.*

*Proof.* Let $G \cong \mathrm{A}_n$ for some $n \geq 14$. Let $\pi$ be the permutation character of $G$, and $\delta$ be the mapping of $G$ into $0, 1, 2, ...$ such that $\delta(g)$ is the number of 2-cycles in the standard composition of $g$. Let $\eta \in \mathrm{Irr}(G), \eta(1) = n - 1$ and $\eta(g) = \pi(g) - 1$. Set

$$\lambda = \frac{(\pi - 1)(\pi - 2)}{2} - \delta,\ \rho = \frac{\pi(\pi - 3)}{2} + \delta.$$

By [6, Hauptsatz V.20.6]), both $\lambda$ and $\rho$ are irreducible characters of $G$.
For odd $n$, set

$a_1 = (1, ..., n-4)(n-3, n-2, n-1)$,
$a_2 = (1, ..., n-6)(n-5, n-4, n-3, n-2, n-1)$,
$a_3 = (1, ..., n-8)(n-7, n-6, n-5, n-4, n-3, n-2, n-1)$,
$a_4 = (1, ..., n-10)(n-9, n-8, n-7, n-6, n-5, n-4, n-3, n-2, n-1)$,
$a_5 = (1, ..., n-12)(n-11, n-10, n-9, n-8, n-7, n-6, n-5, n-4, n-3, n-2, n-1)$;
$b_1 = (1, ..., n-2)$,

$b_2 = (1, ..., n-4)(n-3, n-2, n-1)$,
$b_3 = (1, ..., n-5)(n-4, n-3)$;
$c_1 = (1, ..., n)$,
$c_2 = (1, ..., n-3)(n-2, n-1)$,
$c_3 = (1, ..., n-6)(n-5, n-4, n-3)$.

We see that $\eta(a_1) = \eta(a_2) = \eta(a_3) = \eta(a_4) = \eta(a_5) = 0$, $\lambda(b_1) = \lambda(b_2) = \lambda(b_3) = 0$ and $\rho(c_1) = \rho(c_2) = \rho(c_3) = 0$. Observe that $a_1, a_2, a_3, a_4, a_5$ (or $b_1, b_2, b_3$, or $c_1, c_2, c_3$) lie in distinct conjugacy classes of $G$, thus $z(G) \geq 8$.

For even $n$, set

$a_1 = (1, ..., n-1)$,
$a_2 = (1, ..., n-5)(n-4, n-3)(n-2, n-1)$,
$a_3 = (1, ..., n-9)(n-8, n-7)(n-6, n-5)(n-4, n-3)(n-2, n-1)$;
$b_1 = a_1 = (1, ..., n-1)$,
$b_2 = (1, ..., n-2)(n-1, n)$,
$b_3 = (1, ..., n-5)(n-4, n-3, n-2)$,
$b_4 = (1, ..., n-8)(n-7, n-6, n-5, n-4)(n-3, n-2, n-1)$,
$b_5 = (1, ..., n-11)(n-10, n-9, n-8, n-7, n-6, n-5)(n-4, n-3, n-2, n-1)$;
$c_1 = (1, ..., n-3)$,
$c_2 = (1, ..., n-3)(n-2, n-1, n)$,
$c_3 = (1, ..., n-4)(n-3, n-2)$.

We see that $\eta(a_1) = \eta(a_2) = \eta(a_3) = 0$, $\lambda(b_1) = \lambda(b_2) = \lambda(b_3) = \lambda(b_4) = \lambda(b_5) = 0$ and $\rho(c_1) = \rho(c_2) = \rho(c_3) = 0$. Observe that $a_1, a_2, a_3$ (or $b_1, b_2, b_3, b_4, b_5$, or $c_1, c_2, c_3$) lie in distinct conjugacy classes of $G$, thus $z(G) \geq 8$. Then we have that $n \leq 13$, and thus $G \cong A_5$ or $A_6$ by [2].

Let $G \cong \mathrm{L}_2(q)$, where $q \geq 4$. Let $q$ be odd.

