ON p-SOLUBLE GROUPS WITH A GENERALIZED p-CENTRAL OR POWERFUL SYLOW p-SUBGROUP

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Abstract. Let \( G \) be a finite \( p \)-soluble group, and \( P \) a Sylow \( p \)-subgroup of \( G \). It is proved that if all elements of \( P \) of order \( p \) (or of order \( \leq 4 \) for \( p = 2 \)) are contained in the \( k \)-th term of the upper central series of \( P \), then the \( p \)-length of \( G \) is at most \( 2m + 1 \), where \( m \) is the greatest integer such that \( p^m - p^{m-1} \leq k \), and the exponent of the image of \( P \) in \( G/O_{p'}(G) \) is at most \( p^m \). It is also proved that if \( P \) is a powerful \( p \)-group, then the \( p \)-length of \( G \) is equal to 1.

1. Introduction

A finite \( p \)-group \( P \) is called \( p \)-central if all its elements of order \( p \) are contained in the centre: \( \Omega_1(P) \leq Z(P) \). Sometimes this definition is modified in the case of \( p = 2 \) to require that all elements of order \( \leq 4 \) belong to \( Z(P) \). Such \( p \)-groups are in many respects dual to powerful \( p \)-groups (and the above-mentioned modification for \( p = 2 \) reflects the definition of powerful 2-groups). Although \( p \)-central \( p \)-groups received less attention in the literature than the very important case of powerful \( p \)-groups, there are several papers devoted to \( p \)-central \( p \)-groups and properties of their embeddings in finite groups; the reader can find relevant references in [4].

González-Sánchez and Weigel [4] initiated the study of more general classes: a finite \( p \)-group \( P \) is called \( p^i \)-central of height \( k \) if all its elements of order dividing \( p^i \) are contained in the \( k \)-th term of the upper central series: \( \Omega_i(P) \leq \zeta_k(P) \). In particular, they proved [4, Theorem E] that if, for an odd prime \( p \), a Sylow \( p \)-subgroup of a finite \( p \)-soluble group \( G \) is \( p \)-central of height \( p - 2 \), then \( G \) has \( p \)-length 1.

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In this note we generalize this result to arbitrary height (including the case \( p = 2 \) with the above-mentioned proviso). Namely, we obtain a bound for the \( p \)-length of a \( p \)-soluble group \( G \) whose Sylow \( p \)-subgroup is \( p \)-central of height \( k \) (Theorem 3.1). This result is derived from a bound for the exponent of a Sylow \( p \)-subgroup of \( G/O_{p',p}(G) \) (Theorem 3.2), which is proved on the basis of Hall–Higman theorems.

We also prove the result “dual” to [4, Theorem E], that if a finite \( p \)-soluble group \( G \) has a powerful Sylow \( p \)-subgroup, then the \( p \)-length of \( G \) is equal to 1 (Theorem 4.1).

2. Preliminaries

We shall need the following well-known property of coprime action by automorphisms. Recall that for a finite \( p \)-group \( P \) by definition \( \Omega_i(P) = \langle g \in P \mid g^{p^i} = 1 \rangle \).

**Lemma 2.1** ([3, Kap. IV, Satz 5.12]). Suppose that a finite \( p' \)-group \( A \) acts by automorphisms on a finite \( p \)-group \( P \). If \( A \) acts trivially on \( \Omega_1(P) \) for \( p \neq 2 \), or on \( \Omega_2(P) \) for \( p = 2 \), then \( A \) acts trivially on \( P \).

Some other well-known properties of coprime actions of groups of automorphisms will be used without special references.

Recall that if a finite group \( G \) acts by automorphisms on an elementary abelian \( p \)-group \( V \), then \( V \) can be regarded as a vector space over the field of \( p \) elements \( \mathbb{F}_p \) and the action of \( G \) by conjugation on \( V \) can be regarded as action by linear transformations of this vector space. The linear transformation of \( V \) induced by an element \( g \in G \) is denoted by \( T(g) \). We use the right operator notation for this action: for \( v \in V \) and \( g \in G \) the image of \( v \) under \( T(g) \) is denoted by \( vT(g) \). For example, if \( V \) is a normal elementary abelian section of \( G \), then \( G \) acts on \( V \) by conjugation and \( vT(g) \) is equal to the image of the group element \( \hat{v}^g \), where \( \hat{v} \) is an inverse image of \( v \) in \( G \). Note that \( v(T(g) - 1_V) \), where \( 1_V \) is the identity transformation of \( V \), is equal to the image of the group commutator \([\hat{v}, g]\) , which is also equal to \([v, g]\) in the natural semidirect product \( V \rtimes G \).

