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Approximate Analytical Solutions of Fuzzy Linear Fredholm Integral Equations by HAM

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Abstract

Integral equations have many applications in the theory of elasticity, engineering, mathematical physics, potential theory, electrostatic and radiative heat transfer problems. In this paper, we present a method for solving fuzzy linear Fredholm integral equations of the second kind, namely "homotopy analysis method" (HAM). It is found that the HAM provides us with a simple way to adjust and control the convergence region of solution series by introducing convergence-control parameter $\hbar$, which is the main advantage of this method.

Keywords : Fuzzy numbers; Fuzzy integral equations; Homotopy analysis method; Convergence-control parameter

1 Introduction

Nonlinear phenomena, which appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in biology, quantum mechanics, mathematical economics and queuing theory can be modeled by integral equations. So, obtaining the solution with high accuracy for such equations is very worth-while.

The topic of fuzzy integral equations which is of growing interest for some time, in particular in relation to fuzzy control, has been rapidly developed in recent years. We

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know that solving fuzzy integral equations requires appropriate and applicable definitions of fuzzy function and fuzzy integral of a fuzzy function. The fuzzy mapping function was introduced by Chang and Zadeh [2]. Later, Dubois and Prade [4] presented an elementary fuzzy calculus, based on the extension principle [18]. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [4]. Alternative approaches were later suggested by Goetschel and Voxman [7], Kaleva [9], Nanda [14] and others. In this paper, we apply a well-known method, namely "homotopy analysis method (HAM)", to obtain approximate solutions of the fuzzy linear Fredholm integral equations of the second kind.

The HAM proposed by Liao [10, 11, 12, 13] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-difference equation. This method is unique among other similar methods as it allows us to effectively control the region of convergence and rate of convergence of a series solution to a nonlinear problem, via control of an initial approximation, an auxiliary linear operator, an auxiliary function and a convergence-control parameter [11]. Recently, Van Gorder et al. [17] discussed the selection of the initial approximation, auxiliary linear operator, auxiliary function and convergence-control parameter in the application of the HAM. They presented methods by which one may select the mentioned items when attempting to solve a nonlinear differential equation by using the HAM. Also, they presented necessary and sufficient conditions for the convergence of series solutions obtained via the HAM. In 2010, Abbasbandy et al. [1] showed that the HAM can be applied to solve linear and nonlinear Fredholm integral equations with high accuracy. Recenly, author [6] applied HAM on the fuzzy linear Volterra integral equations to obtain approximate solutions for these equations. The aim of this paper is to solve fuzzy linear Fredholm integral equations of the second kind by HAM. We prepare our discussion in 5 sections.

In Section 2, we give some definitions and preliminaries. In Section 3, we briefly describe the fuzzy linear Fredholm integral equations of the second kind. In Section 4, we apply the HAM and present a recursive scheme for solving fuzzy linear Fredholm integral equations. In Section 5 we give some numerical examples. Finally, we conclude in Section 6.

2 Preliminaries

In this section, we review the fundamental notations of fuzzy set theory to be used throughout this paper.

Definition 2.1. *A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$; $0 \leqslant r \leqslant 1$ which satisfying the following requirements:*

i. *$\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0,1]$,*

ii. *$\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$,*

iii. *$\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.*

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, \ \ 0 \leqslant r \leqslant 1$. The set of all fuzzy numbers (as given by Definition 2.1) is denoted by E [9].

For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and an arbitrary crisp number k, we define fuzzy addition and scalar multiplication as

1. $(\underline{u+v})(r) = (\underline{u}(r) + \underline{v}(r))$,
2. $(\overline{u+v})(r) = (\overline{u}(r) + \overline{v}(r))$,
3. $(\underline{ku})(r) = k\underline{u}(r), \qquad (\overline{ku})(r) = k\overline{u}(r), \quad k \geqslant 0$,
4. $(\underline{ku})(r) = k\overline{u}(r), \qquad (\overline{ku})(r) = k\underline{u}(r), \quad k < 0$.

We will next define the fuzzy function notation and a metric D in E [7].

Definition 2.2. *For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity*

$$D(u, v) = \sup_{0 \leqslant r \leqslant 1} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}, \tag{2.1}$$

is the distance between u and v.

This metric is equivalent to the one used by Puri and Ralescu [16] and Kaleva [9]. It is shown [15] that (E, D) is a complete metric space.

Definition 2.3. *A function $f : \mathbb{R} \to E$ is called a fuzzy function. Also, if for arbitrary fixed $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\xi > 0$ such that*

$$| x - x_0 | < \xi \Longrightarrow D\,(f(x), f(x_0)) < \varepsilon, \tag{2.2}$$

then f is said to be continuous.

Throughout this work we also consider fuzzy functions which are defined only over a finite interval $[a, b]$ (we simply replace $\mathbb{R}$ by $[a, b]$ in Definition 2.3).

We now follow Goetschel and Voxman [7] and define the integral of a fuzzy function using the Riemann integral concept.

