Duality of $g$-Bessel sequences and some results about RIP $g$-frames

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Abstract

In this paper, first we develop the duality concept for $g$-Bessel sequences and Bessel fusion sequences in Hilbert spaces. We obtain some results about dual, pseudo-dual and approximate dual of frames and fusion frames. We also expand every $g$-Bessel sequence to a frame by summing some elements. We define the restricted isometry property for $g$-frames and generalize some results from (B. G. Bodmann et al, Fusion frames and the restricted isometry property, Num. Func. Anal. Optim. 33 (2012) 770-790) to $g$-frame situation. Finally we study the stability of $g$-frames under erasure of operators.

Keywords: $G$-frames; Fusion frames; Dual frames; Pseudo-dual frames; Approximate dual frames; Bessel sequences.

1 Introduction

Let $\mathcal{H}$, $\mathcal{K}$ be two separable Hilbert spaces and $\{W_i\}_{i \in I}$ be a sequence of closed subspaces of $\mathcal{K}$, where $I$ is a subset of $\mathbb{Z}$. For any frame $\{f_i\}_{i \in I}$ there exists at least one dual frame, i.e., a frame $\{g_i\}_{i \in I}$ for which

$$f = \sum_{i \in I} <f, g_i > f_i \quad \forall f \in \mathcal{H}. \quad (1.1)$$

If $\{f_i\}_{i \in I}$ is a Bessel sequence with bound $B < 1$, how can we find two sequences $\{g_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ such that $\{f_i + g_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ are dual frames, i.e., such that

$$f = \sum_{i \in I} <f, p_i > (f_i + g_i) = \sum_{i \in I} <f, f_i + g_i > p_i,$$

for all $f \in \mathcal{H}$. In this paper we obtain some more general results of the type (1.1). Let $\mathcal{L}(\mathcal{H}, W_i)$ be the collection of all bounded linear operators from $\mathcal{H}$ into $W_i$. Recall that a family of operators $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : i \in I\}$ is said to be a generalized frame, or simply a $g$-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ if there exist constants $0 < C \leq D < \infty$ such that

$$C \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq D \|f\|^2 \quad \forall f \in \mathcal{H}. \quad (1.1)$$

The constants $C$ and $D$ are called $g$-frame bounds and $\sup_{i \in I} \Lambda_i$ is called the multiplicity of the $g$-frame. We call $\Lambda$ a tight $g$-frame if $C = D$ and it is a Parseval $g$-frame if $C = D = 1$. $\Lambda$ is called an $\varepsilon$-$g$-frame for $\mathcal{H}$ if $C = \frac{1}{1+\varepsilon}$ and $D = 1+\varepsilon$ for some $\varepsilon > 0$. If the right-hand side of (1.1) holds, then $\Lambda$ is said a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$. The representation space associated with a $g$-Bessel sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ is defined...
by
\[
(\sum_{i\in I} \oplus W_i)_{\ell^2} = \{g_i\}_{i\in I} | g_i \in W_i, \sum_{i\in I} ||g_i||^2 < \infty \}.
\]

The synthesis operator of \( \Lambda \) is defined by
\[
T_\Lambda : \left( \sum_{i\in I} \oplus W_i \right)_{\ell^2} \to \mathcal{H}
\]
\[
T_\Lambda (\{g_i\}_{i\in I}) = \sum_{i\in I} \Lambda^*_i g_i.
\]

The adjoint operator of \( T_\Lambda \), which is called the analysis operator also obtain as follows
\[
T^*_\Lambda : \mathcal{H} \to \left( \sum_{i\in I} \oplus W_i \right)_{\ell^2}
\]
\[
T^*_\Lambda f = \{\Lambda_i f\}_{i\in I}.
\]

By composing \( T_\Lambda \) with its adjoint \( T^*_\Lambda \), we obtain the fusion frame operator
\[
S_\Lambda : \mathcal{H} \to \mathcal{H}
\]
\[
S_\Lambda f = T_\Lambda T^*_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f,
\]
which is a bounded, self-adjoint, positive and invertible operator and \( C I_\mathcal{H} \leq S_\Lambda \leq D I_\mathcal{H} \). The canonical dual g-frame for \( \{\Lambda_i\}_{i\in I} \) is defined by \( \{\Lambda_i\}_{i\in I} \) with \( \Lambda_i = \Lambda_i S_i^{-1} \), which is also a g-frame for \( \mathcal{H} \) with g-frame bounds \( \frac{1}{2} \) and \( \frac{1}{2} \), respectively. Also we have
\[
f = \sum_{i\in I} \Lambda_i^* \Lambda_i f \quad \forall f \in \mathcal{H}.
\]

For more details about the theory and applications of frames we refer the readers to [1, 8, 9, 10, 11] and for fusion frames to [2, 4, 5, 7], about g-frames to [3, 12, 13].

The paper is organized as follows: Section 2, contains an extension of g-Bessel sequences to dual g-frames. In this Section, we consider the dual, pseudo-dual and approximate dual frames, fusion frames and we obtain several characterizations of all this dual frames. In Section 3, we generalize the restricted isometry property to the g-frame situation. In Section 4, we study the conditions which under removing some element from a g-frame, again we obtain another g-frame.

## 2 Dual, approximate dual and pseudo-dual of g-frames

Let \( \Lambda = \{\Lambda_i\}_{i\in I} \) and \( \Gamma = \{\Gamma_i\}_{i\in I} \) be g-Bessel sequences for \( \mathcal{H} \) with synthesis operators \( T_\Lambda \) and \( T_\Gamma \) respectively. Then we say that \( \Lambda \) and \( \Gamma \) are dual g-frames for \( \mathcal{H} \) if \( T_\Lambda T_\Gamma^* = I_\mathcal{H} \) or \( T_\Gamma T_\Lambda^* = I_\mathcal{H} \).

In the following we show that any pair of g-Bessel sequences can be extended to pair of dual g-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim [9] to the situation of g-frames.

**Theorem 2.1** Let \( \Lambda = \{\Lambda_i\}_{i\in I} \) and \( \Gamma = \{\Gamma_i\}_{i\in I} \) be two g-Bessel sequences for \( \mathcal{H} \) with respect to \( \{W_i\}_{i\in I} \). Then there exist g-Bessel sequences \( \{\Xi_j\}_{j\in J} \) and \( \{\Omega_j\}_{j\in J} \) for \( \mathcal{H} \) with respect to \( \{V_j\}_{j\in J} \), such that \( \{\Lambda_i\}_{i\in I} \cup \{\Xi_j\}_{j\in J} \) and \( \{\Gamma_i\}_{i\in I} \cup \{\Omega_j\}_{j\in J} \) form a pair of dual g-frames for \( \mathcal{H} \) with respect to \( \{W_i\}_{i\in I} \cup \{V_j\}_{j\in J} \).