We also recall Theorem B from the celebrated Hall–Higman paper [5].

**Theorem 2.2** ([5, Theorem B]). Let \( H \) be a \( p \)-soluble linear group over a field of characteristic \( p \), with no normal \( p \)-subgroup greater than 1. If \( h \) is an element of order \( p^m \) in \( H \), then the minimal equation of \( h \) is \((x - 1)^r = 0\), where \( r = p^m \), unless there is an integer \( m_0 \), not greater than \( m \), such that \( p^{m_0} - 1 \) is a power of a prime \( q \) for which a Sylow \( q \)-subgroup of \( H \) is non-abelian, in which case, if \( m_0 \) is the least such integer, \( p^m - p^{m-m_0} \leq r \leq p^m \).

We shall only need the fact that we always have \( p^m - p^{m-1} \leq r \leq p^m \).

When an element \( g \in GL(V) \) of order \( p^m \) acts as a linear transformation on a vector space \( V \) over a field of characteristic \( p \), its minimal polynomial always has the form \((x - 1)^r = 0\), because \( x^{p^m} - 1 = (x - 1)^{p^m} \) in characteristic \( p \). It follows that \( V \) has a basis in which the matrix of \( g \) has Jordan normal form, since the only eigenvalue is 1. The maximum size of Jordan blocks is \( p^m \times p^m \).
It is well known that the natural semidirect product \( V\langle g \rangle \) of groups \( V \) and \( \langle g \rangle \) contains an element of order \( p^{m+1} \) if and only if there is at least one Jordan block of size \( p^m \times p^m \).

### 3. Generalized \( p \)-central Sylow \( p \)-subgroup

Recall that \( O_{p'}(G) \) is the maximal normal \( p' \)-subgroup of a finite group \( G \); then \( O_{p',p}(G) \) is the full inverse image of the maximal normal \( p' \)-subgroup of \( G/O_{p'}(G) \), and so on, defining by induction the terms of the upper \( p \)-series \( O_{p',p',p',...}(G) \). A finite group \( G \) is \( p \)-soluble if \( G = O_{p',p',p',...}(G) \) and the minimum number of symbols \( p \) in this equation is called the \( p \)-length of \( G \).

**Theorem 3.1.** Let \( P \) be a Sylow \( p \)-subgroup of a finite \( p \)-soluble group \( G \). Suppose that \( \Omega_1(P) \leq \zeta_k(P) \) for \( p \neq 2 \), or \( \Omega_2(P) \leq \zeta_k(P) \) for \( p = 2 \). Then the \( p \)-length of \( G \) is at most \( 2m + 1 \), where \( m \) is the maximum integer such that \( p^m - p^{m-1} \leq k \).

In particular, as a rough estimate, \( m < 1 + \log_p k \), so that the \( p \)-length is at most \( 3 + 2 \log_p k \).

Theorem 3.1 will follow from a bound for the exponent of a Sylow \( p \)-subgroup of \( G/O_{p',p}(G) \).

**Theorem 3.2.** Let \( P \) be a Sylow \( p \)-subgroup of a finite \( p \)-soluble group \( G \). Suppose that \( \Omega_1(P) \leq \zeta_k(P) \) for \( p \neq 2 \), or \( \Omega_2(P) \leq \zeta_k(P) \) for \( p = 2 \). Then the exponent of a Sylow \( p \)-subgroup of \( G/O_{p',p}(G) \) is at most \( p^m \), where \( m \) is the maximum integer such that \( p^m - p^{m-1} \leq k \).