Definition 2.4. *Let $f : [a, b] \to E$. For each partition $p = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and for arbitrary $\xi_i : x_{i-1} \leqslant \xi_i \leqslant x_i$, $1 \leqslant i \leqslant n$ let*

$$R_p = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}). \tag{2.3}$$

The definite integral of $f(x)$ over $[a, b]$ is

$$\int_a^b f(x)dx = \lim R_p, \qquad \max_{1 \leqslant i \leqslant n} \{|x_i - x_{i-1}|\} \to 0, \tag{2.4}$$

provided that this limit exists in the metric D.

If the fuzzy function $f(x)$ is continuous in the metric D, its definite integral exists [7]. Furthermore,

$$\underline{(\int_a^b f(x;r)dx)} = \int_a^b \underline{f}(x;r)dx, \qquad \overline{(\int_a^b f(x;r)dx)} = \int_a^b \overline{f}(x;r)dx. \tag{2.5}$$

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [9]. However, if $f(x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eq. (2.5) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [7, 9].

3 Fuzzy integral equations

The integral equations which are discussed in this section, are the Fredholm equations of the second kind. The Fredholm integral equation of the second kind is [8]

$$F(x) = f(x) + \lambda \int_a^b K(x,t)F(t)dt, \tag{3.6}$$

where $\lambda > 0$, $K(x,t)$ is an arbitrary kernel function over the square $a \leqslant x, t \leqslant b$ and $f(x)$ is a function of $x : a \leqslant x \leqslant b$. If $f(x)$ is a crisp function then the solutions of Eq. (3.6) are crisp as well. However, if $f(x)$ is a fuzzy function this equation may only possess a fuzzy solution. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind (FFIE-2), i.e., Eq. (3.6) where $f(x)$ is a fuzzy function, are given in [3].

Now, we introduce parametric form of a FFIE-2 with respect to Definition 2.1. Let $(\underline{f}(x;r), \overline{f}(x;r))$ and $(\underline{F}(x;r), \overline{F}(x;r))$, $0 \leqslant r \leqslant 1$ and $x \in [a,b]$ be parametric forms of $f(x)$ and $F(x)$, respectively. Then, parametric form of FFIE-2 is as follows:

$$\begin{cases} \underline{F}(x;r) = \underline{f}(x;r) + \lambda \int_a^b \underline{K(x,t)F(t;r)}\, dt, \\ \overline{F}(x;r) = \overline{f}(x;r) + \lambda \int_a^b \overline{K(x,t)F(t;r)}\, dt, \end{cases} \tag{3.7}$$

where

$$\underline{K(x,t)F(t;r)} = \begin{cases} K(x,t)\underline{F}(t;r), & K(x,t) \geqslant 0, \\ K(x,t)\overline{F}(t;r), & K(x,t) < 0, \end{cases} \tag{3.8}$$

and

$$\overline{K(x,t)F(t;r)} = \begin{cases} K(x,t)\overline{F}(t;r), & K(x,t) \geqslant 0, \\ K(x,t)\underline{F}(t;r), & K(x,t) < 0, \end{cases} \tag{3.9}$$

for each $0 \leqslant r \leqslant 1$ and $a \leqslant x \leqslant b$. We can see that (3.7) is a system of linear Fredholm integral equations in crisp case for each $0 \leqslant r \leqslant 1$.

4 The homotopy analysis method

In this section, we apply homotopy analysis method (HAM) for the system (3.7) and obtain a recursive scheme for it.

Prior to applying HAM for the system (3.7), we suppose that the kernel $K(x,t)$ is non-negative for $a \leqslant t \leqslant c$ and non-positive for $c \leqslant t \leqslant b$. Therefore, we rewrite system (3.7) in the following form

$$\begin{cases} \underline{F}(x;r) = \underline{f}(x;r) + \lambda \int_a^c K(x,t)\underline{F}(t;r)\,dt + \lambda \int_c^b K(x,t)\overline{F}(t;r)\,dt, \\ \overline{F}(x;r) = \overline{f}(x;r) + \lambda \int_a^c K(x,t)\overline{F}(t;r)\,dt + \lambda \int_c^b K(x,t)\underline{F}(t;r)\,dt. \end{cases} \tag{4.10}$$

Eq. (4.10) is a system of linear Fredholm integral equations in crisp case for each $0 \leqslant r \leqslant 1$.

