Non-linear Control of Distillation Columns Using Approximate Feedback Linearization

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ABSTRACT
Distillation column control has been a challenging problem for numerous research activities. Review of past research reveals that among the advanced control methods applied to this system, little attention has been paid to feedback linearization, even in absolutely theoretical cases. This is due to the singularity of the system in steady-state, which invalidates the linearized control law. In this paper, to overcome this problem, a new idea has been suggested that is based on approximate feedback linearization. Based on this idea, an approximate non-linear model has been introduced for the distillation column which can exactly be linearized while reserving the structural properties of the basic system. The control law is derived from the approximate model and then applied to the basic system. The introduced error between the approximate and basic systems is then fed back by two single loop PID controllers to improve the performance of the system. Further, in this approach disturbances due to changes of feed flow and composition have been perfectly decoupled using the disturbance decoupling theory.

Keywords: Distillation column, Non-linear control, Approximate feedback linearization, singularity, Disturbance rejection

Introduction
Distillation is the most common approach for the separation of mixtures in chemical industries. More than 95% of separation processes are using distillation [1]. Distillation, as the last step in many chemical processes, has a considerable influence on the quality of the final products. Non-linearity, existence of strong interactions between inputs and outputs, presence of delay in the system dynamics and measuring instruments, large number of variables, presence of non-measurable disturbances and parameters uncertainty are some characteristics of distillation columns which make them a challenging problem in control theory. With the development of control theory and advent of modern industrial control systems in the last two decades, considerable efforts has been undertaken to involve modern control approaches in chemical processes, specially in distillation columns. Some of these methods are Model Based Control (MBC) [2-4], Model Predictive Control (MPC) [5-8], General Predictive Control (GPC) [9], Robust Control [10], Adaptive Control [11] and Self-Tuning Control [12]. The Differential Geometric methods, approach which may result in feedback linearization and coordinate transformation, is a common approach
to control non-linear systems. In this method, the non-linear system is transformed to a linear one and consequently, controlled by means of linear control methods. It is of great importance to take into account that linearization using feedback, in spite of local linearization, is a global approach. In this case, if the operating point of the system is changed, the controller will still operate properly as the control law has not been designed for specific point. This kind of control laws not only transforms non-linear systems to linear ones but also results in some SISO systems from a square MIMO one. Literature survey reveals that in distillation column control - even in chemical processes control as a whole little attention has been paid to feedback linearization compared with other methods. This is due to the following factors:

- This method is rather theoretical and can be implemented only when the system model is exactly known. There is no method to compensate for the effect of model mismatch.
- It’s impossible to put any constraints on the amplitude and variation rate of the control signal while in other approaches like MPC or MBC these constraints can be applied by changing the control problem to non-linear programming.
- Linearization is not applicable to a broad range of systems. In fact, there are certain necessary conditions required for global linearization.

The last item reveals that many process control problems cannot even theoretically be solved. In a binary distillation column the system steady state is a singular point. It means that feedback linearization law is valid in steady state. In other words, feedback linearization is not applicable to distillation columns.

To overcome this problem in distillation column control, several approaches have been suggested. In [13], a control law proportional to the value of $x_1 - k_2(x_2)$ has been used in the neighborhood of steady state that ensures the elimination of singularity in the control law. Actually, this method is a kind of switching control. The difficulty with this method is the determination of switching surface for which a trade-off has to be achieved between precision and amplitude of the control signal.

In [14], an aggregated model has been used and a local linearizing control law has been defined in the neighborhood of equilibrium points. Also, it has been proved that the internal dynamics of the system under control of the designed law is stable.

In [15], the output of a distillation column has been changed to $y=[x_2,x_3]'$ to overcome the singularity problem, which is a straightforward method in theory. But, one of its disadvantages is that the desired result is not obtained if the redefined output function is physically meaningless.

In [16], the $(D,V)$ model has been involved instead of the classical $(L,V)$ model, which has been identified by means of its step response. Instead of reflux flow, the flow of the distillate product has been used as the manipulated variable.

In all mentioned approaches, the control problem has been investigated from a chemical engineering point of view. In the present paper, authors try to deal with this problem from control theory point of view. With this in mind, one of the problems in control of binary distillation columns is the existence of singular sub-manifolds. This problem is of great importance in input-output linearization method. The existence of singular points changes the feedback linearization control law from global to a local one. The importance of this problem arises of the fact that the foremost advantage of feedback linearization is being global.

Different researches have overcome this problem in control theory. In [17] and [18], for the output function of an affine SISO system the concept of degree of singularity at time $t$ has been defined. This concept is in conjunction with the tangency of the state trajectory while entering a singular submanifold.

In [17], this problem has been solved assuming that the initial state of the system belongs to the singular submanifold but it has been also assumed that the state trajectory will never return to the same singular submanifold.

In [18], a family of output functions have been determined for the system that enables it for global output tracking. These functions are defined such that the state trajectory does not enter a singular submanifold while being tracked. Applying these results requires that:

1. The state trajectory never returns to the same singular submanifold.
2. Output function to be defined such that the state trajectory does not meet the singular submanifold.
3. The control signal be discontinuous.

It is obvious that neither of these conditions are satisfied for a binary distillation column as it has to operate in steady state, which is singular. Also, the control signal of this system is physically continuous.

Although the above results are important from the theoretical point of view, but to have an exact tracking of a "trackable" output function, the initial conditions and the exact model of the system have to be known. It is also assumed that the system is not influenced by any kinds of disturbances. Also, these results are only applicable to SISO systems. To overcome these shortcomings in [19] a method has been introduced that can be applied to MIMO systems under the effect of disturbances. In this method, output tracking is achieved approximately. The most important disadvantage of this method is that the designed control law is discrete while in a broad category of physical systems, like distillation columns, the control signals (manipulated variables) are continuous.