Suppose that $q \equiv 1 \pmod 4$, and write $q = 4r + 1$. Then $G$ has a cyclic subgroup $C$ of order $2r$. If $r \geq 3$, then $C$ contains elements $a$, $b$, $c$ of orders 2, $r$, $2r$, respectively, so no two of them are conjugate in $G$. Furthermore, every irreducible character of $G$ of degree $q-1$ vanishes on $a$, $b, c$ (see [11]). If $q \geq 17$, then it follows by [7, XI, Theorem 5.6 and Theorem 5.7] that $G$ has at least four irreducible characters of degree $q-1$, thus $z(G) \geq 8$. Hence $q \leq 16$, we can conclude that $G \cong \mathrm{L}_2(5) \cong \mathrm{A}_5$ or $\mathrm{L}_2(9) \cong \mathrm{A}_6$ by [2].

Suppose that $q \equiv 3 \pmod 4$ and we follow a similar argument. Set $q = 4r + 3$. Then $G$ has a cyclic subgroup $C$ of order $2(r+1)$. If $r \geq 2$, then $C$ contains elements $a$, $b$, $c$ of orders 2, $r+1$, $2(r+1)$, respectively, so no two of them are conjugate in $G$. Furthermore, every irreducible character of $G$ of degree $q-1$ vanishes on $a$, $b$, $c$ (see [11]). If $q \geq 19$, then it follows by [7, XI, Theorem 5.6 and Theorem 5.7] that $G$ has at least four irreducible characters of degree $q-1$, thus $z(G) \geq 8$. Hence $q \leq 18$, we can see that $G \cong \mathrm{L}_2(7)$ by [2].

Let $q$ be even. That is, $G \cong \mathrm{L}_2(2^f)$ where $f \geq 2$. Then $|G| = (2^f - 1)2^f(2^f + 1)$ and $G$ has two cyclic subgroups whose orders are $2^f + 1$ and $2^f - 1$, respectively (see [6, Hauptsatz II. 8.27]). Suppose that $2^f + 1$ is not of prime power, and let $p$ and $q$ be two distinct prime divisors of $2^f + 1$. In this case,

$G$ contains elements of orders $p$, $q$ and $pq$. By [7, XI, Theorem 5.5], $G$ has $2^{f-1}-1$ $\beta_i$ of degree $2^f+1$. Note that every $\beta_i$ is both $p$-defect zero and $q$-defect zero. Therefore we have that $k_G(\upsilon(\beta_i)) \geq 3$ (see [9, Theorem 8.17]). The hypothesis implies that $2^{f-1}-1 \leq 3$. Then we can conclude that $2^f+1=5$ or 9, which contradicts the assumption that $2^f+1$ is not of prime power. Similarly, $2^f-1$ is of prime power. Since both $2^f+1$ and $2^f-1$ are of prime power, we can easily conclude that $f=2$ or 3. Hence either $G \cong \mathrm{L}_2(4) \cong \mathrm{A}_5$ or $G \cong \mathrm{L}_2(8)$. But by [2], $z(L_2(8)) > 7$. The proof is complete. □

**Proposition 2.4.** *If $G \cong \mathrm{Sz}(2^{2m+1})$ where $m \geq 1$, then $z(G) > 7$.*

*Proof.* Let $G \cong \mathrm{Sz}(2^{2m+1})$ where $m \geq 1$. We claim that $z(G) > 7$. Otherwise, $z(G) \leq 7$. Note that $\pi_e(G) = \{2, 4, \text{all factors of } (2^{2m+1}-1), (2^{2m+1}-2^{m+1}+1) \text{ and } (2^{2m+1}+2^{m+1}+1)\}$. We have

$$(2^{2m+1}+2^{m+1}+1)(2^{2m+1}-2^{m+1}+1) = 2^{4m+2}+1.$$

Hence $5 \in \pi_e(G)$ by $2^{4m+2}+1 \equiv 0 \pmod 5$. Now recall that $2^{2m+1}-1$, $2^{2m+1}-2^{m+1}+1$ and $2^{2m+1}+2^{m+1}+1$ are mutually co-prime. On the other hand, by [16, Corollary], we have that there exists $\chi \in \mathrm{Irr}_1(G)$ such that $\chi$ is of $p$-defect zero for any prime factor $p$ of $|G|$. For such $\chi$, we have that $\{x \in G : p \,|\, o(x)\} \subseteq \upsilon(\chi)$ (see [9, Theorem 8.17]), and thus $k_G(\{x \in G : p \,|\, o(x)\}) \leq k_G(\upsilon(\chi))$. In addition, for any nonidentity element $g$ of $G$, there exists $\chi \in \mathrm{Irr}_1(G)$ such that $\chi(g) = 0$.