For solving system (4.10) by homotopy analysis method, we construct the zero-order deformation equation

$$\begin{cases} (1-p)\ \mathcal{L}[\,\underline{U}(x,p;r) - \underline{w}_0(x;r)\,] & = p\,\hbar\,[\,\underline{U}(x,p;r) - \underline{f}(x;r) \\ & \quad -\lambda \int_a^c K(x,t)\underline{U}(t,p;r)\,dt \\ & \quad -\lambda \int_c^b K(x,t)\overline{U}(t,p;r)\,dt\,], \\ (1-p)\ \mathcal{L}[\,\overline{U}(x,p;r) - \overline{w}_0(x;r)\,] & = p\,\hbar\,[\,\overline{U}(x,p;r) - \overline{f}(x;r) \\ & \quad -\lambda \int_a^c K(x,t)\overline{U}(t,p;r)\,dt \\ & \quad -\lambda \int_c^b K(x,t)\underline{U}(t,p;r)\,dt\,], \end{cases} \tag{4.11}$$

where $p \in [0,1]$ is called the homotopy parameter [13], $\hbar$ is non-zero auxiliary parameter which is called the convergence-parameter [13], $\mathcal{L}$ is an auxiliary linear operator, $\underline{w}_0(x;r)$ and $\overline{w}_0(x;r)$ are initial guesses of $\underline{F}(x;r)$ and $\overline{F}(x;r)$, respectively and $\underline{U}(x,p;r)$ and $\overline{U}(x,p;r)$ are unknown functions on independent variable p.

Using the above zero-order deformation equation, on the assumption that $\mathcal{L}[u] = u$, we have

$$\begin{cases} (1-p)[\,\underline{U}(x,p;r) - \underline{w}_0(x;r)\,] & = p\,\hbar\,[\,\underline{U}(x,p;r) - \underline{f}(x;r) \\ & \quad -\lambda \int_a^c K(x,t)\underline{U}(t,p;r)\,dt \\ & \quad -\lambda \int_c^b K(x,t)\overline{U}(t,p;r)\,dt\,], \\ (1-p)[\,\overline{U}(x,p;r) - \overline{w}_0(x;r)\,] & = p\,\hbar\,[\,\overline{U}(x,p;r) - \overline{f}(x;r) \\ & \quad -\lambda \int_a^c K(x,t)\overline{U}(t,p;r)\,dt \\ & \quad -\lambda \int_c^b K(x,t)\underline{U}(t,p;r)\,dt\,]. \end{cases} \tag{4.12}$$

Obviously, when $p = 0$ and $p = 1$, it holds

$$\begin{cases} \underline{U}(x,0;r) = \underline{w}_0(x;r), \\ \overline{U}(x,0;r) = \overline{w}_0(x;r), \end{cases} \tag{4.13}$$

and

$$\begin{cases} \underline{U}(x,1;r) = \underline{f}(x;r) + \lambda \int_a^c K(x,t)\underline{U}(t,1;r)\,dt + \lambda \int_c^b K(x,t)\overline{U}(t,1;r)\,dt, \\ \overline{U}(x,1;r) = \overline{f}(x;r) + \lambda \int_a^c K(x,t)\overline{U}(t,1;r)\,dt + \lambda \int_c^b K(x,t)\underline{U}(t,1;r)\,dt, \end{cases} \tag{4.14}$$

respectively. Thus, as p increases from 0 to 1, the solution $(\underline{U}(x,p;r), \overline{U}(x,p;r))$ varies from the initial guess $(\underline{w}_0(x;r), \overline{w}_0(x;r))$ to the solution $(\underline{F}(x;r), \overline{F}(x;r))$. Expanding $\underline{U}(x,p;r)$ and $\overline{U}(x,p;r)$ in Taylor series with respect to p, we have

$$\begin{cases} \underline{U}(x,p;r) = \underline{w}_0(x;r) + \sum_{m=1}^{\infty} \underline{u}_m(x;r)\, p^m, \\ \overline{U}(x,p;r) = \overline{w}_0(x;r) + \sum_{m=1}^{\infty} \overline{u}_m(x;r)\, p^m, \end{cases} \tag{4.15}$$

where

$$\begin{cases} \underline{u}_m(x;r) = \frac{1}{m!} \frac{\partial^m \underline{U}(x,p;r)}{\partial p^m}\Big|_{p=0}, \\ \overline{u}_m(x;r) = \frac{1}{m!} \frac{\partial^m \overline{U}(x,p;r)}{\partial p^m}\Big|_{p=0}. \end{cases} \tag{4.16}$$

It should be noted that $\underline{U}(x,0;r) = \underline{w}_0(x;r)$ and $\overline{U}(x,0;r) = \overline{w}_0(x;r)$. Differentiating the zero-order deformation equation (4.12) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have

$$\begin{cases} \underline{u}_m(x;r) & = \alpha_m \,\underline{u}_{m-1}(x;r) + \hbar\,[\,\underline{u}_{m-1}(x;r) - \beta_m \underline{f}(x;r) \\ & \quad -\lambda \int_a^c K(x,t)\underline{u}_{m-1}(t;r)\,dt \\ & \quad -\lambda \int_c^b K(x,t)\overline{u}_{m-1}(t;r)\,dt\,], \\ \overline{u}_m(x;r) & = \alpha_m \,\overline{u}_{m-1}(x;r) + \hbar\,[\,\overline{u}_{m-1}(x;r) - \beta_m \overline{f}(x;r) \\ & \quad -\lambda \int_a^c K(x,t)\overline{u}_{m-1}(t;r)\,dt \\ & \quad -\lambda \int_c^b K(x,t)\underline{u}_{m-1}(t;r)\,dt\,], \end{cases} \tag{4.17}$$

where $m \geqslant 1$ and

$$\alpha_m = \begin{cases} 0, & m = 1, \\ 1, & m \neq 1, \end{cases} \qquad \beta_m = \begin{cases} 1, & m = 1, \\ 0, & m \neq 1, \end{cases}$$

and $\underline{u}_0(x;r) = \underline{w}_0(x;r)$ and $\overline{u}_0(x;r) = \overline{w}_0(x;r)$.