Another method to overcome this problem is to change the output function. The method given in [15] is of this type. This is a classic approach to overcome the singularity problem [20]. Unfortunately, in this method tracking problem cannot be solved easily, which can be considered a disadvantage of this approach.

Using approximate linearization approaches is another alternative for control of non-linear systems. Some of these approaches have been derived by extending the Jacobian linearization idea. Some examples are extended linearization in [21], [22] and [23] and pseudo-linearization in [24]. The basic idea is to extend the linearization around one operating point to that around multiple operating points.

Another idea is to find an approximate feedback linearization or an approximate coordinate transformation that may result in an approximate linear system. The designed control law in this approach is global, which is known to be an important advantage of it. The idea behind this method is to use approximate non-linear models, rather than the exact ones, that satisfy the necessary conditions for being linearized. This idea is applicable to both input state [25] and input-output linearization [26-27]. In this method, if a linearized feedback cannot be designed for the system $(f, g, h)$, an approximate system $(\tilde{f}, \tilde{g}, \tilde{h})$ will be defined such that it is an appropriate approximation to the exact one and it is feedback linearizable, at the same time. The control law is derived based on the approximate model and then applied to the exact one. This method has been used in [27]. The results show that this approach is much more effective than those of extended-and pseudo-linearization, but one of its disadvantages is that there is no feedback of the error between the exact and the approximate models. In [28], it has been shown that considering this error will considerably improve the performance of the controller.

In this paper, the idea of approximate feedback linearization has been used and the idea given in [28] has been extended to use PID controllers to feedback the error resulting from the difference between the exact and the approximate models.

An important point in approximate linearization approach is the method by which the approximate non-linear model is derived. In [26] and [27], two methods have been introduced for this purpose. The method given in [27] is based on differentiation from output and elimination of higher-order terms that prohibit linearization. In fact, only the output function of the system is approximated but in the method given in [10], the system $(f, g)$ is approximated by $(\tilde{f}, \tilde{g})$ such that the involutivity condition is satisfied for the approximate system.

Unfortunately, none of these two methods can be applied to distillation column models [29]. To use the method given in [26], the linearized model around the equilibrium point should be controllable which is not satisfied in a distillation column model, as proved in [29]. Also, the output function of distillation column does not contain any higher-order terms. Hence, the approach given in [27] will also be inapplicable to this system.

In the current paper, the input-output linearization by means of feedback - and disturbance decoupling is first introduced. Then, it is shown that this approach cannot be applied to distillation columns. Then, the idea of approximate linearization by means of an approximate non-linear model is discussed. Finally, the results of this method are compared with that of a PID controller.
Distillation Column Control Using Exact Feedback Linearization

In this section, the singularity problem of a binary distillation column will be investigated using disturbance decoupling theory. It is essential to notice that disturbance decoupling in non-linear systems is in fact an extention of the linearization approach applied to non-linear systems under the influence of disturbances. Therefore, in this paper these two approaches are considered to be equal, unless otherwise specified.

The Dynamic Model of a Binary Distillation Column

A variety of dynamic models have been given for different distillation columns [14-16 and 30-31]. In this paper, a binary distillation column will be considered. A model for such a system is as follows [14]:

\[
\begin{align*}
H_i \dot{x}_i &= (k(x_z) - x_i)V, \text{ Condenser} \\
H_j \dot{x}_j &= (x_{j-1} - x_j)L + (k(x_{j+1}) - k(x_j))V, \ j = 2, ..., j_f \\
H_j \dot{x}_j &= (x_{j-1} - x_j)L + (k(x_{j+1}) - k(x_j))V, \ j = 1, ..., n-1 \\
H_n \dot{x}_n &= (x_{n-1} - x_n)(L + F) + (x_n - k(x_n))V, \text{ Reboiler}
\end{align*}
\]  

(1)

Where \( H_i \) is the liquid hold-up in the \( i \)th tray, \( x_i \) is the liquid phase composition in the \( i \)th tray, \( y_j \) is the vapor composition in the \( j \)th tray, \( N \) is the number of trays, \( V, R \) and \( F \) are the flow rate of vapor, reflux, and unit feed, respectively and \( z_F \) is the feed composition. This model has been derived using mass-balance equations under the following assumptions:

1. Liquid and vapor phases are in thermodynamic equilibrium on each tray and are perfectly mixed.

2. Liquid hold-up is constant on each tray. Vapor hold-up in each tray is ignored. Also, pressure is constant and its distribution is uniform.

3. The flow rate of incoming liquid on each tray is equal to that of the outgoing one. This rule also satisfies the flow of the incoming and outgoing vapor in each tray (except trays 1 and \( n \)). Also, the system feed is assumed to be a saturated liquid.

The equations define a non-linear system in the following form:

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i + \sum_{j=1}^{N} d_j(x)w_j \\
y_i &= h_i(x), \ i = 1, ..., m
\end{align*}
\]  

(2)

A common approach to control of these systems is the application of disturbance decoupling theory (Appendix A). If all the necessary conditions are met, a static feedback control law can be designed such that:

1. Disturbances \( w_j \) have no effect on outputs \( y_j \).

2. A suitable reference trajectory can be tracked. In other words, it will be possible to assign a linear relation between input and output (Appendix A).