**Proof.** Assume that \( \{\Phi_j\}_{j\in J} \) and \( \{\Psi_j\}_{j\in J} \) are any pair of dual g-frames for \( \mathcal{H} \) with respect to \( \{V_j\}_{j\in J} \) and let \( \Theta = I_\mathcal{H} - T_\Theta T_\Lambda^* \). Then for any \( f \in \mathcal{H} \) we have
\[
f = \Theta f + T_\Gamma T_\Lambda^* f \leq \sum_{j\in J} \Psi_j^* \Phi_j \Theta f + \sum_{i\in I} \Gamma_i^* \Lambda_i f.
\]

If we set \( \Xi_j = \Phi_j \Theta \) and \( \Omega_j = \Psi_j \) for all \( j \in J \). Then \( \{\Lambda_i\}_{i\in I} \cup \{\Xi_j\}_{j\in J} \) and \( \{\Gamma_i\}_{i\in I} \cup \{\Omega_j\}_{j\in J} \) are dual g-frames for \( \mathcal{H} \) with respect to \( \{W_i\}_{i\in I} \cup \{V_j\}_{j\in J} \).

**Theorem 2.2** Let \( \mathcal{F} \) be a Bessel sequence for \( \mathcal{H} \) with Bessel bound \( B < 1 \) and let \( \mathcal{E} \) be Parseval frame for \( \mathcal{H} \). Then there exists a Bessel sequence \( \mathcal{G} \) for \( \mathcal{H} \) such that \( \mathcal{F} + \mathcal{E} \) and \( \mathcal{G} + \mathcal{E} \) are dual frames.

Let \( \mathcal{F} = \{f_i\}_{i\in I} \) and \( \mathcal{E} = \{e_i\}_{i\in I} \). Since \( B < 1 \), \( I_{\mathcal{H}} + T_\mathcal{F} T_\mathcal{E}^* \) is an invertible operator in \( \mathcal{L}(\mathcal{H}) \). If we define
\[
\Theta = -(I_{\mathcal{H}} + T_\mathcal{F} T_\mathcal{E}^*)^{-1} T_\mathcal{F} T_\mathcal{E}^*
\]
and \( g_i = \Theta^* e_i \) for all \( i \in I \). Then \( \mathcal{G} = \{g_i\}_{i\in I} \) is a
Bessel sequence for $\mathcal{H}$ and for all $f \in \mathcal{H}$ we have

$$ f = (I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{F}}^\ast) \theta f + T_{\mathcal{F}} T_{\mathcal{F}}^\ast f + T_{\mathcal{F}} T_{\mathcal{F}}^\ast f $$

$$ = \sum_{i \in I} < \Theta f, e_i > e_i + \sum_{i \in I} < f, e_i > e_i $$

$$ + \sum_{i \in I} < \Theta f, e_i > f_i + \sum_{i \in I} < f, e_i > f_i $$

$$ = \sum_{i \in I} < f, e_i > e_i > (f_i + e_i), $$

which this finishes the proof. The following corollaries are generalizations of Theorem 2.2 to the $g$-frames situation. We leave the proofs to interested readers.

**Corollary 2.1** Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ with $g$-Bessel bound $B < 1$. Then there exists $g$-Bessel sequence $\{\Gamma_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, such that $\{\Xi_i + \Lambda_i\}_{i \in I}$ and $\{\Xi_i + \Gamma_i\}_{i \in I}$ are dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, where $\{\Xi_i\}_{i \in I}$ is a Parseval $g$-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$.

**Corollary 2.2** For every $g$-Bessel sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ with Bessel bound $B < 1$ and each Parseval $g$-frame $\Xi = \{\Xi_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, there exists $g$-Bessel sequence $\{\Gamma_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ such that $\{\Lambda_i + \Xi_i\}_{i \in I}$ and $\{\Gamma_i\}_{i \in I}$ are dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$.

**Corollary 2.3** For every $g$-Bessel sequence $\{\Lambda_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ there exist $g$-Bessel sequence $\{\Gamma_i\}_{i \in I}$ and a tight $g$-frame $\{\Xi_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ such that $\{\Lambda_i + \Xi_i\}_{i \in I}$ and $\{\Gamma_i\}_{i \in I}$ are dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$.

Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces in $\mathcal{H}$, and let $\Lambda = \{\alpha_i\}_{i \in I}$ be a family of weights, i.e., $\alpha_i > 0$ for all $i \in I$. A sequence $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$ is a fusion frame, if there exist real numbers $0 < C \leq D < \infty$ such that for all $f \in \mathcal{H}$:

$$ C\|f\|^2 \leq \sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \quad (2.2) $$

where $\pi_{W_i}$ is the orthogonal projection from $\mathcal{H}$ onto $W_i$. The constant $C, D$ are called the fusion frame bounds. If the right-hand inequality of (2.2) holds, then we say that $\mathcal{W}_\alpha$ is a Bessel fusion sequence with Bessel fusion bound $D$. Moreover if $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i}$ is a frame for $W_i$ for all $i \in I$. Then $\mathcal{W} = \{(W_i,\alpha_i,\mathcal{F}_i)\}_{i \in I}$ is called a fusion frame system for $\mathcal{H}$. The constants $A, B$ are called the local frame bounds if they are the common frame bounds for the local frame $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i}$ for all $i \in I$. A collection of dual frames $\mathcal{G}_i = \{g_{ij}\}_{j \in J_i}$, $i \in I$ associated with the local frames is called local dual frames. By Theorem 3.2 from [7], if $\mathcal{W} = \{(W_i,\alpha_i,\mathcal{F}_i)\}_{i \in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds $C, D$ and local frame bounds $A, B$, then $\mathcal{F} = \{\alpha_i f_{ij}\}_{i \in I, j \in J_i}$ is a frame for $\mathcal{H}$ with frame bounds $AC$ and $BD$. Also if $\mathcal{F} = \{\alpha_i f_{ij}\}_{i \in I, j \in J_i}$ is a frame for $\mathcal{H}$ with frame bounds $C$ and $D$, then $\mathcal{W} = \{(W_i,\alpha_i,\mathcal{F}_i)\}_{i \in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds $\frac{C}{\alpha}$ and $\frac{D}{\alpha}$.

**Definition 2.1** Let $\mathcal{W}_\alpha = \{(W_i,\alpha_i)\}_{i \in I}$ and $\mathcal{Z}_\beta = \{(Z_i,\beta_i)\}_{i \in I}$ be Bessel fusion sequences for $\mathcal{H}$ with synthesis operators $T_{\mathcal{W}_\alpha}$ and $T_{\mathcal{Z}_\beta}$ respectively. Then

(i) $\mathcal{W}_\alpha, \mathcal{Z}_\beta$ are dual fusion frames for $\mathcal{H}$ if $T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^* = I_{\mathcal{H}}$ or $T_{\mathcal{Z}_\beta} T_{\mathcal{W}_\alpha}^* = I_{\mathcal{H}}$.