If we take $\underline{w}_0(x;r) = \overline{w}_0(x;r) = 0$, then we have

$$\begin{cases} \underline{u}_1(x;r) & = -\hbar\ \underline{f}(x;r), \\ \overline{u}_1(x;r) & = -\hbar\ \overline{f}(x;r), \\ \underline{u}_m(x;r) & = (1+\hbar)\,\underline{u}_{m-1}(x;r) - \hbar\lambda\,[\int_a^c K(x,t)\underline{u}_{m-1}(t;r)\,dt \\ & \quad + \int_c^b K(x,t)\overline{u}_{m-1}(t;r)\,dt\,], \\ \overline{u}_m(x;r) & = (1+\hbar)\,\overline{u}_{m-1}(x;r) - \hbar\lambda\,[\int_a^c K(x,t)\overline{u}_{m-1}(t;r)\,dt \\ & \quad + \int_c^b K(x,t)\underline{u}_{m-1}(t;r)\,dt\,], \end{cases} \tag{4.18}$$

where $m \geq 2$.

Hence, the solution of Eq. (4.10) in series form is obtained as

$$\begin{cases} \underline{F}(x;r) = \lim_{p\to 1} \underline{U}(x,p;r) = \sum_{m=1}^{\infty} \underline{u}_m(x;r), \\ \overline{F}(x;r) = \lim_{p\to 1} \overline{U}(x,p;r) = \sum_{m=1}^{\infty} \overline{u}_m(x;r). \end{cases} \tag{4.19}$$

We denote the nth-order approximation to solution $\underline{F}(x;r)$ with

$$\underline{F}_n(x;r) = \sum_{m=1}^{n} \underline{u}_m(x;r),$$

and $\overline{F}(x;r)$ with

$$\overline{F}_n(x;r) = \sum_{m=1}^{n} \overline{u}_m(x;r).$$

5 Text examples

Example 5.1. *Consider the fuzzy Fredholm integral equation with*

$$\underline{f}(x;r) = \frac{1}{2}x^2(r+1),$$

$$\overline{f}(x;r) = \frac{1}{2}x^2(3-r),$$

and kernel

$$K(x,t) = t\,x^2, \qquad -1 \leqslant x, t \leqslant 1,$$

and $a=-1$, $b=1$, $\lambda=1$. The exact solution in this case is given by

$$\underline{F}(x;r) = x^2 r,$$

$$\overline{F}(x;r) = x^2(2-r).$$

In this example, $K(x,t) \leqslant 0$ for each $-1 \leqslant t \leqslant 0$ and $K(x,t) \geqslant 0$ for each $0 \leqslant t \leqslant 1$. By Eq. (4.18), we can see that, some first terms of HAM series are as follows:

$$\begin{aligned}
\underline{u}_1(x;r) &= -\tfrac{1}{2}\hbar x^2[\,(1+r)\,],\\
\underline{u}_2(x;r) &= -\tfrac{1}{4}\hbar x^2\,[\,2(r+1)+\hbar(r+3)\,],\\
\underline{u}_3(x;r) &= -\tfrac{1}{8}\hbar x^2\,[\,4(r+1)+4\hbar(r+3)+\hbar^2(r+7)\,],\\
\underline{u}_4(x;r) &= -\tfrac{1}{16}\hbar x^2\,[\,8(r+1)+12\hbar(r+3)+6\hbar^2(r+7)+\hbar^3(r+15)\,],\\
\underline{u}_5(x;r) &= -\tfrac{1}{32}\hbar x^2\,[\,16(r+1)+32\hbar(r+3)+24\hbar^2(r+7)+8\hbar^3(r+15)+\hbar^4(r+31)\,],\\
\underline{u}_6(x;r) &= -\tfrac{1}{64}\hbar x^2\,[\,32(r+1)+80\hbar(r+3)+80\hbar^2(r+7)+40\hbar^3(r+15)\\
&\quad +10\hbar^4(r+31)+\hbar^5(r+63)\,],\\
\underline{u}_7(x;r) &= -\tfrac{1}{128}\hbar x^2\,[\,64(r+1)+192\hbar(r+3)+240\hbar^2(r+7)+160\hbar^3(r+15)\\
&\quad +60\hbar^4(r+31)+12\hbar^5(r+63)+\hbar^6(r+127)\,],
\end{aligned}$$