In [7], the application of this theory in the control of the binary distillation column defined by (1) has been investigated. It has been shown that applying this theory leads to the singularity problem. In fact to investigate the possibility of disturbance decoupling, first the non-singularity conditions of decoupling matrix \( A(x) \) has to be checked (Appendix A). Considering model (1) and relation (AS), \( A(x) \) can be written as:

\[
A(x) = \begin{bmatrix}
0 & k_z(x_z) - x_i \\
x_{n-1} - x_n & x_n - k_z(x_n) \\
\end{bmatrix}
\]  

(3)

Further, in equilibrium the following relation can be deduced from equation (1):

\[
0 = [k_z(x_z) - x_i] \bar{V}
\]  

(4)

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where \( \bar{V} \) and \( \bar{x} \) are the steady-state values of the system. This relation shows that in steady state the decoupling matrix can be stated as follow:

\[
A(\bar{x}) = \begin{bmatrix}
- \bar{x}_i - \bar{x}_i^* & 0 \\
0 & - \bar{x}_i - \bar{x}_i^* \\
H_a & 0 \\
H_a & H_a
\end{bmatrix}
\]  

(5)

Therefore, \( A(\bar{x}) \) is singular and as a result, disturbance decoupling method cannot be applied. As the system reaches the steady state, the control signal amplitude goes to infinity. In other words, the system singularity set is the same as the equilibrium point of the system. In practice, if such a control law is designed, the control signal will be saturated as the system reaches the steady state and this means that the designed control law is invalid.

These results lead to development of a new method to derive an approximate non-linear model for system (1), which is the subject of the next section.

**The Approximate Non-linear Model**

As stated earlier, the approach given in [26] and [27] cannot be used to derive the approximate linearizable model of a distillation column (1). In this section, a new idea is introduced to define the approximate model of distillation columns. The proposed idea is based on the distillation column model to satisfy Rosenbrock theorem.

Based on different assumptions in a binary distillation column, two dynamic models have been proposed in [14] and [32]. It has been shown in these papers that both models satisfy the Rosenbrock theorem (Appendix B). In addition, in [14] it has been shown that using Tikhonov theorem in two-scale systems yields a reduced-order model that satisfies the Rosenbrock theorem.

Considering the above points, the following idea is proposed:

Consider the following system that satisfies the Rosenbrock theorem:

\[
\dot{\xi} = \phi(\xi)
\]  

(6)

in which \( \xi \in \Omega \subseteq \mathbb{R}^n \) and \( \Omega \) is a bounded, closed and convex set. Further, \( \phi = (\phi_i)_{i=1...n} \) and its partial derivatives are continuous functions of \( \xi \).

Now, an approximate system is defined as follows:

\[
\dot{\tilde{\xi}} = \tilde{\phi}(\tilde{\xi})
\]  

(7)

where \( \tilde{\xi} \in \tilde{\Omega} \subseteq \mathbb{R}^n \) and \( \tilde{\Omega} \) is a bounded, closed and convex set. Also, \( \tilde{\phi} = (\tilde{\phi}_i)_{i=1...n} \) and its partial derivatives are continuous functions of \( \tilde{\xi} \).

Moreover, system (7) must satisfy the Rosenbrock theorem.

If system (7) is defined such that \( \tilde{\Omega} = \Omega \), then defining the difference between the two systems as \( e = \xi - \tilde{\xi} \) and also considering that \( \tilde{\xi}(t; t_0; \tilde{\xi}_0) \in \tilde{\Omega} \), \( \xi(t; t_0; \xi_0) \in \Omega \), and \( \Omega = \tilde{\Omega} \), one can write:

\[
\forall t_0 \in \mathbb{R}^n, \forall \xi_0 \in \Omega, \forall \tilde{\xi}_0 \in \tilde{\Omega}, \exists \varepsilon \quad \forall t \geq t_0, e(t; t_0; \xi_0; \tilde{\xi}_0) \in \mathbb{R}^n, \forall t \geq t_0, e(t; t_0; \xi_0; \tilde{\xi}_0) \in \text{Br}
\]

(8)

The statements (8) state that the error \( e \) is bounded over a known region. Moreover, as the steady states of systems (6) and (7) are unique (according to Rosenbrock theorem), a one-to-one mapping can be found between the steady states of the exact and approximate systems.

According to the above idea, the suggested method has the following steps:

1. Addition of a term to the exact model (1) so that the approximate model can be exactly linearized.
2. Verification of the Rosenbrock theorem satisfaction for the approximate model.
3. Verification of the equality \( \tilde{\Omega} = \Omega \).

The approximate non-linear model is suggested as follows:
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\[
H_j \dot{x}_j = (1 - x_j) \dot{L} + (k(\bar{x}_j) - x_j) \dot{V}
\]
\[
L_j \ddot{x}_j = (\bar{x}_j - x_j) \ddot{L} + (k(\bar{x}_j) - k(\bar{x}_j)) \ddot{V}, \quad j = 2, \ldots, J
\]
\[
H_j \dddot{x}_j = (\bar{x}_j - x_j) \dddot{L} + (k(\bar{x}_j) - k(\bar{x}_j)) \dddot{V}, \quad j = 2, \ldots, J
\]
\[
H_n \dddot{x}_n = (\bar{x}_n - x_n) \dddot{L} + (k(\bar{x}_n) - k(\bar{x}_n)) \dddot{V}
\]

(9)

It is obvious in equations (9) that the term \((1 - \bar{x}_j) \dddot{L}\) has been added to the condenser equation. This provides the possibility to have an exact linearization.

In [14], it has been proved that the model (1) satisfies the Rosenbrock theorem if the following assumptions are not violated:

**Assumption (1):** Input and disturbance functions \((L(t), V(t), F(t), z_p(t))\) are continuous functions from \([0, +\infty)\) to \([0, +\infty)\) such that:

\[
\forall t \in [0, +\infty) : \begin{cases} L(t) < V(t) < L(t) + V(t) \\
z_p(t) \in (0, 1) \end{cases}
\]

(10)

Assumption (2): Thermodynamic equilibrium function \(y = k(x)\) and its derivatives are continuous from \([0, 1]\) to \([0, 1]\) and also:

\[
k(0) = 0, \quad k(1) = 1
\]
\[
\forall x \in [0, 1] : \frac{dk(x)}{dx} > 0
\]

(11)

Assumption (1) states that the flow rates of distillation column products are always positive, which is physically always true. Assumption (2) is valid for binary mixtures. For more general cases, the reader is referred to [33]. In practice, thermodynamic equilibrium functions are derived by solving the non-linear algebraic equations describing the thermodynamic equilibrium. The thermodynamic equilibrium equations depend on the type of the thermodynamic model and are usually solved numerically. For instance, in [14] the model given in [34] has been used for a depropanizer unit. Another method is to use the following relation:

\[
y_j = k_j(x_j) = \frac{\alpha_j x_j}{1 + (\alpha_j - 1)x_j}
\]

(12)

where \(\alpha_j\) stands for relative volatility in the \(j\)th tray.