(ii) $\mathcal{W}_\alpha, \mathcal{Z}_\beta$ are approximate dual fusion frames for $\mathcal{H}$ if $\|I_{\mathcal{H}} - T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^*\| < 1$ or $\|I_{\mathcal{H}} - T_{\mathcal{Z}_\beta} T_{\mathcal{W}_\alpha}^*\| < 1$.

(iii) $\mathcal{W}_\alpha, \mathcal{Z}_\beta$ are called pseudo-dual fusion frames for $\mathcal{H}$ if $T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^* = 0$, $T_{\mathcal{Z}_\beta} T_{\mathcal{W}_\alpha}^* = 0$ and $T_{W_i} T_{Z_j}^* = 0$ for all $i, j \in I$.

**Theorem 2.3** For each $i \in I$ let $\alpha_i > 0$ and $J_i = J_{i1} \cup J_{i2}$ be a partition of $J_i$ and let $\mathcal{W} = \{(W_i,\alpha_i,\mathcal{F}_{ij})\}_{i \in I}$ and $\mathcal{Z} = \{(Z_i,\beta_i,\mathcal{G}_{ij})\}_{j \in J_i}$ be two fusion frame system for $\mathcal{H}$. Define

$$ u_{ij} = \begin{cases} \frac{1}{\sqrt{\alpha_i}} f_{ij} & j \in J_{i1} \\ \frac{1}{\sqrt{\beta_i}} \pi_{W_i} g_{ij} & j \in J_{i2} \end{cases} $$

and

$$ v_{ij} = \begin{cases} \frac{1}{\sqrt{\beta_i}} Z_i \tilde{f}_{ij} & j \in J_{i1} \\ \frac{1}{\sqrt{\alpha_i}} \pi_{Z_i} \tilde{g}_{ij} & j \in J_{i2} \end{cases} $$

for all $i \in I, j \in J_i$. Then the following conditions are equivalent:

1. $\mathcal{W}_\alpha = \{(W_i,\alpha_i)\}_{i \in I}$ and $\mathcal{Z}_\beta = \{(Z_i,\beta_i)\}_{i \in I}$ are (dual, pseudo-dual, approximate dual) fusion frames.
(2) \( \{\alpha_{ij}\}_{i \in I, j \in J_i} \) and \( \{\beta_{ij}\}_{i \in I, j \in J_i} \) are (dual, pseudo-dual, approximate dual) frames for \( \mathcal{H} \).

**Proof.** This claim follows immediately from the fact that for \( f \in \mathcal{H} \) we have

\[
\sum_{i \in I} \sum_{j \in J_i} <f, \beta_{ij}> \alpha_{ij} > \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> > \alpha_i u_{ij} \\
= \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> > \alpha_i u_{ij} \\
+ \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> > \alpha_i u_{ij} \\
= \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> > \alpha_i u_{ij} \\
+ \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> > \alpha_i u_{ij} \\
= \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> > \alpha_i u_{ij}.
\]

\[
\sum_{i \in I} \sum_{j \in J_i} <f, \beta_{ij}> \alpha_{ij} = \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> = \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> = \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}>.
\]

\[
\sum_{i \in I} \sum_{j \in J_i} <f, \beta_{ij}> \alpha_{ij} = \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> = \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}> = \sum_{i \in I} \alpha_i \sum_{j \in J_i} <f, \beta_{ij}>.
\]

**Theorem 2.4** Let \( \{\{W_i, \alpha_i, \{f_{ij}\}_{j \in J_i}\}\}_{i \in I} \) be a fusion frame system and let \( \mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I} \) be a fusion Bessel sequence for \( \mathcal{H} \). Put \( g_{ij} = \pi_{Z_i}(f_{ij}) \) for all \( i \in I, j \in J_i \). Then the following conditions are equivalent:

(1) \( \mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I} \) and \( \mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I} \) are (dual, pseudo-dual, approximate dual) frames.

(2) \( \mathcal{F} = \{\alpha_{ij} f_{ij}\}_{i \in I, j \in J_i} \) and \( \mathcal{G} = \{\beta_{ij} g_{ij}\}_{i \in I, j \in J_i} \) are (dual, pseudo-dual, approximate dual) frames for \( \mathcal{H} \).

**Proof.** First we prove that \( \mathcal{G} \) is a Bessel sequence for \( \mathcal{H} \). Let \( D \) be the Bessel fusion bound of \( \mathcal{Z}_\beta \) and \( A, B \) be the local frame bounds of \( \{\{W_i, \alpha_i, \{f_{ij}\}_{j \in J_i}\}\}_{i \in I} \), then for all \( f \in \mathcal{H} \) we have

\[
\sum_{i \in I} \sum_{j \in J_i} |<f, \beta_{ij}>|^2 = \sum_{i \in I} \sum_{j \in J_i} \beta_{ij}^2 |<f, \pi_{Z_i}(f)>|^2 = \sum_{i \in I} \beta_{ij}^2 \sum_{j \in J_i} |<\pi_{Z_i}(f), f_{ij}>|^2 \\
\leq \sum_{i \in I} \frac{\beta_{ij}^2}{A} \|\pi_{Z_i}(f)\|^2 \leq \frac{D}{A} \|f\|^2.
\]

Let \( T_F \) and \( T_G \) be the synthesis operators for \( \mathcal{F} \) and \( \mathcal{G} \) respectively. Then for all \( f \in \mathcal{H} \) we obtain

\[
T_{\mathcal{W}_a} T_{\mathcal{Z}_\beta}^* (f) = \sum_{i \in I} \alpha_i \beta_i \pi_{Z_i}(f) \\
- \sum_{i \in I} \alpha_i \beta_i \sum_{j \in J_i} <\pi_{Z_i}(f), f_{ij}> = \sum_{i \in I} \sum_{j \in J_i} <\pi_{Z_i}(f), f_{ij}> = T_F T_G^* (f).
\]

This finishes the proof.

**Theorem 2.5** Let \( \mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I} \) and \( \mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I} \) be Bessel fusion sequences for \( \mathcal{H} \) and let \( T \in B(\mathcal{H}) \) be a bounded invertible operator such that \( T^* T \mathcal{W}_i \subseteq \mathcal{W}_i \), \( T^* T \mathcal{Z}_i \subseteq \mathcal{Z}_i \). Then

(1) \( \mathcal{W}_\alpha \) and \( \mathcal{Z}_\beta \) are (dual, pseudo-dual) fusion frames if and only if \( T \mathcal{W}_\alpha = \{(T W_i, \alpha_i)\}_{i \in I} \) and \( T \mathcal{Z}_\beta = \{(T Z_i, \beta_i)\}_{i \in I} \) are (dual, pseudo-dual) fusion frame for \( \mathcal{H} \).