and

$$\begin{aligned}
\overline{u}_1(x;r) &= -\tfrac{1}{2}\hbar x^2[\,(3-r)\,],\\
\overline{u}_2(x;r) &= -\tfrac{1}{4}\hbar x^2\,[\,2(3-r)+\hbar(5-r)\,],\\
\overline{u}_3(x;r) &= -\tfrac{1}{8}\hbar x^2\,[\,4(3-r)+4\hbar(5-r)+\hbar^2(9-r)\,],\\
\overline{u}_4(x;r) &= -\tfrac{1}{16}\hbar x^2\,[\,8(3-r)+12\hbar(5-r)+6\hbar^2(9-r)+\hbar^3(17-r)\,],\\
\overline{u}_5(x;r) &= -\tfrac{1}{32}\hbar x^2\,[\,16(3-r)+32\hbar(5-r)+24\hbar^2(9-r)+8\hbar^3(17-r)+\hbar^4(33-r)\,],\\
\overline{u}_6(x;r) &= -\tfrac{1}{64}\hbar x^2\,[\,32(3-r)+80\hbar(5-r)+80\hbar^2(9-r)+40\hbar^3(17-r)\\
&\quad +10\hbar^4(33-r)+\hbar^5(65-r)\,],\\
\overline{u}_7(x;r) &= -\tfrac{1}{128}\hbar x^2\,[\,64(3-r)+192\hbar(5-r)+240\hbar^2(9-r)+160\hbar^3(17-r)\\
&\quad +60\hbar^4(33-r)+12\hbar^5(65-r)+\hbar^6(129-r)\,]
\end{aligned}$$

Then we approximate $\underline{F}(x;r)$ with

$$\begin{aligned}
\underline{F}_7(x;r) &= \sum_{m=1}^{7}\underline{u}_m(x;r)\\
&= -\frac{1}{128}\hbar x^2\,[\,448(r+1)+672\hbar(r+3)+560\hbar^2(r+7)+280\hbar^3(r+15)\\
&+ 84\hbar^4(r+31)+14\hbar^5(r+63)+\hbar^6(r+127)\,],
\end{aligned}$$

and $\overline{F}(x;r)$ with

$$\begin{aligned}\overline{F}_7(x;r) &= \sum_{m=1}^{7} \overline{u}_m(x;r) \\ &= -\frac{1}{128}\hbar x^2 \,[\,448(3-r) + 672\hbar(5-r) + 560\hbar^2(9-r) + 280\hbar^3(17-r) \\ &+ 84\hbar^4(33-r) + 14\hbar^5(65-r) + \hbar^6(129-r)\,].\end{aligned}$$

It has been proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of the exact solutions [11]. So it is important to ensure that the solution series (4.19) is convergent. Note that the solution series (4.19) contains the convergence-parameter $\hbar$, which provides us with a simple way to adjust and control the convergence of the solution series. In general, by means of the so-call $\hbar$-curve (a curve of a versus $\hbar$), it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed by Liao [11], the valid region of $\hbar$ is a horizontal line segment. In Figure 1, we plot the $\hbar$-curves of $\underline{F}(0.5;0.5)$ and $\overline{F}(0.5;0.5)$ given by 7th-order approximate solution ,i.e., $\underline{F}_7(0.5;0.5)$ and $\overline{F}_7(0.5;0.5)$, respectively. From the Fig. 1, we could find that if $\hbar$ is about in area $[-1.6,-0.4]$ the result is convergent.

On the other hand, it is clear that $\underline{F}_7(x;r)$ and $\overline{F}_7(x;r)$ are continuous, increasing and decreasing with respect to r, for any $x \in [-1,1]$ and $\hbar \in [-1.6,-0.4]$, respectively. Also, we can show that

$$\underline{F}_7(x;1) \leq \overline{F}_7(x;1),$$

for any $x \in [-1,1]$ and $\hbar \in [-1.6,-0.4]$. Therefore

$$(\underline{F}_7(x;r), \overline{F}_7(x;r)),$$

is the parametric form of a fuzzy number, for any $x \in [-1,1]$ and $\hbar \in [-1.6,-0.4]$.

We compare results with exact solutions using metric of Definition 2.2 for different values of $\hbar$ by 7th-order approximate solution in Table 1. The results reveal that the homotopy analysis method can provide us with a convenient way to adjust and control the convergence region and rate of approximation series by introducing convergence-control parameter $\hbar$.

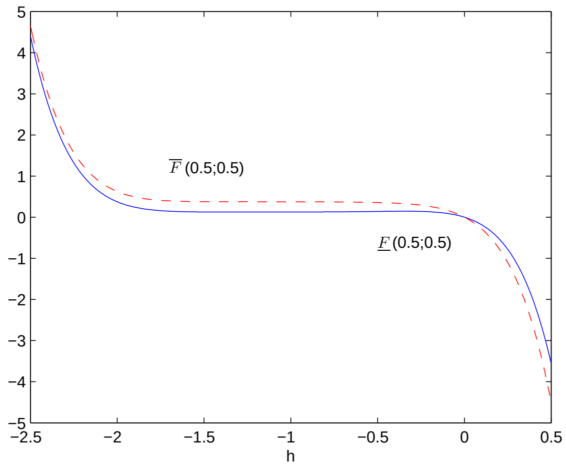

Fig. 1. $\hbar$-curves of $\underline{F}(0.5; 0.5)$ and $\overline{F}(0.5; 0.5)$ given by the 7th-order approximate solution.