Considering assumptions (1) and (2) it can be verified that the approximate model (9) satisfies the Rosenbrock theorem. As a result, it is possible to introduce a theorem similar to those given in [14]:

**Theorem (1) [29]:** Assume that assumptions (1) and (2) are valid. Then:

1. For each initial condition \(\bar{x}^0 \in [0, 1]^n\), the response of the approximate model (9) on the time interval of \([0, +\infty)\) is defined, and:

\[
\bar{x}(t, \bar{x}^0) \in [0, 1]^n, \quad \forall t \in [0, +\infty]
\]

(13)

in which \(\bar{x}(t, \bar{x}^0)\) is the response to \(\bar{x}^0\).

2. For constant \((L(t), V(t), F(t), z_p(t))\), a unique steady state \(\bar{x}\) exists on \((0, 1)^n\) which is the unique solution of \(\bar{f}(\bar{x}, \bar{u}, \bar{w}) = 0\). In addition, if the thermodynamic equilibrium functions satisfy the condition \(k(\bar{x}) > \bar{x}\) on \((0, 1)\), then:

\[
1 > \bar{x}_1 > \bar{x}_2 > \cdots > \bar{x}_{n-1} > \bar{x}_n > 0
\]

(14)

3. If \((L(t), V(t), F(t), z_p(t))\) are constant and if \(\bar{x}^0 \in [0, 1]^n\), then the approximate system (9) is stable in the sense of Liapunov and its response converges to the unique steady state associated with \((L(t), V(t), F(t), z_p(t))\).

**Proof of (1):** To prove theorem (1), it is enough to show that the vector field \(\bar{f}\) in (9) is oriented inward on \(\partial D\), the boundary of \(D = [0, 1]^n\), where:

\[
\partial D = \{ \bar{x} \in \mathbb{R}^n | \exists j \in [1, \ldots, n] : \bar{x}_j = 0 \lor \exists \bar{y}_j = 1 \}
\]

(15)

that is, it must be proved that:

\[
\forall \bar{x} \in \partial D : \begin{cases} \bar{x}_j = 0 \Rightarrow \bar{x}_j > 0 \\
\bar{x}_j = 1 \Rightarrow \bar{x}_j < 0 \end{cases}, \quad j = 1, 2, \ldots, n
\]

(16)
If $\bar{x}_i = 0$, then $\ddot{x}_i = -\frac{1}{H}$, and if $\bar{x}_i = 1$, then $\ddot{x}_i = -\frac{k(\bar{x}_i)}{H_i} \bar{v}$. Repeating this for $j = 2, 3, \ldots, n$, it can be shown that the vector field $\vec{f}(\bar{x}, \bar{u}, w)$ is oriented inward on $\partial \bar{D}$.

**Proof of (2) and (3):** To prove the existence, unity and global asymptotic stability of the system response, the Rosenbrock theorem is used. First, it is shown that the approximate model (9) satisfies the conditions of Rosenbrock theorem. For the approximate model of distillation column, $\vec{\xi}_i = H \bar{x}_i$, $k = 1, 2, \ldots, n$, and the vector field $\vec{\phi}(\vec{\xi})$ is defined based on (9).

**Rosenbrock first condition:** Referring back to proof (1), it can be concluded that this condition is satisfied.

**Rosenbrock second condition:** Using relation (B2), the functions $\vec{\psi}_i(\vec{\xi})$ are as follows:

$$\vec{\psi}_i(\vec{\xi}) = \frac{\bar{v}}{H}, \quad \vec{\psi}_i(\vec{\xi}) = \frac{1}{H} \left( \bar{L} + F - \bar{v} \right)$$

Considering assumption (1), $\vec{\psi}_i(\vec{\xi}) > 0$, and $\vec{\psi}_i(\vec{\xi}) > 0$. Hence, the rosenbrock second condition is also satisfied.

**Rosenbrock third condition:** Considering the structure of system (9) the Jacobian matrix is in a tridiagonal form. On the other hand this condition doesn’t put any constraints on the elements of the main diagonal. So the validity of this condition can be investigated only for the upper and lower elements of the main diagonal. The lower elements of the main diagonal are derived as follow:

$$\frac{\partial \phi_i}{\partial \bar{x}_j} = \frac{\bar{L}}{H_i}, \quad i = 2, \ldots, j - 1$$

$$\frac{\partial \phi_i}{\partial \bar{x}_j} = \left( \frac{\bar{L} + F}{H} \right) \bar{v}, \quad i = j, \ldots, n - 1$$

The upper elements are derived from the following relations:

$$\frac{\partial \phi_i}{\partial \bar{x}_j} = \left( \frac{\bar{L} + F}{H} \right) \bar{v}, \quad i = 2, \ldots, n$$

Also, $\frac{\partial \phi_i}{\partial \bar{x}_j} = 0, (i \neq k, k \neq i \pm 1)$ for the other elements. Considering assumptions (1) and (2), it is obvious that the Rosenbrock third condition is also satisfied.

