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ON p-SOLUBLE GROUPS WITH A GENERALIZED p-CENTRAL OR POWERFUL SYLOW p-SUBGROUP

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Abstract. Let G be a finite p-soluble group, and P a Sylow p-subgroup of G. It is proved that if all elements of P of order p (or of order $\leqslant 4$ for $p = 2$) are contained in the k-th term of the upper central series of P, then the p-length of G is at most $2m+1$, where m is the greatest integer such that $p^m - p^{m-1} \leqslant k$, and the exponent of the image of P in $G/O_{p',p}(G)$ is at most p^m. It is also proved that if P is a powerful p-group, then the p-length of G is equal to 1.

1. Introduction

A finite p-group P is called *p-central* if all its elements of order p are contained in the centre: $\Omega_1(P) \leqslant Z(P)$. Sometimes this definition is modified in the case of $p = 2$ to require that all elements of order $\leqslant 4$ belong to $Z(P)$. Such p-groups are in many respects dual to powerful p-groups (and the above-mentioned modification for $p = 2$ reflects the definition of powerful 2-groups). Although p-central p-groups received less attention in the literature than the very important case of powerful p-groups, there are several papers devoted to p-central p-groups and properties of their embeddings in finite groups; the reader can find relevant references in [4].

González-Sánchez and Weigel [4] initiated the study of more general classes: a finite p-group P is called *p^i-central of height k* if all its elements of order dividing p^i are contained in the k-th term of the upper central series: $\Omega_i(P) \leqslant \zeta_k(P)$. In particular, they proved [4, Theorem E] that if, for an odd prime p, a Sylow p-subgroup of a finite p-soluble group G is p-central of height $p-2$, then G has p-length 1.

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In this note we generalize this result to arbitrary height (including the case $p = 2$ with the above-mentioned proviso). Namely, we obtain a bound for the p-length of a p-soluble group G whose Sylow p-subgroup is p-central of height k (Theorem 3.1). This result is derived from a bound for the exponent of a Sylow p-subgroup of $G/O_{p',p}(G)$ (Theorem 3.2), which is proved on the basis of Hall–Higman theorems.

We also prove the result "dual" to [4, Theorem E], that if a finite p-soluble group G has a powerful Sylow p-subgroup, then the p-length of G is equal to 1 (Theorem 4.1).

2. Preliminaries

We shall need the following well-known property of coprime action by automorphisms. Recall that for a finite p-group P by definition $\Omega_i(P) = \langle g \in P \mid g^{p^i} = 1\rangle$.

Lemma 2.1 ([6, Kap. IV, Satz 5.12]). *Suppose that a finite p'-group A acts by automorphisms on a finite p-group P. If A acts trivially on $\Omega_1(P)$ for $p \neq 2$, or on $\Omega_2(P)$ for $p = 2$, then A acts trivially on P.*

Some other well-known properties of coprime actions of groups of automorphisms will be used without special references.

Recall that if a finite group G acts by automorphisms on an elementary abelian p-group V, then V can be regarded as a vector space over the field of p elements $\mathbb{F}_p$ and the action of G by conjugation on V can be regarded as action by linear transformations of this vector space. The linear transformation of V induced by an element $g \in G$ is denoted by $T(g)$. We use the right operator notation for this action: for $v \in V$ and $g \in G$ the image of v under $T(g)$ is denoted by $vT(g)$. For example, if V is a normal elementary abelian section of G, then G acts on V by conjugation and $vT(g)$ is equal to the image of the group element $\hat{v}^g$, where $\hat{v}$ is an inverse image of v in G. Note that $v(T(g) - \mathbf{1}_V)$, where $\mathbf{1}_V$ is the identity transformation of V, is equal to the image of the group commutator $[\hat{v}, g]$, which is also equal to $[v, g]$ in the natural semidirect product $V \rtimes G$.

We also recall Theorem B from the celebrated Hall–Higman paper [5].

Theorem 2.2 ([5, Theorem B]). *Let H be a p-soluble linear group over a field of characteristic p, with no normal p-subgroup greater than 1. If h is an element of order p^m in H, then the minimal equation of h is $(x-1)^r = 0$, where $r = p^m$, unless there is an integer m_0, not greater than m, such that $p^{m_0} - 1$ is a power of a prime q for which a Sylow q-subgroup of H is non-abelian, in which case, if m_0 is the least such integer, $p^m - p^{m-m_0} \leqslant r \leqslant p^m$.*

We shall only need the fact that we always have $p^m - p^{m-1} \leqslant r \leqslant p^m$.

When an element $g \in GL(V)$ of order p^m acts as a linear transformation on a vector space V over a field of characteristic p, its minimal polynomial always has the form $(x-1)^r = 0$, because $x^{p^m} - 1 = (x-1)^{p^m}$ in characteristic p. It follows that V has a basis in which the matrix of g has Jordan normal form, since the only eigenvalue is 1. The maximum size of Jordan blocks is $p^m \times p^m$.