Table 1
The error in the solution obtained by HAM for various $\hbar$ by 7th-order approximate solution.

x	$\hbar = -1.4$	$\hbar = -1.3$	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar = -1$
-1	$1.8571e-003$	$8.6209e-004$	$1.6512e-003$	$3.7368e-003$	$7.8125e-003$
-0.8	$1.1885e-003$	$5.5174e-004$	$1.0568e-003$	$2.3915e-003$	$5.0000e-003$
-0.6	$6.6856e-004$	$3.1035e-004$	$5.9443e-004$	$1.3452e-003$	$2.8125e-003$
-0.4	$2.9714e-004$	$1.3793e-004$	$2.6419e-004$	$5.9789e-004$	$1.2500e-003$
-0.2	$7.4284e-005$	$3.4484e-005$	$6.6048e-005$	$1.4947e-004$	$3.1250e-004$
0	0	0	0	0	0
0.2	$7.4284e-005$	$3.4484e-005$	$6.6048e-005$	$1.4947e-004$	$3.1250e-004$
0.4	$2.9714e-004$	$1.3793e-004$	$2.6419e-004$	$5.9789e-004$	$1.2500e-003$
0.6	$6.6856e-004$	$3.1035e-004$	$5.9443e-004$	$1.3452e-003$	$2.8125e-003$
0.8	$1.1885e-003$	$5.5174e-004$	$1.0568e-003$	$2.3915e-003$	$5.0000e-003$
1	$1.8571e-003$	$8.6209e-004$	$1.6512e-003$	$3.7368e-003$	$7.8125e-003$

Example 5.2. *[5] Consider the fuzzy Fredholm integral equation with*

$$\underline{f}(x;r) = \sin(\frac{x}{2})\,[\frac{13}{15}(r^2+r) + \frac{2}{15}(4-r-r^3)],$$

$$\overline{f}(x;r) = \sin(\frac{x}{2})\,[\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r-r^3)],$$

and kernel

$$K(x,t) = 0.1\,\sin(t)\sin(\frac{x}{2}), \qquad 0 \leqslant x,t \leqslant 2\pi,$$

and $a = 0$, $b = 2\pi$, $\lambda = 1$. The exact solution in this case is given by

$$\underline{F}(x;r) = \sin(\frac{x}{2})\,[r^2+r],$$

$$\overline{F}(x;r) = \sin(\frac{x}{2})\,[4-r-r^3].$$

In this example, $K(x,t) \geqslant 0$ for each $0 \leqslant t \leqslant \pi$ and $K(x,t) \leqslant 0$ for each $\pi \leqslant t \leqslant 2\pi$. By Eq. (4.18), Some first terms of HAM series are:

$$\begin{aligned}
\underline{u}_1(x;r) &= \tfrac{1}{15}\hbar \sin(\tfrac{x}{2})\,[\,2r^3 - 13r^2 - 11r - 8\,],\\
\underline{u}_2(x;r) &= \tfrac{1}{225}\hbar \sin(\tfrac{x}{2})\,[\,30r^3 - 195r^2 - 165r - 120\,]\\
&\quad + \tfrac{1}{225}\hbar^2 \sin(\tfrac{x}{2})\,[\,52r^3 - 173r^2 - 121r - 208\,],\\
\underline{u}_3(x;r) &= \tfrac{1}{3375}\hbar \sin(\tfrac{x}{2})\,[\,450r^3 - 2925r^2 - 2475r - 1800\,]\\
&\quad + \tfrac{1}{3375}\hbar^2 \sin(\tfrac{x}{2})\,[\,1560r^3 - 5190r^2 - 3630r - 6240\,]\\
&\quad + \tfrac{1}{3375}\hbar^3 \sin(\tfrac{x}{2})\,[\,1022r^3 - 2353r^2 - 1331r - 4088\,],\\
\underline{u}_4(x;r) &= \tfrac{1}{50625}\hbar \sin(\tfrac{x}{2})\,[\,6750r^3 - 43875r^2 - 37125r - 27000\,]\\
&\quad + \tfrac{1}{50625}\hbar^2 \sin(\tfrac{x}{2})\,[\,35100r^3 - 116775r^2 - 81675r - 140400\,]\\
&\quad + \tfrac{1}{50625}\hbar^3 \sin(\tfrac{x}{2})\,[\,45990r^3 - 105885r^2 - 59895r - 183960\,]\\
&\quad + \tfrac{1}{50625}\hbar^4 \sin(\tfrac{x}{2})\,[\,17992r^3 - 32633r^2 - 14641r - 71968\,],\\
\underline{u}_5(x;r) &= \tfrac{1}{759375}\hbar \sin(\tfrac{x}{2})\,[\,101250r^3 - 658125r^2 - 556875r - 405000\,]\\
&\quad + \tfrac{1}{759375}\hbar^2 \sin(\tfrac{x}{2})\,[\,702000r^3 - 2335500r^2 - 1633500r - 2808000\,]\\
&\quad + \tfrac{1}{759375}\hbar^3 \sin(\tfrac{x}{2})\,[\,1379700r^3 - 3176550r^2 - 1796850r - 5518800\,]\\
&\quad + \tfrac{1}{759375}\hbar^4 \sin(\tfrac{x}{2})\,[\,1079520r^3 - 1957980r^2 - 878460r - 4318080\,]\\
&\quad + \tfrac{1}{759375}\hbar^5 \sin(\tfrac{x}{2})\,[\,299162r^3 - 460213r^2 - 161051r - 1196648\,],
\end{aligned}$$