Let $\chi_2 \in \mathrm{Irr}_1(G)$ such that $\chi_2$ is of 2-defect zero. Note that all elements of order 4 in $G$ form two conjugacy classes of $G$, which can be easily verified by [7, XI, Theorem 3.10]. So we have that $k_G(\upsilon(\chi_2)) \geq 3$.

**Case 1.** Suppose that both $2^{2m+1}-2^{m+1}+1$ and $2^{2m+1}+2^{m+1}+1$ are prime.

In this case, $2^{2m+1}-2^{m+1}+1=5$, and $G \cong \mathrm{Sz}(8)$, but $\mathrm{Sz}(8)$ does not satisfy the hypothesis by [2]. Therefore this case is excluded.

**Case 2.** Suppose that $2^{2m+1}-2^{m+1}+1$ is prime and $2^{2m+1}+2^{m+1}+1$ is a composite number.

In this case, $2^{2m+1}-2^{m+1}+1 \neq 5$, thus $5 \,|\, (2^{2m+1}+2^{m+1}+1)$. Applying Lemma 2.2, all elements of orders $2^{2m+1}-2^{m+1}+1$ in $G$ form at least two conjugacy classes of $G$. So we have that $z(G) \geq 3$. If $|\pi(2^{2m+1}+2^{m+1}+1)| \geq 3$, then we can easily see that $z(G) > 7$, a contradiction. Thus we have that $|\pi(2^{2m+1}+2^{m+1}+1)| = 2$ and $2^{2m+1}+2^{m+1}+1 = 5p, 5p^2$ or $25p$, where $p$ is an odd prime and $p \neq 5$.

**Case 2.1.** $2^{2m+1}+2^{m+1}+1 = 5p^2$ or $25p$, where $p$ is an odd prime and $p \neq 5$.

In this case, let $\chi_5, \chi_p \in \mathrm{Irr}_1(G)$ such that $\chi_5$ is of 5-defect zero and $\chi_p$ is of $p$-defect zero, respectively. It follows by [9, Theorem 8.17] that $k_G(\upsilon(\chi_2)) + k_G(\upsilon(\chi_5)) + k_G(\upsilon(\chi_p)) \geq 10$. By the hypothesis, it is easy to see that $2^{2m+1}-1$ is prime. Lemma 2.2 implies that all elements of orders $2^{2m+1}-1$ and $2^{2m+1}-2^{m+1}+1$ in $G$ form at least two conjugacy classes of $G$, respectively. Hence $z(G) \geq 8$, a contradiction.

**Case 2.2.** $2^{2m+1}+2^{m+1}+1 = 5p$, where $p$ is an odd prime and $p \neq 5$.

In this case, we can easily conclude that $2^{2m+1}-1$ is $r^2$, $rs$ or $r$, where $r$ and $s$ are two distinct primes.

Suppose that $2^{2m+1}-1=r^2$. Set $r=2k+1$ (note that $r$ is odd). Then we have

$$2^{2m+1}=4k^2+4k+2.$$

Hence we easily obtain a contradiction, which shows that the case does not occur.

Suppose that $2^{2m+1}-1=r$. Lemma 2.2 implies all elements of orders $2^{2m+1}-1$ and $2^{2m+1}-2^{m+1}+1$ in $G$ form at least two conjugacy classes of $G$, respectively. In this case, we have that $z(G)\geq 6$. Note that $\pi_e(G)=\{1,2,4,5,p,5p,r,2^{2m+1}-2^{m+1}+1\}$. By [15], we may conclude that $k(G)>12$, implying $k(G)-|\pi_e(G)|\geq 5$, and thus $z(G)>7$, a contradiction.

Suppose that $2^{2m+1}-1=rs$, and arguing as in above paragraph, we see that $z(G)>7$, a contradiction.

When $2^{2m+1}-2^{m+1}+1$ is a composite number and $2^{2m+1}+2^{m+1}+1$ is prime, or both $2^{2m+1}-2^{m+1}+1$ and $2^{2m+1}+2^{m+1}+1$ are composite numbers, and arguing as in Case 2, we conclude that $z(G)>7$, a contradiction.