**Rosenbrock fourth condition:** Calculation of $\vec{\psi}_i$ for system (9) leads to relations (17), which show that:

$$\forall i \in \{2, \ldots, n - 1\}: \psi_i = 0$$

Because of the tridiagonal structure of (9) and according to (18) and (19), by choosing $j = i + 1$, it can be observed that $\frac{\partial \phi_i}{\partial \bar{x}_i} = 0$. Now, if $i = n - 1$, then $\vec{\psi}_j = \vec{\psi}_{j+1} = \vec{\psi}_n = 0$. Hence, the Rosenbrock fourth condition is satisfied, but if $i < n - 1$, then $\vec{\psi}_j = 0$. In the latter case, $k$ has been selected such that $k = j + 1$. The tridiagonal structure of (9) and relations (18) and (19) will again confirm that $\frac{\partial \phi_i}{\partial \bar{x}_i} = 0$, and if $k = n$, then $\psi_i = \psi_n = 0$, but if $k < n$, again $\psi_n = 0$. Repeating this algorithm finally leads to an index $l = n$ such that $\vec{\psi}_l = \vec{\psi}_n = 0$ (It is also possible to follow this algorithm with index reduction). So, the Rosenbrock fourth condition is also satisfied.

Now, it must be proved that the steady state of the approximate system (9) lies within the open interval $(0, 1)^n$. Since $\vec{x} \in [0, 1]^n$ satisfies the relation $\vec{f}(\vec{x}, \bar{u}, w) = 0$, then:

$$k(\vec{x}, \bar{u}, w) = \vec{x} + (\vec{l} + F - \bar{v}) \bar{x}$$

$$F \vec{x} + \vec{l} = \bar{v} \bar{x} + (\vec{l} + F - \bar{v}) \bar{x}$$

$$k(\vec{x}, \bar{u}, w) = \bar{x} + \bar{v} \bar{x}$$
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\[
\tilde{x}_{i,j} = \frac{\tilde{V}}{L + F} k(\tilde{x}_i) + \left(1 - \frac{\tilde{V}}{L + F}\right) \tilde{x}_i, \quad j = j_1, \ldots, n
\]  
(23)

Equation (21) is derived by adding up the equations (9) in steady state. The addition of the first \( j \) equations yields (22) while adding up the last \( n - j + 1 \) equations results in (23). If \( \tilde{x}_i = 0 \), using (22) it is concluded that \( \tilde{x}_{j_f} = 0 \).

Substituting this in (23) yields \( \tilde{x}_i = 0 \), which is in contradiction with (21). So, \( \tilde{x}_i = 0 \). Also, if \( \tilde{x}_i = 1 \), then (22) shows that \( k(x_i) = \ldots = k(x_{j_f}) = 1 \) and hence, \( x_i = \ldots = x_{j_f} = 1 \). Substitution of these result in the feed tray equation in equilibrium leads to the following relation:

\[
F(\tilde{x}_i - 1) + (k(\tilde{x}_{j_f}) - 1) \tilde{V} = 0
\]

(24)

which is valid only for \( k(x_{j_f}) > 1 \). This contradicts with assumption (2). So, \( \tilde{x}_i \neq 1 \). Following the same procedure, it can be proved that \( \tilde{x} \in (0,1) \).

Considering that thermodynamic equilibrium is described by increasing functions, the inequality (14) can also be proved easily. For example:

\[
0 = (1 - \tilde{x}_i)L + (k(\tilde{x}_i) - \tilde{x}_i) \tilde{V} \Rightarrow k(\tilde{x}_i) > \tilde{x}_i \Rightarrow \tilde{x}_i < \tilde{x}_i \\
0 = (\tilde{x}_i - \tilde{x}_i)L + (k(\tilde{x}_i) - k(\tilde{x}_i)) \tilde{V} \Rightarrow k(\tilde{x}_i) < k(\tilde{x}_i) \Rightarrow \tilde{x}_i < \tilde{x}_2
\]

\[
\ldots
\]

(25)

**Result (1):** Considering that \( \tilde{x} \in \tilde{D} = D = [0,1] \) and also assumptions (1) and (2) yields:

\[
\forall x, \tilde{x} \in [0,1], \exists r \in B_r(x - \tilde{x} \in B_r)
\]

(26)

\[
B_r = \left\{ \varepsilon \in \mathbb{R}^n \mid \|\varepsilon\|_2 < r \right\}
\]

which reveals that the error between the exact and approximate systems is bounded. The error range in relation (26) is wide but the simulation results show that this error is much less than that expressed in this relation.

**Remark (1):** It is obvious that error minimization has not been involved as a criterion to define the approximate model. Hence, it cannot be claimed that this model is optimal. On the other hand, considering error minimization problem alone, could lead to an approximate model whose decoupling matrix is singular in steady state. To overcome this problem and to define an optimal approximate model, constraints such as \( \text{rank } A(\tilde{x}) = 2 \) or \( |\det A(\tilde{x})| > \varepsilon > 0 \) \( \forall \tilde{x} \in [0,1] \) can be defined.

**Remark (2):** The subject of approximate model is of no concern to modeling of chemical processes. Approximate model is in fact a kind of mapping.

**Lemma (1) [29]:** The mapping \( \tilde{x} = \tau(x) \) is one-to-one.

**Proof:** In [14], using Rosenbrock theorem it has been shown that a one-to-one mapping exists between \( (L, V, F, z_F) \) and \( \tilde{x} \). Also theorem (1) shows that the mapping between \( (L, \tilde{V}, F, \tilde{z}_F) \) and \( \tilde{x} \) is one-to-one. Now, if \( (L, \tilde{V}, F, \tilde{z}_F) = (L, V, F, z_F) \), then a one-to-one mapping will also exist between the steady states of the exact and approximate systems.