It is well known that the natural semidirect product $V\langle g\rangle$ of groups V and $\langle g\rangle$ contains an element of order p^{m+1} if and only if there is at least one Jordan block of size $p^m \times p^m$.

3. Generalized p-central Sylow p-subgroup

Recall that $O_{p'}(G)$ is the maximal normal p'-subgroup of a finite group G; then $O_{p',p}(G)$ is the full inverse image of the maximal normal p-subgroup of $G/O_{p'}(G)$, and so on, defining by induction the terms of the *upper p-series* $O_{p',p,p',p,\dots}(G)$. A finite group G is *p-soluble* if $G = O_{p',p,p',p,\dots,p,p'}(G)$ and the minimum number of symbols p in this equation is called the *p-length* of G.

Theorem 3.1. *Let P be a Sylow p-subgroup of a finite p-soluble group G. Suppose that $\Omega_1(P) \leqslant \zeta_k(P)$ for $p \neq 2$, or $\Omega_2(P) \leqslant \zeta_k(P)$ for $p = 2$. Then the p-length of G is at most $2m+1$, where m is the maximum integer such that $p^m - p^{m-1} \leqslant k$.*

In particular, as a rough estimate, $m < 1 + \log_p k$, so that the p-length is at most $3 + 2\log_p k$.

Theorem 3.1 will follow from a bound for the exponent of a Sylow p-subgroup of $G/O_{p',p}(G)$.

Theorem 3.2. *Let P be a Sylow p-subgroup of a finite p-soluble group G. Suppose that $\Omega_1(P) \leqslant \zeta_k(P)$ for $p \neq 2$, or $\Omega_2(P) \leqslant \zeta_k(P)$ for $p = 2$. Then the exponent of a Sylow p-subgroup of $G/O_{p',p}(G)$ is at most p^m, where m is the maximum integer such that $p^m - p^{m-1} \leqslant k$.*

Proof. We can obviously assume that $O_{p'}(G) = 1$.

Let Q be a Hall p'-subgroup of $O_{p,p'}(G)$, so that $O_{p,p'}(G) = O_p(G)Q$. By the generalized Frattini argument,

$$G = O_{p,p'}(G)N_G(Q) = O_p(G)N_G(Q),$$

so we need to obtain a bound for the exponent of the image of a Sylow p-subgroup of $N_G(Q)$ in $\bar{G} = G/O_p(G)$. We use bars to denote images of elements or subsets in $\bar{G}$.

Let g be an element of a Sylow p-subgroup of $N_G(Q)$, so that $\bar{g}$ is its image in $\bar{G} = G/O_p(G)$. Let $|\bar{g}| = p^n$. We must show that $p^n - p^{n-1} \leqslant k$. We can of course assume that $n \geqslant 1$.

Since $O_p(\bar{G}) = 1$ and $\bar{Q} = O_{p'}(\bar{G})$, we have $C_{\bar{G}}(\bar{Q}) \leqslant \bar{Q}$ (see, for example, [2, Theorem 6.3.2]). Hence the p-element $\bar{g}$ acts faithfully on $\bar{Q}$; in other words, $[\bar{Q}, \bar{g}^{p^{n-1}}] \neq 1$. Clearly, $\bar{g}$ also acts on Q itself, and $[Q, \bar{g}^{p^{n-1}}] = [\bar{Q}, \bar{g}^{p^{n-1}}] \neq 1$.

Let Ω denote $\Omega_1(O_p(G))$ if $p \neq 2$, and $\Omega_2(O_p(G))$ if $p = 2$.

Consider a series of normal subgroups of G

$$1 = U_0 < U_1 < \cdots < U_n = \Omega \tag{3.1}$$

in which each factor U_{i+1}/U_i is an elementary abelian p-group contained in the centre of $O_p(G)/U_i$. Then the action of the semidirect product $Q\langle\bar{g}\rangle$ on each factor U_{i+1}/U_i is well defined.

Since $O_{p'}(G) = 1$, the p'-subgroup $[Q, \bar{g}^{p^{n-1}}] \neq 1$ acts faithfully on $O_p(G)$. By Lemma 2.1, moreover, $[Q, \bar{g}^{p^{n-1}}]$ acts faithfully on Ω. Since the action is coprime, we obtain that $[Q, \bar{g}^{p^{n-1}}]$ acts nontrivially on at least one of the factors V of the series (3.1). Let H denote the image of $Q\langle\bar{g}\rangle$ in the group of linear transformations of the vector space V over $\mathbb{F}_p$, which consists of elements $T(u)$ for $u \in Q\langle\bar{g}\rangle$ in accordance with our notation.