Therefore our claim is true, that is, $z(\mathrm{Sz}(2^{2m+1}))>7$, where $m\geq 1$. The proof is complete. □

**Theorem 2.5.** *Let $G$ be a nonabelian simple group. Then $z(G)\leq 7$ if and only if $G$ is isomorphic to $A_5$, $L_2(7)$ or $A_6$.*

*Proof.* Suppose $G\cong \mathrm{A}_n$ where $n\geq 5$. Applying Proposition 2.3, we have that $G\cong \mathrm{A}_5$ or $\mathrm{A}_6$.

Suppose that $G$ is isomorphic to one of the sporadic simple groups. Then it follows by [2] that $z(G)>8$. By the classification theorem of the finite simple groups, now suppose that $G$ is a simple group of Lie type.

By [16, Corollary], we have that there exists $\chi\in \mathrm{Irr}_1(G)$ such that $\chi$ is of $p$-defect zero for any prime factor $p$ of $|G|$. For such $\chi$, we have that $\{x\in G : p\,|\,o(x)\}\subseteq \upsilon(\chi)$ ([9, Theorem 8.17]), and thus $k_G(\{x\in G : p\,|\,o(x)\})\leq k_G(\upsilon(\chi))$. In addition, for any nonidentity element $g$ of $G$, there exists $\chi\in \mathrm{Irr}_1(G)$ such that $\chi(g)=0$. Let $P\in \mathrm{Syl}_2(G)$.

First, we suppose that $P$ is abelian.

Since $P$ is abelian, it follows by [7, XI, Theorem 13.7] that $G$ is one of the following groups: $\mathrm{L}_2(2^f)$; $\mathrm{L}_2(q)$ where $q\equiv 3,5 \pmod 8$; ${}^2G_2(q)$ where $q=3^{2n+1}(n\geq 1)$.

Suppose that $G\cong{}^2 G_2(q), q=3^{2n+1}(n\geq 1)$. Then $G$ has an involution $j$ such that $C_G(j)=<j> \times L_2(q)$ (see [7, XI, Theorem 13.4]). Note that $|\mathrm{L}_2(q)|=(q+1)q(q-1)/2$ where $q=3^{2n+1}(n\geq 1)$. We can easily conclude that $|\mathrm{L}_2(q)|$ has at least 3 odd prime divisors, thus $G$ contains elements $a$, $b$, $c$ of orders $2r$, $2s$, $2t$, respectively, where $r$, $s$ and $t$ are three distinct odd primes. In addition, it follows by [13] that $\mathrm{L}_2(q)$ contains elements of order $uv$, where $u$ and $v$ are two distinct odd primes (note that $q=3^{2n+1}(n\geq 1)$). Lemma 2.1 implies that $k(G)-|\pi_e(G)|\geq 2$. Recall that every nonidentity element of $G$ is a vanishing element of $G$. In this case, we can easily conclude that $z(G)\geq 8$, a contradiction.

Suppose that $G\cong \mathrm{L}_2(q)$ or $\mathrm{L}_2(2^f)$, where $q\equiv 3,5 \pmod 8$ and $f\geq 2$. Applying Proposition 2.3, we have that $G\cong \mathrm{A}_5$.

Now, we suppose that $P$ is nonabelian.

In this case, all 2-elements in $G$ form at least two conjugacy classes of $G$. Take $\chi_1 \in \mathrm{Irr}_1(G)$ such that $\chi_2$ is of 2-defect zero, then $k_G(\upsilon(\chi_2)) \geq 2$.

Suppose that $G$ is a simple $\mathrm{K}_3$-group. Then by [10], $G$ is isomorphic to one of the following groups: $\mathrm{A}_5 \cong \mathrm{L}_2(4)$, $\mathrm{A}_6 \cong \mathrm{L}_2(9)$, $\mathrm{L}_2(7)$, $\mathrm{L}_2(8)$, $\mathrm{L}_2(17)$, $\mathrm{L}_3(3)$, $\mathrm{U}_3(3)$, $\mathrm{U}_4(2)$. By [2], $G \cong \mathrm{L}_2(7)$ or $G \cong \mathrm{A}_6 \cong \mathrm{L}_2(9)$.

If $|\pi(G)| = 4$, then by [10], [2], Proposition 2.3 and Proposition 2.4, we see that this case does not occur. In the following, let $|\pi(G)| \geq 5$.

Suppose that every nonidentity element of $G$ of composite order is of prime power order. As $|\pi(G)| \geq 5$, $G$ is isomorphic to one of the following groups: $\mathrm{L}_3(4)$, $\mathrm{Sz}(8)$ or $\mathrm{Sz}(32)$ (see [13]). But they do not satisfy the hypothesis by [2]. Hence $G$ contains an element whose order is not of prime power.