**Proof.** We can obviously assume that \( O_{p'}(G) = 1 \).

Let \( Q \) be a Hall \( p' \)-subgroup of \( O_{p,p'}(G) \), so that \( O_{p,p'}(G) = O_p(G)Q \). By the generalized Frattini argument,

\[ G = O_{p,p'}(G)N_G(Q) = O_p(G)N_G(Q), \]

so we need to obtain a bound for the exponent of the image of a Sylow \( p \)-subgroup of \( N_G(Q) \) in \( G = G/O_p(G) \). We use bars to denote images of elements or subsets in \( G \).

Let \( g \) be an element of a Sylow \( p \)-subgroup of \( N_G(Q) \), so that \( \bar{g} \) is its image in \( \bar{G} = G/O_p(G) \). Let \( |\bar{g}| = p^n \). We must show that \( p^n - p^{n-1} \leq k \). We can of course assume that \( n \geq 1 \).

Since \( O_p(\bar{G}) = 1 \) and \( Q = O_{p'}(\bar{G}) \), we have \( C_{\bar{G}}(\bar{Q}) \leq \bar{Q} \) (see, for example, [2, Theorem 6.3.2]). Hence the \( p \)-element \( \bar{g} \) acts faithfully on \( \bar{Q} \); in other words, \( [\bar{Q}, \bar{g}^{p^{n-1}}] \neq 1 \). Clearly, \( \bar{g} \) also acts on \( Q \) itself, and \( [Q, \bar{g}^{p^{n-1}}] = [\bar{Q}, \bar{g}^{p^{n-1}}] \neq 1 \).

Let \( \Omega \) denote \( \Omega_1(O_p(G)) \) if \( p \neq 2 \), and \( \Omega_2(O_p(G)) \) if \( p = 2 \).

Consider a series of normal subgroups of \( G \)

\[ 1 = U_0 < U_1 < \cdots < U_n = \Omega \]

in which each factor \( U_{i+1}/U_i \) is an elementary abelian \( p \)-group contained in the centre of \( O_p(G)/U_i \). Then the action of the semidirect product \( Q\langle g \rangle \) on each factor \( U_{i+1}/U_i \) is well defined.

Since \( O_{p'}(G) = 1 \), the \( p' \)-subgroup \( [Q, \bar{g}^{p^{n-1}}] \neq 1 \) acts faithfully on \( O_p(G) \). By Lemma 2.1, moreover, \( [Q, \bar{g}^{p^{n-1}}] \) acts faithfully on \( \Omega \). Since the action is coprime, we obtain that \( [Q, \bar{g}^{p^{n-1}}] \) acts nontrivially on at least one of the factors \( V \) of the series \( \{1\} \). Let \( H \) denote the image of \( Q\langle g \rangle \) in the group of
linear transformations of the vector space $V$ over $\mathbb{F}_p$, which consists of elements $T(u)$ for $u \in Q(\bar{g})$ in accordance with our notation.

Since the subgroup $[Q, g^{p^{n-1}}]$ acts non-trivially on $V$, we must have $O_p(H) = 1$. Indeed, otherwise $T(\bar{g})^{p^n-1}$ would be in $O_p(H)$ and then the image of $[Q, g^{p^{n-1}}]$ would be in $O_2(H) \cap O_p(H) = 1$ and therefore trivial, contrary to the assumption. For the same reasons, $T(\bar{g})$ has the same order $p^n$.

By the Hall–Higman Theorem 2.2, the minimal polynomial of $T(\bar{g})$ is $(x-1)^r = 0$, where $p^n - p^{n-1} \leq r \leq p^n$. Therefore there is $v \in V$ such that

$$v(T(\bar{g}) - 1_v)^{p^n - p^{n-1} - 1} \neq 0.$$  

Since the image of an element $u \in V$ under the linear transformation $T(\bar{g}) - 1_V$ is equal to the group commutator $[u, \bar{g}]$, it follows from (3.2) that

$$[[...[[[v, \bar{g}], \bar{g}], ..., \bar{g}] \neq 1.$$  

But by the hypothesis of the theorem we have $\Omega \leq \Omega_1(P) \leq \zeta_k(P)$ for $p \neq 2$ (or $\Omega \leq \Omega_2(P) \leq \zeta_k(P)$ for $p = 2$). Therefore we must also have

$$[[...[[[v, \bar{g}], \bar{g}], ..., \bar{g}] = 1.$$  

It follows that $p^n - p^{n-1} - 1 < k$, as required.