(2) If \( \mathcal{W}_\alpha \) and \( \mathcal{Z}_\beta \) are approximate dual fusion frames and \( \|T\| = 1 \) then \( T \mathcal{W}_\alpha = \{(T W_i, \alpha_i)\}_{i \in I} \) and \( T \mathcal{Z}_\beta = \{(T Z_i, \beta_i)\}_{i \in I} \) are also approximate dual fusion frames for \( \mathcal{H} \).

**Proof.** (1) Since \( T \) is invertible and \( T^* T \mathcal{W}_i \subseteq \mathcal{W}_i \), \( T^* T \mathcal{Z}_i \subseteq \mathcal{Z}_i \) hence for all \( i \in I \) \( \pi_{T \mathcal{W}_i} = T \pi_{\mathcal{W}_i} T^{-1} \), \( \pi_{T \mathcal{Z}_i} = T \pi_{\mathcal{Z}_i} T^{-1} \). This implies that \( T \mathcal{W}_\alpha T \mathcal{Z}_\beta = T \mathcal{W}_a T \mathcal{Z}_\beta T^{-1} \), that from this the claim follows immediately.

(2) We have

\[
\|T_H - T \mathcal{W}_a T \mathcal{Z}_\beta^* \| = \|T^{-1} - T \mathcal{W}_a T \mathcal{Z}_\beta T^{-1} \| \\
\leq \|T_H - T \mathcal{W}_a T \mathcal{Z}_\beta^* \|.
\]
From this the result follows at once.

**Theorem 2.6** Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$ be a fusion frame and let $\mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i \in I}$ be a Bessel fusion sequence for $\mathcal{H}$. Suppose that $T : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded invertible operator such that $T W_i \subseteq Z_i$ for all $i \in I$. Then $\mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i \in I}$ and $T \mathcal{W}_\alpha = \{(T W_i, \alpha_i)\}_{i \in I}$ are pseudo-dual fusion frames for $\mathcal{H}$. Moreover if $T \mathcal{W}_\alpha$ is a Parseval fusion frame then $\mathcal{Z}_\alpha$ and $T \mathcal{W}_\alpha$ are dual fusion frames.

**Proof.** Since $T W_i \subseteq Z_i$ hence $\pi_{T W_i} \pi_{Z_i} = \pi_{T W_i}$ for all $i \in I$. It follows that $T T W_i T^*_{Z_i} = T^*_{Z_i} T W_i = S T W_{\alpha}$ which finishes the proof.

**Definition 2.2** Let $\{W_i\}_{i \in I}$ and $\{\widetilde{W}_i\}_{i \in I}$ be closed subspaces in $\mathcal{H}$ and $\varepsilon > 0$. If for every $f \in \mathcal{H}$ we have

$$\sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\|^2 \leq \varepsilon \|f\|^2.$$

Then we say that $\{(\widetilde{W}_i, \alpha_i)\}_{i \in I}$ is an $\varepsilon$-perturbation of $\{(W_i, \alpha_i)\}_{i \in I}$.

**Theorem 2.7** Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$, $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ be Bessel fusion sequences with Bessel fusion bounds $D_1, D_2$ respectively for $\mathcal{H}$. Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$ be an $\varepsilon$-perturbation of $\mathcal{W}_\alpha$ and $\varepsilon D_2 < 1$. If $\mathcal{W}_\alpha$ and $\mathcal{Z}_\beta$ are dual fusion frames, then $\mathcal{W}_\alpha$ and $\mathcal{Z}_\beta$ are also approximate dual fusion frames for $\mathcal{H}$.

**Proof.** By Proposition 2.4 from [4] $\mathcal{W}_\alpha$ is a Bessel fusion sequence for $\mathcal{H}$. Now for all $f \in \mathcal{H}$ we have

$$\|f - T \mathcal{W}_\alpha T^*_{\mathcal{W}_\alpha}(f)\|^2 = \|T W_i T^*_{Z_i}(f) - T \mathcal{W}_\alpha T^*_{\mathcal{W}_\alpha}(f)\|^2 = \sup_{\|g\|=1} |< T W_i T^*_{Z_i}(f) - T \mathcal{W}_\alpha T^*_{\mathcal{W}_\alpha}(f), g >|^2$$

$$\leq \sup_{\|g\|=1} \left( \sum_{i \in I} \alpha_i \beta_i \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\| \|\pi_{Z_i}(g)\| \right)^2$$

$$\leq \sup_{\|g\|=1} \left( \sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\|^2 \right. \times \left. \sum_{i \in I} \beta_i^2 \|\pi_{Z_i}(g)\|^2 \right) \leq \varepsilon \|f\|^2.$$

From this the result follows at once.

### 3 RIP for g-frames

In this section we generalize the restricted isometry property for g-frames. We denote that $\mathcal{K}$ is a Hilbert space and $\mathcal{H}_N$ is a Hilbert space with dimension $N$ and $\{e_j\}_{j=1}^N$ an orthonormal basis for $\mathcal{H}_N$. Moreover, the Hilbert-Schmidt norm of operator $T \in \mathcal{L}(\mathcal{H}_N, \mathcal{K})$ is defined by

$$\|T\|^2_{HS} = \sum_{j=1}^N \|T e_j\|^2.$$

**Proposition 3.1** Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ with g-frame bounds $A$ and $B$ in $\mathcal{H}$. Then $A \leq \|\Lambda_i\|^2_{HS} \leq B$.

**Proof.** Since $\sum_{i \in I} \|\Lambda_i\|^2_{HS} = \sum_{j=1}^N \langle \Lambda e_j, e_j \rangle$ and $A \|e_j\| \leq \|\Lambda e_j\| \leq B \|e_j\|$, we have

$$A \dim \mathcal{H} = A \sum_{j=1}^N \|e_j\|^2 \leq \sum_{j=1}^N \langle \Lambda e_j, e_j \rangle \leq B \sum_{j=1}^N \|e_j\|^2 = B \dim \mathcal{H}.$$

This yields

$$A \dim \mathcal{H} \leq \sum_{i \in I} \|\Lambda_i\|^2_{HS} \leq B \dim \mathcal{H}.$$

From this the claim follows immediately.

**Theorem 3.1** Let $\Lambda = \{\Lambda_i\}_{i = 1}^M$ be a g-frame for $\mathcal{H}_N$ with respect to $\{W_i\}_{i = 1}^M$. Then

(i) The optimal g-frame bounds of $\Lambda$ are the smallest and biggest eigenvalues of g-frame operator $S_{\Lambda}$.