**Approximate Feedback Linearization**

In this section, an approximate linearized feedback is designed for a binary distillation column using the approximate linearization idea given in [25], [26] and [27] and also the approximate model (9). To do this, the following procedure is involved:

1. Designing an exact feedback linearization for the approximate model
2. Applying the control law designed for the approximate model to the exact one
3. Using the error feedback between the approximate and the exact systems to improve the system operation

Using the relations given in Appendix A and the approximate model (9), the exact feedback linearization for the approximate model is computed as below:

\[
\tilde{u} = \begin{bmatrix} \tilde{L} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} 1 - \tilde{x}_i & k(\tilde{x}_i) - \tilde{y}_1 \\ H_i & \tilde{y}_1 - \tilde{x}_i \\ \tilde{x}_i - \tilde{y}_1 & H_i \\ H_s & \tilde{y}_1 - \tilde{x}_i \\ \tilde{x}_i - k(\tilde{x}_i) & H_i \\ H_s & \tilde{y}_1 - \tilde{x}_i \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}
\]

(27)

(The results of the numerical simulation shows that the determinant of the decoupling matrix in (26) is always non-zero). Applying the control law (26) to approximate model (9) yields:

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The system output at an assigned constant value or to make it track a reference trajectory, a reference signal such as

\[
\hat{y}_d = \hat{v}_1
\]

\[
\hat{y}_s = \hat{v}_2
\]

is defined and a new system input \( \hat{v} \) is selected as follows:

\[
\hat{v}_1 = \hat{y}_r - \alpha_1 (\hat{y}_1 - y_1)
\]

\[
\hat{v}_2 = \hat{y}_r - \alpha_2 (\hat{y}_2 - y_2)
\]

In which \( \alpha_1 \) and \( \alpha_2 \) are the exponential convergence coefficients. Defining the error vector of the system as \( e = \hat{y} - y \) leads to the following linear relation for the system tracking error:

\[
\dot{e}_1 + \alpha_1 e_1 = 0
\]

\[
\dot{e}_2 + \alpha_2 e_2 = 0
\]

If \( \alpha_1 \) and \( \alpha_2 \) are selected such that (30) satisfies the Hurwitz stability criterion, then the tracking error of the system will exponentially approach zero.

According to the idea given in [28], the control law for the basic system is designed as follows:

\[
u = \left[ \begin{array}{c} \frac{1-\hat{x}_1}{\hat{H}_1} \\ \frac{\hat{x}_1 - \hat{x}_2}{\hat{H}_1} \\ \frac{\hat{x}_2 - \hat{x}_3}{\hat{H}_2} \\ \frac{\hat{x}_3 - \hat{H}_3}{\hat{H}_2} \\ \frac{\hat{H}_3}{\hat{H}_2} \end{array} \right] \left[ \begin{array}{c} v_1 \\ v_2 \\ \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{array} \right]
\]

It is obvious from (31) that the state variables of the approximate system have been used for computing the control signal. However, \( v \) is calculated differently. In this paper, according to [29], it is suggested to use the following relations instead of those given in [28] (unity feedback of error):

\[
v_1 = \hat{y}_r - \alpha_1 (\hat{y}_r - y_1) + \int k_{e_1}(\hat{e}_1 + \frac{k_{e_2}}{\alpha_2} \hat{e}_2) dt
\]

\[
v_2 = \hat{y}_r - \alpha_2 (\hat{y}_r - y_2) + \int k_{e_3}(\hat{e}_3 + \frac{k_{e_4}}{\alpha_4} \hat{e}_4) dt
\]

In these relations, \( z \) is the difference between the exact and the approximate systems, i.e.: \( z = \hat{y} - y \)

Selection of coefficients \( k \) in (32) depends on the relative importance of the products and energy consumption of the system. In fact, an objective function has to be defined and parameters \( k \) have to be selected such that the objective function becomes minimum, which yields an optimum behavior for the system. In this paper, a special algorithm has not been introduced for tuning the coefficients of equation (32), but the simulation results show that they can be easily tuned by trial and error. It is of course clear that the controller performance will be significantly improved if a numerical or an analytical algorithm is developed to tune the coefficients. Finally, applying the control laws (31) and (32) leads to the approximate linearization of the basic system. The schematic diagram of the above control system has been depicted in Figure 1.
Simulation Results
A set of simulations were performed using feedback linearization with redefined output, as in [15], approximate feedback linearization without error feedback as in [26], approximate feedback linearization using unity feedback, as in [28], and finally, the proposed method of this paper. For simplicity, these four methods are hereafter called methods 1 to 4. The conditions under which the simulations were performed and the main results obtained are described in this part. The nominal values of the parameters and molar fractions (steady-state) in the column are given in Tables (1) and (2). The reference signal to be tracked, \( y_r(t) \), is the output of a linear reference model described by:

\[
x_M = \begin{bmatrix} -0.0003125 & 0 \\ 0 & -0.0003125 \end{bmatrix} x_M + \begin{bmatrix} 0.0003125 \\ 0 \end{bmatrix} u_M
\]

\[
y_r = x_M, x_M(0) = [0.890.118]^T, \\
\alpha = [0.0003125, 0.0003125]^T
\]

(34)

with state \( x_M \in \mathbb{R}^2 \), input \( u_r \in \mathbb{R}^1 \) and output \( y_r \in \mathbb{R}^2 \).

In Fig. 2, the simulation results for the error between the exact model and the approximate one, \( z = \bar{x} - x \), has been depicted. According to (26), it is obvious that \( z \) is bounded and in this figure, it is illustrated that \( B \), is more bounded than that shown in (26).