**Proof of Theorem 3.2.** Once we know a bound for the exponent $e_p$ of a Sylow $p$-subgroup of $G/O_{p',p}(G)$, we obtain a bound for the $p$-length $l$ of $G/O_{p',p}(G)$. Indeed, for $p \neq 2$ we have $e \geq [(l+1)/2]$ by the Hall–Higman theorem [5, Theorem A], and for $p = 2$ we have $e \geq l$ by Bryukhanova’s theorem [1] (which is the best-possible improvement of the earlier estimate $2e - 2 \geq l$ by Gross [3]). Since $l + 1$ is exactly the $p$-length of $G$, the result follows from Theorem 3.2.

**Remark 3.3.** The Hall–Higman Theorem A gives a better bound $e \geq l$ if $p$ is not a Fermat prime. As noticed in the Hall–Higman paper [5], it follows from the proof that in the Hall–Higman Theorem 2.2 we have $r = p^m$ if $p$ is odd and not a Fermat prime. Thus, the estimates can be further improved in these cases.

**Remark 3.4.** Theorems 3.1 and 3.2 lend further support to the viewpoint that the “correct” definition of $2$-central $2$-groups (also those of height $k$) must involve $\Omega_2$ rather than $\Omega_1$. May be, this definition can also be used to extend to $p = 2$ some other results involving $p$-central $p$-groups of height $k$, which do not hold for $p = 2$ without this amendment.

### 4. Powerful Sylow $p$-subgroup

Recall that a finite $p$-group $P$ is **powerful** if $P^p \geq [P, P]$ for $p \neq 2$, or $P^4 \geq [P, P]$ for $p = 2$. Properties of powerful $p$-groups that we need here are well known since the original paper by Lubotzky and Mann [7]. In particular, if $P$ is a powerful $p$-group, then the subgroups $P^{p^i} = \langle g^{p^i} | g \in P \rangle$ form a central series of $P$, and $P^{p^i} = \langle g^{p^i} | g \in P \rangle$ for all $i$.  

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Theorem 4.1. If a finite p-soluble group $G$ has a powerful Sylow $p$-subgroup, then the $p$-length of $G$ is equal to 1.

Proof. We argue by contradiction. Let $G$ be a finite $p$-soluble group of minimal order with a powerful Sylow $p$-subgroup such that the $p$-length of $G$ is greater than 1. By minimality we must have $O_{p'}(G) = 1$. Since homomorphic images of powerful $p$-groups are powerful, it follows by minimality that $V := O_p(G)$ is an elementary abelian $p$-group. Then $G/V$ acts faithfully on $V$, which we can also regard as an $\mathbb{F}_p(G/V)$-module.

Let $Q$ be a Hall $p'$-subgroup of $O_{p,p'}(G)$. Then $Q$ acts faithfully on $V/C_V(Q)$, since the action is coprime. Clearly, $C_V(Q) = Z(O_{p,p'}(G))$, and therefore $C_V(Q)$ is normal in $G$. By minimality we must have $C_V(Q) = 1$.

By the generalized Frattini argument, $V N_G(Q) = G$. Let $S$ be a Sylow $p$-subgroup of $N_G(Q)$. Then $P := VS$ is a Sylow $p$-subgroup of $G$. Note that $V \cap S = 1$, since $C_V(Q) = 1$.

Choose an element $g \in P$ of maximal possible order $p^n$, so that $p^n$ is the exponent of $P$. From this moment on we consider separately the cases $p \neq 2$ and $p = 2$.

Case $p \neq 2$. Then $n \geq 2$. Indeed, a powerful $p$-group of exponent $p$ is abelian, and if we had $n = 1$, then $P$ would be abelian and the $p$-length of $G$ would be equal to 1, contrary to our assumption.

Hence the element $h = g^{p^{n-2}}$ is well defined. By the properties of powerful $p$-groups, $P^{p^{n-1}} \leq Z(P)$ and $P^{p^{n-2}} \leq \zeta_2(P)$. Therefore, $1 \neq h^p \in Z(P) \leq V$ and $h \in \zeta_2(P)$. Since $V$ is elementary abelian, we also have $h \not\in V$.