(ii) If $\{\lambda_i\}_{i = 1}^N$ is a representation of eigenvalues of $S_{\Lambda}$. Then

$$\sum_{j=1}^N \lambda_j = \sum_{i = 1}^M \|\Lambda_i\|^2_{HS}.$$
and
\[ \lambda_j = \sum_{i=1}^{M} \| \Lambda_i e_j \|^2, \]
where \( \{ e_j \}_{j=1}^N \) is the orthonormal basis consisting of eigenvectors of \( S_A \).

**Proof.** To prove (i), since \( S_A \) is a self-adjoint, \( H_N \) has an orthonormal basis include eigenvectors of \( S_A \). Let \( \{ e_j \}_{j=1}^N \) be an orthogonal basis of \( H_N \) include of eigenvectors of \( S_A \). Let \( \{ \lambda_j \}_{j=1}^N \) be eigenvalues of \( \{ e_j \}_{j=1}^N \). Then for any \( f \in H_N \) we have
\[
\sum_{i=1}^{M} \| \Lambda_i f \|^2 = < S_A f, f > \\
= \sum_{j=1}^{N} < f, e_j > S_A e_j, f > \\
= \sum_{j=1}^{N} < f, e_j > < S_A e_j, f > \\
= \sum_{j=1}^{N} < f, e_j > < \lambda_j e_j, f > \\
= \sum_{j=1}^{N} \lambda_j < f, e_j > ^2.
\]
Now from \( \lambda_{\min} \leq \lambda_i \leq \lambda_{\max}, \quad (1 \leq i \leq N) \)
we obtain
\[
\lambda_{\min} \| f \|^2 \leq \sum_{i=1}^{M} \| \Lambda_i f \|^2 \leq \lambda_{\max} \| f \|^2.
\]
To prove (ii) we have:
\[
\sum_{j=1}^{N} \lambda_j = \sum_{j=1}^{N} < \lambda_j e_j, e_j > \\
= \sum_{j=1}^{N} < S_A e_j, e_j > = \sum_{j=1}^{N} \sum_{i=1}^{M} \| \Lambda_i e_j \|^2 \\
= \sum_{i=1}^{M} \sum_{j=1}^{N} \| \Lambda_i e_j \|^2 = \sum_{i=1}^{M} \| \Lambda_i \|_{HS}^2.
\]
We also have
\[
\lambda_j = < \lambda_j e_j, e_j > = < S_A e_j, e_j > \\
= \sum_{i=1}^{M} \| \Lambda_i e_j \|^2.
\]

**Corollary 3.1** Let \( \{ \Lambda_i \}_{i=1}^{M} \) be an A-tight g-frame for \( H_N \) with respect to \( \{ W_i \}_{i=1}^{M} \) and \( \| \Lambda_i \|_{HS} = 1 \) for all \( 1 \leq i \leq M \). Then \( A = \frac{M}{N} \).

**Proof.** This is a direct result from Proposition 3.1.

**Definition 3.1** Let \( \Lambda_i \in L(H, W_i) \) for all \( i \in I \).

Then
(i) \( \{ \Lambda_i \}_{i\in I} \) is called an orthonormal g-system for \( H \) with respect to \( \{ W_i \}_{i\in I} \), if \( \Lambda_i \Lambda_i^* g_j = \delta_{ij} g_j \) for all \( i, j \in I, g_j \in W_j \).

(ii) If \( H = \{ \Lambda_i^*(W_i) \}_{i\in I} \), then we say that \( \{ \Lambda_i \}_{i\in I} \) is g-complete.

(iii) We say that \( \{ \Lambda_i \}_{i\in I} \) is a g-orthonormal basis for \( H \) with respect to \( \{ W_i \}_{i\in I} \), if it is a g-orthonormal g-complete system for \( H \) with respect to \( \{ W_i \}_{i\in J} \).

(iv) \( \{ \Lambda_i \}_{i\in I} \) is called a g-Riesz basis for \( H \) with respect to \( \{ W_i \}_{i\in I} \), if \( \{ \Lambda_i \}_{i\in I} \) is g-complete and there exist real numbers \( 0 < A \leq B < \infty \) such that:
\[
A \sum_{j\in J} \| g_j \|^2 \leq \| \sum_{j\in J} \Lambda_i^* g_j \|^2 \leq B \sum_{j\in J} \| g_j \|^2,
\]
for all finite subset \( J \subset I \) and \( g_j \in W_j \). Moreover, \( \{ \Lambda_i \}_{i\in I} \) is called an \( \varepsilon \)-g-Riesz basis for \( H \), if \( A = \frac{1}{1+\varepsilon} \) and \( B = 1 + \varepsilon \) for some \( \varepsilon > 0 \). Also \( \{ \Lambda_i \}_{i\in I} \) is an \( \varepsilon \)-g-Riesz sequence if \( \{ \Lambda_i \}_{i\in I} \) is an \( \varepsilon \)-g-Riesz basis for \( \Lambda_i^*(W_i) \) for all \( i \in I \).

The next proposition is similar to a result of Bodmann, Cahill and Casazza [6] to the situation of g-frames.

**Proposition 3.2** Let \( \{ \Lambda_i \}_{i\in I} \) be an \( \varepsilon \)-g-Riesz sequence for \( H \) with respect to \( \{ W_i \}_{i\in I} \) and let \( \{ I_j \}_{j=1}^{L} \) be a partition of \( I \). Then
\[
\frac{1}{1+\varepsilon} \sum_{j=1}^{L} \| \sum_{k\in I_j} \Lambda_k^* g_{jk} \|^2 \leq \sum_{j=1}^{L} \sum_{k\in I_j} \| g_{jk} \|^2 \leq (1+\varepsilon) \sum_{j=1}^{L} \| \sum_{k\in I_j} \Lambda_k^* g_{jk} \|^2,
\]

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for every $1 \leq j \leq L$ and any sequence $\{g_{jk}\}_{k \in I_j} \in (\sum_{k \in I_j} + W_k)_{l^2}$. Also

$$\frac{1}{(1+\varepsilon)^2} \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2 \leq \| \sum_{j=1}^{L} \sum_{k \in I_j} \Lambda_k g_{jk} \|^2 \leq (1+\varepsilon)^2 \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2.$$ 

**Proof.** Let $1 \leq j \leq L$ and $\{g_{jk}\}_{k \in I_j} \in (\sum_{k \in I_j} + W_k)_{l^2}$.

$$\frac{1}{1+\varepsilon} \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2 \leq \frac{1}{1+\varepsilon} \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k g_{jk} \|^2 \leq \| \sum_{j=1}^{L} \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2 \leq (1+\varepsilon) \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2.$$ 

This yields

$$\frac{1}{(1+\varepsilon)^2} \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2 \leq \| \sum_{j=1}^{L} \sum_{k \in I_j} \Lambda_k g_{jk} \|^2 \leq (1+\varepsilon) \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2 \leq (1+\varepsilon)^2 \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2.$$ 

It is known that if $\{\Lambda_i\}_{i \in I}$ is a $\varepsilon$-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ with $\varepsilon$-Riesz constants $A$ and $B$, then $\{\Lambda_i\}_{i \in I}$ is a $g$-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ with same bounds $A$ and $B$. The next lemma is analogous to Lemma 3.3 in [6] to the situation of $g$-frames.