and

$$\begin{aligned}
\overline{u}_1(x;r) &= \tfrac{1}{15}\hbar \sin(\tfrac{x}{2})\,[\,13r^3 - 2r^2 + 11r - 52\,]\\
\overline{u}_2(x;r) &= \tfrac{1}{225}\hbar \sin(\tfrac{x}{2})\,[\,195r^3 - 30r^2 + 165r - 780\,]\\
&\quad + \tfrac{1}{225}\hbar^2 \sin(\tfrac{x}{2})\,[\,173r^3 - 52r^2 + 121r - 692\,],\\
\overline{u}_3(x;r) &= \tfrac{1}{3375}\hbar \sin(\tfrac{x}{2})\,[\,2925r^3 - 450r^2 + 2475r - 11700\,]\\
&\quad + \tfrac{1}{3375}\hbar^2 \sin(\tfrac{x}{2})\,[\,5190r^3 - 1560r^2 + 3630r - 20760\,]\\
&\quad + \tfrac{1}{3375}\hbar^3 \sin(\tfrac{x}{2})\,[\,2353r^3 - 1022r^2 + 1331r - 9412\,],
\end{aligned}$$

$$\begin{aligned}\overline{u}_4(x;r) &= \tfrac{1}{50625}\hbar\,\sin(\tfrac{x}{2})\,[\,43875r^3-6750r^2+37125r-175500\,]\\ &+\tfrac{1}{50625}\hbar^2\sin(\tfrac{x}{2})\,[\,116775r^3-35100r^2+81675r-467100\,]\\ &+\tfrac{1}{50625}\hbar^3\sin(\tfrac{x}{2})\,[\,105885r^3-45990r^2+59895r-423540\,]\\ &+\tfrac{1}{50625}\hbar^4\sin(\tfrac{x}{2})\,[\,32633r^3-17992r^2+14641r-130532\,],\\ \overline{u}_5(x;r) &= \tfrac{1}{759375}\hbar\,\sin(\tfrac{x}{2})\,[\,658125r^3-101250r^2+556875r-2632500\,]\\ &+\tfrac{1}{759375}\hbar^2\sin(\tfrac{x}{2})\,[\,2335500r^3-702000r^2+1633500r-9342000\,]\\ &+\tfrac{1}{759375}\hbar^3\sin(\tfrac{x}{2})\,[\,3176550r^3-1379700r^2+1796850r-12706200\,]\\ &+\tfrac{1}{759375}\hbar^4\sin(\tfrac{x}{2})\,[\,1957980r^3-1079520r^2+878460r-7831920\,]\\ &+\tfrac{1}{759375}\hbar^5\sin(\tfrac{x}{2})\,[\,460213r^3-299162r^2+161051r-1840852\,].\end{aligned}$$

Then we approximate $\underline{F}(x;r)$ with

$$\begin{aligned}\underline{F}_5(x;r) &= \sum_{m=1}^{5}\underline{u}_m(x;r)\\ &= \frac{1}{759375}\hbar\,\sin(\frac{x}{2})\,[\,506250r^3-3290625r^2-2784375r-2025000\,]\\ &+ \frac{1}{759375}\hbar^2\sin(\frac{x}{2})\,[\,1755000r^3-5838750r^2-4083750r-7020000\,]\\ &+ \frac{1}{759375}\hbar^3\sin(\frac{x}{2})\,[\,2299500r^3-5294250r^2-2994750r-9198000\,]\\ &+ \frac{1}{759375}\hbar^4\sin(\frac{x}{2})\,[\,1349400r^3-2447475r^2-1098075r-5397600\,]\\ &+ \frac{1}{759375}\hbar^5\sin(\frac{x}{2})\,[\,299162r^3-460213r^2-161051r-1196648\,],\end{aligned}$$