Since $P = VS$, we can represent $h$ as $h = vs$ for $v \in V$ and $s \in S$. Then $|s| = p$, because $s^p \in V \cap S = 1$. At the same time, $|vs| = |h| = p^2$. Hence the Jordan normal form of the linear transformation $T(s)$ of $V$ induced by the action of $s$ by conjugation must have a block of size $p \times p$. Therefore there is a vector $x \in V$ such that

$$x(T(s) - 1_V)^{p-1} \neq 0.$$ 

In terms of group commutators, this means that

$$\underbrace{[[[x, s], s], \ldots, s]}_{p-1} \neq 1.$$ 

But the action of $s$ on $V$ coincides with the action of $h = vs$. Therefore,

$$\underbrace{[[[x, h], h], \ldots, h]}_{p-1} \neq 1.$$ 

This contradicts the inclusion $h \in \zeta_2(P)$, since $p \geq 3$.

Case $p = 2$. Then $n \geq 3$. Indeed, a powerful 2-group of exponent 4 is abelian, and if we had $n \leq 2$, then $P$ would be abelian and the $p$-length of $G$ would be equal to 1, contrary to our assumption.

Hence the element $h = g^{2^{n-3}}$ is well defined. By the properties of powerful 2-groups, $P^{2^{n-1}} \leq Z(P)$, $P^{2^{n-2}} \leq \zeta_2(P)$, and $P^{2^{n-3}} \leq \zeta_3(P)$. Therefore, $1 \neq h^4 \in Z(P) \leq V$ and $h \in \zeta_3(P)$. Since $V$ is elementary abelian, we also have $h^2 \not\in V$. 
We again represent $h$ as $h = vs$ for $v \in V$ and $s \in S$. Then $|s| = 4$, since $s^4 \in V \cap S = 1$. At the same time, $|vs| = |h| = 8$. Hence the Jordan normal form of the linear transformation $T(s)$ of $V$ induced by the action of $s$ by conjugation must have a block of size $4 \times 4$. Therefore there is a vector $x \in V$ such that

$$x(T(s) - 1_V)^3 \neq 0.$$ 

In terms of group commutators, this means that

$$[[[x, s], s], s] \neq 1.$$ 

Since the action of $s$ on $V$ coincides with the action of $h = vs$, we also have

$$[[[x, h], h], h] \neq 1.$$ 

This contradicts the inclusion $h \in \zeta_3(P)$. \hfill \square

**Remark 4.2.** It is not immediately clear how to generalize the definition of powerful $p$-groups “dually” to the definition of $p$-central $p$-groups of height $k$. Probably, such a definition would also allow to prove a bound for the $p$-length of $p$-soluble group $G$ with a Sylow $p$-subgroup satisfying this definition. A rough bound for the $p$-length would follow by Hall–Higman theorems if such generalized “$k$-powerful” $p$-groups had the following property: if the exponent is $p^m$, then the nilpotency class is bounded by a function of $n$ and $k$ that is subexponential (even linear) in $n$. This would of course generalize the property of powerful $p$-groups, where the nilpotency class is at most $n$. Indeed, let $p^m$ be the exponent of the image of a Sylow $p$-subgroup $P$ of $G$ in $G/O_{p',p}(G)$. Let $V$ be the Frattini quotient of $O_{p',p}(G)/O_{p'}(G)$ regarded as an $F_p(G/O_{p',p}(G))$-module. As we saw in the proof of Theorem 3.2, then by Hall–Higman theorems there are elements $v \in V$ and $g \in P$ such that

$$[[[v, g], g], \ldots, g] \neq 1.$$ 

On the other hand, we would have

$$[[[v, g], g], \ldots, g] = 1$$

with the hypothetical function $f(k, n)$ bounding the nilpotency class. Hence,

$$p^m - p^{m-1} - 1 \leq f(k, m).$$

Provided the function $f(k, m)$ is subexponential in $m$ (and it is most likely and natural to have this function being linear in $m$), an estimate for $m$ would follow, which would in turn give an estimate for the $p$-length.
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