**Lemma 3.1.** Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be an $\varepsilon$-$g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$. Then for all $n \in \mathbb{N}$

$$\frac{1}{(1+\varepsilon)^n} I_{\mathcal{H}} \leq S_{\Lambda}^n \leq (1+\varepsilon)^n I_{\mathcal{H}}$$

**Proof.** Since $\{\Lambda_i\}_{i \in I}$ is an $\varepsilon$-$g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, so this family is a $g$-frame for $\mathcal{H}$ with bounds $\frac{1}{1+\varepsilon}, 1+\varepsilon$ respectively. Hence $\frac{1}{1+\varepsilon} \leq ||S_\Lambda|| \leq (1+\varepsilon)$ and $\frac{1}{1+\varepsilon} \leq ||S_\Lambda^{-1}|| \leq (1+\varepsilon)$. On the other hand for any $f \in \mathcal{H}$ and $n \in \mathbb{N}$ we have $||S_{\Lambda}^{-1}||^{-n} ||f|| \leq ||S_{\Lambda} f|| \leq ||S_{\Lambda}||^{n} ||f||$. From this we have $||S_{\Lambda}^{-1}||^{-n} I_{\mathcal{H}} \leq S_{\Lambda}^n I_{\mathcal{H}}$. Consequently

$$\frac{1}{(1+\varepsilon)^n} I_{\mathcal{H}} \leq ||S_{\Lambda}^{-1}||^{-n} I_{\mathcal{H}} \leq S_{\Lambda}^n I_{\mathcal{H}} \leq (1+\varepsilon)^n I_{\mathcal{H}}.$$ 

This shows that $\frac{1}{(1+\varepsilon)^n} I_{\mathcal{H}} \leq S_{\Lambda}^n \leq (1+\varepsilon)^n I_{\mathcal{H}}$ and so $\frac{1}{(1+\varepsilon)^n} I_{\mathcal{H}} \leq S_{\Lambda}^{-n} \leq (1+\varepsilon)^n I_{\mathcal{H}}$.

**Proposition 3.3** Let $\{\Lambda_i\}_{i \in I}$ be an $\varepsilon$-$g$-Riesz sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$. Then

$$\langle f, g \rangle \geq |\langle f, \lambda \psi \rangle| = |\langle \varphi, \lambda \psi \rangle| = \frac{2}{\|\varphi\|^2 - 1} \leq \frac{(1+\varepsilon)^2}{2} \sum_{i \in F_1 \cup F_2} \|g_i\|^2 - 1$$

for all partition $\{I_1, I_2\}$ of $I$ and $f \in \{\Lambda_i(W_i)\}_{i \in I_1}, g \in \{\Lambda_i^* (W_i)\}_{i \in I_2}$ with $||f|| = ||g|| = 1$.

**Proof.** Let $F_1 \subseteq I_1, F_2 \subseteq I_2$ be arbitrary finite subsets, $g_i \in W_i (i \in F_1 \cup F_2)$ and $\varphi = \sum_{i \in F_1} \Lambda_i^* g_i$ and $\psi = \sum_{i \in F_2} \Lambda_i^* g_i$ with conditions $||\varphi|| = ||\psi|| = 1$. Then for any $|\lambda| = 1$ we have

$$\langle \varphi, \lambda \psi \rangle = \frac{2}{\|\varphi\|^2 - 1} \leq \frac{(1+\varepsilon)^2}{2} \sum_{i \in F_1 \cup F_2} \|g_i\|^2 - 1$$

This yields

$$|\langle f, \varphi \rangle| = \max_{|\lambda| = 1} \langle f, \lambda \varphi \rangle \leq 2 \varepsilon + \varepsilon^2,$$

which implies that $|\langle f, g \rangle| \leq 2 \varepsilon + \varepsilon^2$.

**Definition 3.2** For every $1 \leq i \leq M$, let $\Lambda_i \in \mathcal{L}(H_N, W_i)$. Then we say that the family $\{\Lambda_i\}_{i=1}^M$ has the restricted isometry property with constant $0 < \varepsilon < 1$ for sets of size $s \leq N$, if for every $I \subseteq \{1, 2, \ldots, M\}$ with $|I| \leq s$, the family $\{\Lambda_i\}_{i \in I}$ is an $\varepsilon$-$g$-Riesz sequence for $H_N$ with respect to $\{W_i\}_{i \in I}$.
The next theorem is a generalization of Theorem 4.2 in [6] to the g-frames situation.

**Theorem 3.2** Let \( \{\Lambda_i\}_{i=1}^{M} \) be a tight g-frame for \( \mathcal{H}_N \) with respect to \( \{W_i\}_{i=1}^{M} \) with the restricted isometry property with constant \( 0 < \varepsilon < 1 \) for sets of size \( s \leq N \). Suppose that \( \{I_j\}_{j=1}^{L} \) is an arbitrary partition of \( \{1, 2, \ldots, M\} \) with \( |I_j| \leq s \). Define \( V_j = \{\Lambda_i(W_i)\}_{i \in I_j} \) for all \( 1 \leq j \leq L \), then \( \{V_j\}_{j=1}^{L} \) is a fusion frame for \( \mathcal{H}_N \) with fusion frame bounds 

\[
\sum_{i=1}^{M} ||\Lambda_i||_{HS}^2, \quad \left(1 + \varepsilon\right) \sum_{i=1}^{M} ||\Lambda_i||_{HS}^2
\]

and

\[
\frac{1}{1 + \varepsilon} \sum_{i \in I_j} ||\Lambda_i f||^2 \leq ||\pi_{V_j} f||^2 \leq (1 + \varepsilon) \sum_{i \in I_j} ||\Lambda_i f||^2.
\]