We claim that $G$ must contain an element of order $2p$, where $p$ is an odd prime. If else, then $C_G(t)$ is 2-group for any involution $t$. By Proposition 2.3 and since $P$ is nonabelian, it follows from [14] that $G$ is one of the following groups: $\mathrm{S}_z(q)$, $q = 2^{2m+1}$, $\mathrm{M}_{10}$. Note that $\mathrm{M}_{10}$ does not satisfy the hypothesis by [2]. Hence $G \cong \mathrm{Sz}(2^{2n+1})$ where $n \geq 1$. Proposition 2.4 implies $z(\mathrm{Sz}(2^{2n+1})) > 7$, a contradiction. Hence our claim is true. Thus we have that $k_G(\upsilon(\chi_2)) \geq 3$. Recall $P$ is nonabelian and $|\pi(G)| \geq 5$. Lemma 2.1 implies that $k(G) - |\pi_e(G)| \geq 2$. Recall that every nonidentity element of $G$ is a vanishing element of $G$. It follows that $z(G) \geq 3$. In this case, by the hypothesis, we now break the proof into three cases.

**Case 1.** Suppose that the set of composite numbers in $\pi_e(G)$ is $\{4, 8, 2p\}$, $\{4, 8, 16, 2p\}$ or $\{4, 2p, r^2, s^2\}$, where $p, r$ and $s$ are three distinct odd primes. By [5], [2] and Proposition 2.3, we conclude that this case does not occur.

**Case 2.** Suppose that the set of composite numbers in $\pi_e(G)$ is $\{4, 2p, rs\}$, $\{4, 2p, 2q\}$ or $\{4, 2p, pq\}$, where $p, r$ and $s$ are three distinct odd primes, and $p$ and $q$ are two distinct odd primes.

In this case, by [3, Theorem A], we have that $|\pi(G)| \leq 6$. Note that $|\pi(G)| \geq 5$.

If $|\pi(G)| = 5$, by [2] and Proposition 2.3, it follows from [10] that $G$ is one of the following groups: $\mathrm{L}_3(q)$ where $\pi((q^2-1)(q^3-1)) = 4$, $\mathrm{U}_3(q)$ where $\pi((q^2-1)(q^3+1)) = 4$, $\mathrm{O}_5(q)$ where $\pi(q^4-1) = 4$, $\mathrm{R}(q)$ where $q$ is an odd power of 3 and $\pi(q^2-1) = 3$ and $\pi(q^2-q+1) = 1$. Note that the number of the connected components of the prime graph of $G$ is 3. If $q$ is odd, this case does not occur (see [17]). If $q$ is even, we see that this case is excluded by [8].

If $|\pi(G)| = 6$, by [2] and Proposition 2.3, it follows from [13] that $G$ is one of the following groups: $\mathrm{L}_3(q)$ where $\pi((q^2-1)(q^3-1)) = 5$, $\mathrm{L}_4(q)$ where $\pi((q^2-1)(q^3-1)(q^4-1)) = 5$, $\mathrm{U}_3(q)$ where $\pi((q^2-1)(q^3+1)) = 5$, $\mathrm{U}_4(q)$ where $\pi((q^2-1)(q^3+1)(q^4-1)) = 5$, $\mathrm{O}_5(q)$ where $\pi(q^4-1) = 5$, $\mathrm{G}_2(q)$ where $\pi(q^6-1) = 5$, $\mathrm{R}(3^{2m+1})$ where $\pi((3^{2m+1}-1)(3^{6m+3}+1)) = 5$. Note that the number of the connected components of the prime graph of $G$ is 4. If $q$ is odd, this case does not occur (see [17]). If $q$ is even, we see that this case is excluded by [8].

**Case 3.** Suppose that the set of composite numbers in $\pi_e(G)$ is $\{4, 2p, p^2, 2p^2\}$, where $p$ is an odd prime.

In this case, by [3, Theorem A], we have that $|\pi(G)| \leq 7$.

If $|\pi(G)| = 5$ or $|\pi(G)| = 6$, arguing as in Case 2, we conclude that this case does not occur.

If $|\pi(G)| = 7$, by [3, Theorem A], [2] and Proposition 2.3, this case is excluded. The proof is complete. □

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