| Table 1. Nominal Values of Parameters of Binary Distillation Column, [15] |
|-----------------|------------------|
| Number of Trays | 9                |
| Feed Tray No.   | 5                |
| Feed Flow       | 1.815 mol/sec    |
| Feed Composition| 0.5              |
| Distillate Product Initial Set-point | 0.89 |
| Bottom Product Initial Set-point   | 0.118            |
| Reflux Initial Flow  | 0.463 mol/sec   |
| Vapor Initial Flow    | 1.323 mol/sec    |

| Table 2. Initial Values and Parameters of Binary Distillation Column, [15] |
|-----------------|----------------|
| Tray No.        | Liquid Holdup (mol) | Relative Volatility Coefficient | Initial state |
| 1 (Condenser)   | 10000            | -                              | 0.890         |
| 2               | 1000             | 2.989                          | 0.719         |
| 3               | 1000             | 2.989                          | 0.612         |
| 4               | 1000             | 3.473                          | 0.545         |
| 5               | 1000             | 3.637                          | 0.505         |
| 6               | 1000             | 3.773                          | 0.490         |
| 7               | 1000             | 3.914                          | 0.443         |
| 8               | 1000             | 4.598                          | 0.315         |
| 9 (Reboiler)    | 20000            | 6.267                          | 0.118         |
The distillation model is first allowed to stabilize around the equilibrium point shown in Table (3); that is, no changes are made in the dynamics of the reference model. Simultaneously, the disturbance signals are kept constant at their values (i.e., no disturbances are introduced) during the first 20,000 seconds. After this time, the step changes $\Delta u_y$, $\Delta L_f$ and $\Delta x_f$ of Table (1) are introduced, obtaining the results shown in Figure 2.

| Table 3. Values and Time Duration of Step Changes in Setpoints and disturbance |
|--------------------------------------------------|---------------|-----------------|
| Step Change                                      | Value         | Application     |
| Distillate Product Set-point                     | +5%           | 2000            |
| Bottom Product Set-point                         | -25%          | 2000            |
| Feed Flow                                        | +20%          | 12000           |
| Feed Composition                                 | +20%          | 12000           |

**Figure 2. State Difference Between Exact and Approximate Models**

It is essential to mention that in [15] feedback linearization is done using redefined output function $\psi = \begin{bmatrix} x_1; x_0 \end{bmatrix}^T$. So, the set-point of the distillation column in this approach is different from the above mentioned values. These values are given in Table (3). Considering the relation $\tilde{X}_1 = k_1(\tilde{x}_2)$ and that $\tilde{x}_2 = 0.82685$ leads to $\tilde{x}_1 = 0.9345$, which shows that both methods are in the same experimental conditions.

In Figure 3, the simulation results for both methods 1 and 2 have been depicted. Since the approximate model has been used, the system has a steady-state error but its tracking characteristics for the upper products of the column have been improved from the point of view of amplitude and the rate at which the system reaches the steady state. In contrast, the tracking error of the distillation column bottom products has increased. It must be noticed that the increase of error is only 0.1% while the reduction of error for the upper products is 1.5%.
**Non-linear Control of Distillation Columns**

Figure 3. Simulation Results for Methods 1 (- -) and 2(-)

Figure 4. Simulation Results for Methods 1(-), 2 (- -) and 3 ( - - )
In Figure 4, the simulation results for the methods 1, 3 and 4 can be observed. Feedback parameters of (32) have been tuned based on the values given in Table (5).

To compare these methods, single loop PID controllers have been used [29]. The response of the system using the PID controller has been shown in Fig. 5. The simulation results show that the performance of the PID controller is better than that of method 1 in that its error amplitude is less with reduced time constant. But, an important point to consider is that PID controllers cannot completely reject disturbances. In fact, the main advantage of model-based controllers over sensor-based ones is in disturbance rejection. To better investigate this problem, it is assumed that the feed flow of the distillation column changes according to Figures 6.A and 7.A. The responses of the PID controller and method 4 have been depicted in Figures 6 and 7. It is observed that method 4 can completely decouple the disturbance while it is not so for the PID controller.
Another important advantage of approximate linearization method over PID is its proper operation against measurement delay (i.e., when an analyzer is used). In Figures 8 and 9, the effect of time delay on PID controllers has been shown. In Figure 8, the amount of time delay is $T = 42$ sec. It is obvious from this figure that when the set-point is changed or disturbance is applied, the system starts oscillation. A time delay of 42 sec. leads to an instability of the system in Figure 9.

The approximate linearization approach has a satisfactory performance against the measuring time delay, as well in Fig. 10, the response of the system to a time delay of $T = 600$ sec has been shown. It is observed that the system operation has improved in comparison with that in case of the PID controller. In this case, the coefficients of relation (32) has been tuned as $k_{p,1} = 9e-4$, $k_{i,1} = 5e-8$, and $k_{d,1} = 1$ for the distillate product loop and $k_{p,2} = 3.125e-4$, $k_{i,2} = 1e-8$, and $k_{d,2} = 0.5$ for the bottom product loop.

A point that differentiates the linearization with feedback from the other linearization approaches around an operating point is that the linearized
system in the first approach can operate in a wide range of operating conditions. In this case, the approximate linearity of the system can be investigated in a wide range of operating conditions. To do this, it is assumed that the system is in the initial condition given in Table (2). Then in time intervals of 3000 seconds the upper products composition set-point is increased by 0.01, i.e. from 0.89 to 0.99. The tracking error in this case has been depicted in Figure 11. It is observed that the tracking error remains approximately unchanged although the operating point of the system has a considerable change.

![Figure 10. Approximate Feedback Linearization in The Presence of Time Delay](image)

![Figure 11. Approximate Global Linearity Investigation](image)

**Discussion and Conclusion**

In this section, the results of the new method, are compared with that of output function variation.
- Since the upper product of the column are controlled directly, the tracking error related to set-point changes is considerably decreased.
- In the approximate linearization method the disturbances in the flow rate and composition of the feed are decoupled.
- There is a steady-state error. When error feedback is not used (method 2).
- Methods 3 and 4 are highly influenced by error feedback and its coefficients.

Considering that the approximate model is practically a simulator and that the control law is derived from the approximate model variables (i.e., simulator) by means of relation (31), it can be concluded that for the implementation of this approach, it is only necessary to measure either
directly or indirectly the products composition in the top and bottom of the column and there is no need to measure the compositions in internal trays. In [14], temperature measurement has been used to obtain the compositions in the internal trays. Of course in the new method the feed composition has to be measured, which can be considered a disadvantage for this approach. This shortcoming can be ignored if the feed composition changes within a small range or is somehow constant. Also, the problem will be overcome if feed analyzers with little time delay are used.