Since the subgroup $[Q, \bar{g}^{p^{n-1}}]$ acts non-trivially on V, we must have $O_p(H) = 1$. Indeed, otherwise $T(\bar{g})^{p^{n-1}}$ would be in $O_p(H)$ and then the image of $[Q, \bar{g}^{p^{n-1}}]$ would be in $O_q(H) \cap O_p(H) = 1$ and therefore trivial, contrary to the assumption. For the same reasons, $T(\bar{g})$ has the same order p^n.

By the Hall–Higman Theorem 2.2 the minimal polynomial of $T(\bar{g})$ is $(x-1)^r = 0$, where $p^n - p^{n-1} \leqslant r \leqslant p^n$. Therefore there is $v \in V$ such that

$$v(T(\bar{g}) - \mathbf{1}_V)^{p^n - p^{n-1} - 1} \neq 0. \tag{3.2}$$

Since the image of an element $u \in V$ under the linear transformation $T(\bar{g}) - \mathbf{1}_V$ is equal to the group commutator $[u, \bar{g}]$, it follows from (3.2) that

$$[...[[v, \underbrace{\bar{g}], \bar{g}], \ldots, \bar{g}}_{p^n - p^{n-1} - 1}] \neq 1.$$

But by the hypothesis of the theorem we have $\Omega \leqslant \Omega_1(P) \leqslant \zeta_k(P)$ for $p \neq 2$ (or $\Omega \leqslant \Omega_2(P) \leqslant \zeta_k(P)$ for $p = 2$). Therefore we must also have

$$[...[[v, \underbrace{\bar{g}], \bar{g}], \ldots, \bar{g}}_{k}] = 1.$$

It follows that $p^n - p^{n-1} - 1 < k$, as required. □

Proof of Theorem 3.2. Once we know a bound for the exponent p^e of a Sylow p-subgroup of $G/O_{p',p}(G)$, we obtain a bound for the p-length l of $G/O_{p',p}(G)$. Indeed, for $p \neq 2$ we have $e \geqslant [(l+1)/2]$ by the Hall–Higman theorem [5, Theorem A], and for $p = 2$ we have $e \geqslant l$ by Bryukhanova's theorem [1] (which is the best-possible improvement of the earlier estimate $2e - 2 \geqslant l$ by Gross [3]). Since $l + 1$ is exactly the p-length of G, the result follows from Theorem 3.2. □

Remark 3.3. The Hall–Higman Theorem A gives a better bound $e \geqslant l$ if p is not a Fermat prime. As noticed in the Hall–Higman paper [5], it follows from the proof that in the Hall–Higman Theorem 2.2 we have $r = p^m$ if p is odd and not a Fermat prime. Thus, the estimates can be further improved in these cases.

Remark 3.4. Theorems 3.1 and 3.2 lend further support to the viewpoint that the "correct" definition of 2-central 2-groups (also those of height k) must involve Ω_2 rather than Ω_1. May be, this definition can also be used to extend to $p = 2$ some other results involving p-central p-groups of height k, which do not hold for $p = 2$ without this amendment.

4. **Powerful Sylow p-subgroup**

Recall that a finite p-group P is *powerful* if $P^p \geqslant [P, P]$ for $p \neq 2$, or $P^4 \geqslant [P, P]$ for $p = 2$. Properties of powerful p-groups that we need here are well known since the original paper by Lubotzky and Mann [7]. In particular, if P is a powerful p-group, then the subgroups $P^{p^i} = \langle g^{p^i} \mid g \in P\rangle$ form a central series of P, and $P^{p^i} = \{g^{p^i} \mid g \in P\}$ for all i.

Theorem 4.1. *If a finite p-soluble group G has a powerful Sylow p-subgroup, then the p-length of G is equal to* 1.

Proof. We argue by contradiction. Let G be a finite p-soluble group of minimal order with a powerful Sylow p-subgroup such that the p-length of G is greater than 1. By minimality we must have $O_{p'}(G) = 1$. Since homomorphic images of powerful p-groups are powerful, it follows by minimality that $V := O_p(G)$ is an elementary abelian p-group. Then G/V acts faithfully on V, which we can also regard as an $\mathbb{F}_p(G/V)$-module.

Let Q be a Hall p'-subgroup of $O_{p,p'}(G)$. Then Q acts faithfully on $V/C_V(Q)$, since the action is coprime. Clearly, $C_V(Q) = Z(O_{p,p'}(G))$, and therefore $C_V(Q)$ is normal in G. By minimality we must have $C_V(Q) = 1$.

By the generalized Frattini argument, $VN_G(Q) = G$. Let S be a Sylow p-subgroup of $N_G(Q)$. Then $P := VS$ is a Sylow p-subgroup of G. Note that $V \cap S = 1$, since $C_V(Q) = 1$.