**Proof.** By the hypothesis \( \{\Lambda_i\}_{i \in I_j} \) is a g-frame for \( V_j \) with respect to \( \{W_i\}_{i \in I_j} \) for all \( 1 \leq j \leq L \) with g-frame bounds \( \frac{1}{1 + \varepsilon}, 1 + \varepsilon \) respectively. Let \( S_j \) be g-frame operator of \( \{\Lambda_i\}_{i \in I_j} \) and \( \{e_i\}_{i=1}^{N} \) be the orthonormal basis of eigenvectors of \( S_j \) with eigenvalues \( \{\lambda_i\}_{i=1}^{N} \), then \( \lambda_i = 0 \) for all \( |J_i| < s \) and \( \frac{1}{1 + \varepsilon} \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|I_j|} \leq 1 + \varepsilon \). Since \( \{e_i\}_{i \in I_j} \) is an orthonormal basis for \( V_j \), hence

\[ \pi_{V_j} f = \sum_{i=1}^{N} <f, e_i> e_i, \text{ for any } f \in \mathcal{H}_N. \]

Now we have

\[ S_j f = S_j \left( \sum_{i=1}^{N} <f, e_i> e_i \right) = \sum_{i=1}^{N} <f, e_i> S_j e_i = \sum_{i=1}^{N} <f, e_i> \lambda_i e_i \]

which implies that

\[ <S_j f, f> = \sum_{i=1}^{N} \lambda_i <f, e_i>^2. \]

Thus we have

\[ \frac{1}{1 + \varepsilon} \sum_{i \in I_j} ||\Lambda_i f||^2 = \frac{1}{1 + \varepsilon} <S_j f, f> \]

\[ = \sum_{i \in I_j} \frac{\lambda_i}{1 + \varepsilon} <f, e_i>^2 \leq ||\pi_{V_j} f||^2 \]

\[ \leq \sum_{i \in I_j} \lambda_i (1 + \varepsilon) <f, e_i>^2 \]

\[ = (1 + \varepsilon) <S_j f, f> = (1 + \varepsilon) \sum_{i \in I_j} ||\Lambda_i f||^2. \]

It follows that

\[
\frac{1}{1 + \varepsilon} \sum_{j=1}^{L} \sum_{i \in I_j} ||\Lambda_i f||^2 \leq \sum_{j=1}^{L} ||\pi_{V_j} f||^2 \leq (1 + \varepsilon) \sum_{j=1}^{L} ||\Lambda_i f||^2.
\]

Now by Proposition 3.1 we have

\[
\frac{1}{1 + \varepsilon} \sum_{i=1}^{M} ||\Lambda_i||_{HS}^2 \leq \sum_{j=1}^{L} ||\pi_{V_j} f||^2 \leq (1 + \varepsilon) \sum_{i=1}^{M} ||\Lambda_i||_{HS}^2
\]

**Corollary 3.2** Under the assumptions of Theorem 3.2 if

\[ \{1, 2, \ldots, L\} \subseteq \{1, 2, \ldots, M\} \]

and there exists a family \( \{J_j\}_{j=1}^{L} \) such that \( \sum_{j=1}^{L} |J_j| \leq s \) and \( J_j \subseteq I_j \) for all \( 1 \leq j \leq L \). Then

\[
\frac{1}{(1 + \varepsilon)^2} \sum_{j=1}^{L} \sum_{i \in I_j} ||\Lambda_i g_i||^2 \leq \sum_{j=1}^{L} \sum_{i \in I_j} ||\Lambda_i g_i||^2 \]

\[ \leq (1 + \varepsilon)^2 \sum_{j=1}^{L} \sum_{i \in I_j} ||\Lambda_i g_i||^2. \]

**Proof.** This follows from the Proposition 3.2. The following theorem will give another method for obtaining a fusion frame from an unit norm tight frame for \( \mathcal{H}_N \) without having the restricted isometry property. Another form of this result can be found in [6] Theorem 4.2.

**Theorem 3.3** Let \( \{f_i\}_{i=1}^{M} \) be an unit norm tight frame of vectors for \( \mathcal{H}_N \) and let \( \{I_j\}_{j=1}^{L} \) be a partition of \( \{1, 2, \ldots, M\} \). Define \( W_j = \{f_i\}_{i \in I_j} \), then the family \( \{W_j\}_{j=1}^{L} \) is a fusion frame for \( \mathcal{H}_N \) with fusion frame bounds \( \frac{AM}{N} \) and \( \frac{BN}{M-N} \) where

\[ A = \min_{j=1}^{L} \min_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^{L} \max_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}} \]

and \( \{\lambda_{jk}\}_{k=1}^{\dim W_j} \) is the family of eigenvalues of frame operator associated to \( \{f_i\}_{i \in I_j} \).

**Proof.** Let \( S_j \) be the frame operator associated to \( \{f_i\}_{i \in I_j} \) and let \( \{e_{jk}\}_{k=1}^{N} \) be the orthonormal
basis for $\mathcal{H}_N$ of eigenvectors of $S_j$ with eigenvalues \( \{\lambda_{jk}\}_{k=1}^{N} \). Then $\lambda_{jk} = 0$ for any $\dim W_j < k \leq N$ and $\{e_{jk}\}_{k=1}^{\dim W_j}$ is an orthonormal basis for $W_j$. Thus
\[
<S_j f, f> = \sum_{k=1}^{\dim W_j} \lambda_{jk}|<f, e_k>|^2.
\]

Now for any $f \in \mathcal{H}_N$ we have
\[
\min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} \sum_{i \in I_j} |<f, f_i>|^2 = \min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} <S_j f, f>
\]
\[
= \sum_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}} \min_{1 \leq k \leq \dim W_j} |<f, e_{jk}>|^2 \leq ||\pi_{W_j}||^2
\]
\[
\leq \sum_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}} \min_{1 \leq k \leq \dim W_j} |<f, e_{jk}>|^2 = \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |<S_j f, f>|^2 = \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} \sum_{i \in I_j} |<f, f_i>|^2.
\]

This yields
\[
\sum_{j=1}^{L} \sum_{i \in I_j} \min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |<f, f_i>|^2 \leq \sum_{j=1}^{L} ||\pi_{W_j}||^2 \leq \sum_{j=1}^{L} \sum_{i \in I_j} \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |<f, f_i>|^2.
\]

Put
\[
A = \min_{j=1}^{L} \min_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^{L} \max_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}.
\]

Then
\[
\frac{AM}{N} \|f\|^2 \leq \sum_{j=1}^{L} ||\pi_{W_j} f\|^2 \leq BM \|f\|^2.
\]

The next corollary generalizes Theorem 3.3 to the g-frames situation which the proof leave to interested readers.