Relation (32) shows “a kind of PID controller” in the proposed method. Now the question is: “Does the existence of this controller show the disability of approximate feedback linearization in the control of system?” The answer to this question clarifies some other points as well, which will be discussed now.

As it is seen in Fig. 3, if a PID controller is not used in the approximate linearization method, the system will have a steady-state error. This problem arises as no constraints are put on the error in deriving the approximate model. But, it should be noted that even in this case, the time constant of the system is reduced, which means less off-specification products during transients. Comparing the approximate linearization with PID approach reveals the considerable advantages of the proposed method. It can be claimed that the approach offered in this paper is a model-based PID and can decouple disturbances completely. In fact, approximate feedback linearization has caused the closed-loop system to show an almost global linear dynamics when disturbance occurs or set point changes. In this case, to have a proper response, it is sufficient to design the PID controller only for one operating point. While conventional PID controllers are designed for only one operating point, as well, if the operating point changes, the performance of the system will degrade.

References
Appendix A: Linearization and Decoupling of Non-linear Systems by Feedback

A common approach to reducing the effects of disturbance or parameter variation is disturbance decoupling. In fact, disturbance decoupling theory in non-linear systems is one of the most important subjects in control theory. Much researches have worked on this subject in the 80’s [35,36] Disturbance decoupling, considered simultaneously with output tracking or input – output linearization, is a challenging subject for researchers [37].

Consider a non-linear system as follows:

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{n} g_i(x)u_i + \sum_{j=1}^{m} d_j(x)w_j, \quad (A1) \\
y_i &= h_i(x), \quad i = 1,2,...,m
\end{align*}
\]

where \( x \in D \subset \mathbb{R}^n \) and \( D \) is an open set. Also \( u_i, y_i \in \mathbb{R}^n \) and \( w \in \mathbb{R}^m \). Vector fields \( f(x), \ G = (g_i(x))_{i=1,n} \) and \( D = (d_j(x))_{j=1,m} \) are smooth and output function \( (h_i(x))_{i=1,m} \) is real and defined on \( D \). It is observed in the system equations that the input and the disturbance are both linear.

Disturbance decoupling is defined as:

Find a static feedback in the following form:

\[
u = a(x) + b(x)v + y(x)v
\]

(A2)

Such that \( a(x), \ b(x), \) and \( y(x) \) are real on \( D \) and the disturbance vector \( v \) has no effects on the output \( y \). Vector \( v \) is the closed-loop system input. The \( j \) th column of matrix \( y(x) \) is in conformance with disturbance \( w_j \). It is obvious that if only the first \( p \) disturbances are measurable, then the last \( n-p \) columns of \( y(x) \) will be all zero. This case in which only some parts of disturbances are measurable is referred to as “disturbance decoupling with partial measurement,” or DDPM. Vector fields \( (a(x),b(x))_{i=1,n} \) are the columns of matrix \( \tilde{D}(x) \).

(A3)

In [36], it has been illustrated that if the relative degrees set \( \{r_1,...,r_m\} \) is defined on \( D \), then

\[
\begin{align*}
a(x) &= -A^{-1}(x)b^{-1}(x) \\
b(x) &= A^{-1}(x) \quad (A4) \\
y(x) &= -A^{-1}(x)\tilde{D}(x)
\end{align*}
\]

where the elements of matrix \( A(x) \) and vector \( b(x) \) can be calculated using the following relations:

\[
\begin{align*}
a_i(x) &= L_{e_i}L_j^{-1}h_i(x) \\
b_i(x) &= L_j^{-1}h_i(x) \quad (A5)
\end{align*}
\]

The components of \( \tilde{D}(x) \) are also computed using (A3) and (A5). If disturbance is perfectly decoupled, then it will be possible to track a smooth reference signal with an acceptable convergence speed. The non-singularity set of system (A1) is defined as follows:

\[
D_s = \{x \in D \mid \det A(x) = 0\}
\]

(A6)

Appendix B: Rosenbrock theorem

Consider the following system:

\[
\dot{\xi} = \phi(\xi)
\]

(B1)

in which \( \xi \in \Omega \subset \mathbb{R}^n \) and \( \Omega \) is a bounded, closed and convex set. Also, \( \phi = (\phi_i)_{i=1,n} \) and its partial derivatives are continuous functions of \( \xi \). Assume that the following conditions are satisfied:

1. For each initial condition in \( \Omega \) the system response remains in \( \Omega \).
2. for each \( i \in \{1,2,...,n\} \) the \( \psi_i(\xi) \) functions are non-negative.

\[
\psi_i(\xi) = -\sum_{k=1}^{n} \frac{\partial \phi_k}{\partial \xi_k} \quad (B2)
\]

3. for each \( i \) and \( k \) in \( \{1,2,...,n\} \), where \( i \neq k \), \( \frac{\partial \phi_k}{\partial \xi_i} \geq 0 \).

4. for each \( i \in \{1,2,...,n\} \), if \( \psi_i = 0 \), then \( j \neq i \) exists in \( \{1,2,...,n\} \) such that \( \frac{\partial \phi_i}{\partial \xi_j} \neq 0 \). If \( \psi_j = 0 \), then \( k \neq i,j \) exists in \( \{1,2,...,n\} \) such that \( \frac{\partial \phi_j}{\partial \xi_k} \neq 0 \).

If \( \psi_i = 0 \), then ... . This algorithm always leads to an \( \xi \in \{1,2,...,n\} \) such that \( \psi_i \neq 0 \).

If the above four conditions are satisfied, then a unique steady state exists in \( \Omega \) such that the system response, starting from any initial state in \( \Omega \), will converge to it. Also, the function \( \sum_{i=1}^{n} \phi_i(\xi) \) is a Liapunov function for the system.

This theorem has been proved by Rosenbrock in 1962 [32].