Choose an element $g \in P$ of maximal possible order p^n, so that p^n is the exponent of P. From this moment on we consider separately the cases $p \neq 2$ and $p = 2$.

Case $\boldsymbol{p \neq 2}$. Then $n \geqslant 2$. Indeed, a powerful p-group of exponent p is abelian, and if we had $n = 1$, then P would be abelian and the p-length of G would be equal to 1, contrary to our assumption.

Hence the element $h = g^{p^{n-2}}$ is well defined. By the properties of powerful p-groups, $P^{p^{n-1}} \leqslant Z(P)$ and $P^{p^{n-2}} \leqslant \zeta_2(P)$. Therefore, $1 \neq h^p \in Z(P) \leqslant V$ and $h \in \zeta_2(P)$. Since V is elementary abelian, we also have $h \notin V$.

Since $P = VS$, we can represent h as $h = vs$ for $v \in V$ and $s \in S$. Then $|s| = p$, because $s^p \in V \cap S = 1$. At the same time, $|vs| = |h| = p^2$. Hence the Jordan normal form of the linear transformation $T(s)$ of V induced by the action of s by conjugation must have a block of size $p \times p$. Therefore there is a vector $x \in V$ such that

$$x(T(s) - \mathbf{1}_V)^{p-1} \neq 0.$$

In terms of group commutators, this means that

$$[...[[x, \underbrace{s], s], \dots, s}_{p-1}] \neq 1.$$

But the action of s on V coincides with the action of $h = vs$. Therefore,

$$[...[[x, \underbrace{h], h], \dots, h}_{p-1}] \neq 1.$$

This contradicts the inclusion $h \in \zeta_2(P)$, since $p \geqslant 3$.

Case $\boldsymbol{p = 2}$. Then $n \geqslant 3$. Indeed, a powerful 2-group of exponent 4 is abelian, and if we had $n \leqslant 2$, then P would be abelian and the p-length of G would be equal to 1, contrary to our assumption.

Hence the element $h = g^{2^{n-3}}$ is well defined. By the properties of powerful 2-groups, $P^{2^{n-1}} \leqslant Z(P)$, $P^{2^{n-2}} \leqslant \zeta_2(P)$, and $P^{2^{n-3}} \leqslant \zeta_3(P)$. Therefore, $1 \neq h^4 \in Z(P) \leqslant V$ and $h \in \zeta_3(P)$. Since V is elementary abelian, we also have $h^2 \notin V$.

We again represent h as $h = vs$ for $v \in V$ and $s \in S$. Then $|s| = 4$, since $s^4 \in V \cap S = 1$. At the same time, $|vs| = |h| = 8$. Hence the Jordan normal form of the linear transformation $T(s)$ of V induced by the action of s by conjugation must have a block of size 4×4. Therefore there is a vector $x \in V$ such that

$$x(T(s) - \mathbf{1}_V)^3 \neq 0.$$

In terms of group commutators, this means that

$$[[[x, s], s], s] \neq 1.$$

Since the action of s on V coincides with the action of $h = vs$, we also have

$$[[[x, h], h], h] \neq 1.$$

This contradicts the inclusion $h \in \zeta_3(P)$. $\square$

Remark 4.2. It is not immediately clear how to generalize the definition of powerful p-groups "dually" to the definition of p-central p-groups of height k. Probably, such a definition would also allow to prove a bound for the p-length of p-soluble group G with a Sylow p-subgroup satisfying this definition. A rough bound for the p-length would follow by Hall–Higman theorems if such generalized "k-powerful" p-groups had the following property: if the exponent is p^n, then the nilpotency class is bounded by a function of n and k that is subexponential (even linear) in n. This would of course generalize the property of powerful p-groups, where the nilpotency class is at most n. Indeed, let p^m be the exponent of the image of a Sylow p-subgroup P of G in $G/O_{p',p}(G)$. Let V be the Frattini quotient of $O_{p',p}(G)/O_{p'}(G)$ regarded as an $\mathbb{F}_p(G/O_{p',p}(G))$-module. As we saw in the proof of Theorem 3.2, then by Hall–Higman theorems there are elements $v \in V$ and $g \in P$ such that

$$[...[[v, \underbrace{g], g], \dots, g}_{p^m - p^{m-1} - 1}] \neq 1.$$

On the other hand, we would have

$$[...[[v, \underbrace{g], g], \dots, g}_{f(k,m)}] = 1$$

with the hypothetical function $f(k, n)$ bounding the nilpotency class. Hence,

$$p^m - p^{m-1} - 1 \leqslant f(k, m).$$

Provided the function $f(k, m)$ is subexponential in m (and it is most likely and natural to have this function being linear in m), an estimate for m would follow, which would in turn give an estimate for the p-length.

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