and $\overline{F}(x;r)$ with

$$\begin{aligned}\overline{F}_5(x;r) &= \sum_{m=1}^{5}\overline{u}_m(x;r)\\ &= \frac{1}{759375}\hbar\,\sin(\frac{x}{2})\,[\,3290625r^3-506250r^2+2784375r-13162500\,]\\ &+ \frac{1}{759375}\hbar^2\sin(\frac{x}{2})\,[\,5838750r^3-1755000r^2+4083750r-23355000\,]\\ &+ \frac{1}{759375}\hbar^3\sin(\frac{x}{2})\,[\,5294250r^3-2299500r^2+2994750r-21177000\,]\\ &+ \frac{1}{759375}\hbar^4\sin(\frac{x}{2})\,[\,2447475r^3-1349400r^2+1098075r-9789900\,]\\ &+ \frac{1}{759375}\hbar^5\sin(\frac{x}{2})\,[\,460213r^3-299162r^2+161051r-1840852\,].\end{aligned}$$

As pointed out earlier, the convergence region and rate of approximation strongly depend on the choice of the values of the convergence-contol parameter $\hbar$ for the HAM. We should therefore focus on the choice of $\hbar$ by plotting of $\hbar$-curves. Fig. 2, shows the $\hbar$-curves of $\underline{F}(\pi; 0.5)$ and $\overline{F}(\pi; 0.5)$ given by 5th-order approximate solutions,i.e., $\underline{F}_5(\pi; 0.5)$ and $\overline{F}_5(\pi; 0.5)$, respectively. It is seen that convergent results can be obtained when $\hbar \in [-1.5, -0.6]$. Similar to the Example 5.1, we can show that

$$(\underline{F}(x; r), \overline{F}(x; r)),$$

is the parametric form of a fuzzy number, for any $x \in [0, 2\pi]$ and $\hbar \in [-1.5, -0.6]$.

Table 2
The error in the solution obtained by HAM for various $\hbar$ by 5th-order approximate solution.

x	$\hbar = -1.3$	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar = -1$	$\hbar = -0.9$
0	0	0	0	0	0
$\pi/5$	$1.5575e-003$	$2.1408e-004$	$1.7311e-004$	$8.3340e-003$	$2.8142e-003$
$2\pi/5$	$2.9626e-003$	$4.0720e-004$	$3.2928e-004$	$1.5852e-003$	$5.3530e-003$
$3\pi/5$	$4.0776e-003$	$5.6046e-004$	$4.5322e-004$	$2.1819e-003$	$7.3678e-003$
$4\pi/5$	$4.7935e-003$	$6.5886e-004$	$5.3279e-004$	$2.5650e-003$	$8.6614e-003$
π	$5.0402e-003$	$6.9276e-004$	$5.6021e-004$	$2.6970e-003$	$9.1071e-003$
$6\pi/5$	$4.7935e-003$	$6.5886e-004$	$5.3279e-004$	$2.5650e-003$	$8.6614e-003$
$7\pi/5$	$4.0776e-003$	$5.6046e-004$	$4.5322e-004$	$2.1819e-003$	$7.3678e-003$
$8\pi/5$	$2.9626e-003$	$4.0720e-004$	$3.2928e-004$	$1.5852e-003$	$5.3530e-003$
$9\pi/5$	$1.5575e-003$	$2.1408e-004$	$1.7311e-004$	$8.3340e-004$	$2.8142e-003$
2π	$4.8986e-016$	$4.8986e-016$	$4.8986e-016$	$4.8986e-016$	$4.8986e-016$

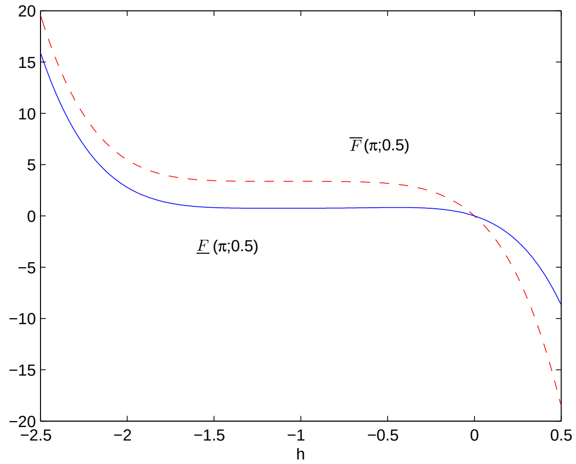

Fig. 2. $\hbar$-curves of $\underline{F}(\pi; 0.5)$ and $\overline{F}(\pi; 0.5)$ given by the 5th-order approximate solution.

We compare results with exact solutions using metric of Definition 2.2 for different values of $\hbar$ by 5th-order approximate solution in Table 2. The results reveal that the HAM is very effective and simple.

6 Conclusion

In this paper, the homotopy analysis method (HAM) was successfully applied for solving fuzzy linear Fredholm integral equations of the second kind. It wa illustrated that the HAM provides a convenient way to adjust and control the convergence of approximation series, which is the main advantage of this method. The results revealed the validity and the great potential of HAM in solving fuzzy integral equations.

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