**Corollary 3.3** Let $\{\Lambda_i\}_{i=1}^{M}$ be a tight g-frame for $\mathcal{H}_N$ with respect to $\{W_i\}_{i=1}^{M}$ and let $\{I_j\}_{j=1}^{L}$ be a partition of $\{1, 2, \cdots, M\}$. Define
\[
V_j = \{\Lambda_i^*(W_i)\}_{i \in I_j}.
\]

Then the family $\{V_j\}_{j=1}^{L}$ is a fusion frame for $\mathcal{H}_N$ with fusion frame bounds
\[
\frac{A \sum_{i=1}^{M} \|\Lambda_i\|^2_H}{N} \quad \text{and} \quad \frac{B \sum_{i=1}^{M} \|\Lambda_i\|^2_H}{N},
\]
where
\[
A = \min_{j=1}^{L} \min_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^{L} \max_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}
\]
and $\{\lambda_{jk}\}_{k=1}^{\dim V_j}$ is the family of eigenvalues of g-frame operator associated to $\{\Lambda_i\}_{i \in I_j}$.

### 4 Stability of g-frames

Our purpose of this section is to study the conditions which under removing some element from a g-frame, again we obtain another g-frame. The next theorem gives an erasure result of g-frames so that Theorem 4.3 obtained in [5] is a special case of it.

**Theorem 4.1** Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ with g-frame bounds $A$ and $B$ and let $J \subset I$. Then $\{\Lambda_i\}_{i \in I-J}$ is a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I-J}$ with bounds
\[
\frac{A^2}{B} ||(I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda^*_i \Lambda_i)^{-1}||^{-2} \quad \text{and} \quad B,
\]
if and only if $I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda^*_i \Lambda_i$ is a bounded invertible operator on $\mathcal{H}$.

**Proof.** For any $f \in \mathcal{H}$ we have
\[
f = \sum_{i \in I} S_{\Lambda_i}^{-1} \Lambda^*_i \Lambda_i f = \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda^*_i \Lambda_i f + \sum_{i \in I-J} S_{\Lambda_i}^{-1} \Lambda^*_i \Lambda_i f.
\]

Thus
\[
I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda^*_i \Lambda_i = \sum_{i \in I-J} S_{\Lambda_i}^{-1} \Lambda^*_i \Lambda_i.
\]
Moreover we have

\[
\| (I_H - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f) \| = \left\| \sum_{i \in I - J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f \right\|
\]

\[
= \sup_{\|g\| = 1} \left| \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f, g > \right|
\]

\[
= \sup_{\|g\| = 1} \left| \sum_{i \in I - J} < \Lambda_i f, \Lambda_i S_{\Lambda_i}^{-1} g > \right|
\]

\[
\leq \sup_{\|g\| = 1} \left( \sum_{i \in I - J} \| \Lambda_i f \| \| \Lambda_i S_{\Lambda_i}^{-1} g \| \right)
\]

\[
\leq \sup_{\|g\| = 1} \left( \sum_{i \in I - J} \| \Lambda_i f \|^2 \right)^{1/2} \left( \sum_{i \in I - J} \| \Lambda_i S_{\Lambda_i}^{-1} g \|^2 \right)^{1/2}
\]

\[
\leq \frac{\sqrt{B}}{A} \left( \sum_{i \in I - J} \| \Lambda_i f \|^2 \right)^{1/2}.
\]

Now if \( I_H - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i \) is invertible on \( \mathcal{H} \). Then

\[
\frac{A^2}{B} \|(I_H - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i)^{-1}\|^{-2} \|f\|^2 \leq \frac{A^2}{B} \|(I_H - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f)\|^2 \]

\[
\leq \sum_{i \in I - J} \| \Lambda_i f \|^2.
\]

On the other hand, since \( \Lambda \) is a \( g \)-frame hence \( \{\Lambda_i\}_{i \in I - J} \) is a \( g \)-Bessel sequence. It follows that \( \{\Lambda_i\}_{i \in I - J} \) is a \( g \)-frame. Conversely, suppose that \( \{\Lambda_i\}_{i \in I - J} \) is a \( g \)-frame for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I - J} \), with \( g \)-frame bounds \( A \) and \( B \). We first show that \( I_H - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i \) is injective. Let

\[
(I_H - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i) f = 0 \Rightarrow
\]

\[
S_{\Lambda_i}^{-1} \left( \sum_{i \in I - J} \Lambda_i^* \Lambda_i f \right) = \sum_{i \in I - J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f = 0
\]

hence \( \sum_{i \in I - J} \Lambda_i^* \Lambda_i f = 0 \). It follows that

\[
A \| f \|^2 \leq \sum_{i \in I - J} \| \Lambda_i f \|^2
\]

\[
= \sum_{i \in I - J} < \Lambda_i f, \Lambda_i f >
\]

\[
= \sum_{i \in I - J} \Lambda_i^* \Lambda_i f, f >= 0
\]

which implies that \( f = 0 \). Also, if \( (I_H - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i) f = 0 \) then \( \sum_{i \in I - J} \Lambda_i^* \Lambda_i S_{\Lambda_i}^{-1} f = 0 \) and therefore \( S_{\Lambda_i}^{-1} f = 0 \). It follows that \( f = 0 \). This finishes the proof.

**Corollary 4.1** Let \( \{\Lambda_i\}_{i \in I} \) be a \( g \)-frame for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \) and let \( J \subset I \). If there exists \( 0 \neq f_0 \in \mathcal{H} \) such that \( \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f_0 = f_0 \). Then \( \{\Lambda_i\}_{i \in I - J} \) is not a \( g \)-frame for \( \mathcal{H} \).

**Proof.** If there exists \( 0 \neq f_0 \in \mathcal{H} \) such that \( \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f_0 = f_0 \), then \( \sum_{i \in I - J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f_0 = 0 \), hence \( \sum_{i \in I - J} \Lambda_i^* \Lambda_i f_0 = 0 \). It follows that

\[
\sum_{i \in I - J} \| \Lambda_i f_0 \| = \sum_{i \in I - J} < \Lambda_i f_0, \Lambda_i f_0 >
\]

\[
= \sum_{i \in I - J} \Lambda_i^* \Lambda_i f_0, f_0 >= 0
\]

Therefore \( \{\Lambda_i\}_{i \in I - J} \) is not a \( g \)-frame.

**Corollary 4.2** Let \( \{\Lambda_i\}_{i \in I} \) be a \( g \)-tight \( g \)-frame for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \) and let \( J \subset I \). If there exists \( 0 \neq f_0 \in \mathcal{H} \) such that \( \sum_{i \in J} \Lambda_i^* \Lambda_i f_0 = A f_0 \), then \( \{\Lambda_i\}_{i \in I - J} \) is not a \( g \)-frame for \( \mathcal{H} \).

**5 Conclusion**

In this paper, we proved that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame and computed its optimal bounds. We also showed that a Bessel sequence is an inner summand of a frame and changed every Bessel sequence to a dual frame by summing it with any Parseval frame. Moreover, we proved that any pair of \( g \)-Bessel sequences can be extended to pair of dual \( g \)-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim in [9] to the situation of \( g \)-frames. We defined the restricted isometry property for \( g \)-frames and generalized some results from [6] to \( g \)-frames.

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References


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