

## A Generalization of Clique Polynomials and Graph Homomorphism

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**Abstract.** The clique polynomial of a graph  $G$  is the ordinary generating function of the number of complete subgraphs (cliques) of  $G$ . In this paper, we introduce a new vertex-weighted version of these polynomials. We also show that these weighted clique polynomials have always a real root provided that the weights are non-negative real numbers. As an application, we obtain a no-homomorphism criteria based on the largest real root of our vertex-weighted clique polynomial.

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### 1. Introduction

The dependence polynomial was first introduced by Fisher [1], while working on the problem of counting the number of words of length  $n$  from the alphabet of  $m$  letters so that some pairs of them can commute. Fisher and Solow [2] introduced the dependence polynomial, as follows:

$$f_G(x) = 1 - c_1x + c_2x^2 - c_3x^3 + \cdots + (-1)^\omega c_\omega x^\omega$$

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where  $\omega$  is the size of the largest clique in  $G$  and  $c_i$  denotes the number of complete subgraphs of size  $i$  in  $G$ . Fisher [1], showed that the generating function of the above word-counting problem is  $\frac{1}{f_G(x)}$ .

If we change the sign of all negative coefficients in  $f_G(x)$  to positive signs, we obtain a polynomial which is called the clique polynomial and denoted by  $C(G, x)$ . Hajiabolhasan and Mehrabadi [3] showed that for any simple graph  $G$ , the clique polynomial of  $G$  has always a real root using basic counting techniques, induction and the intermediate value theorem. As an immediate consequence, they obtained a new generating function proof of Mantel's theorem [4, p.41] for *triangle-free* graphs. In this paper, we will continue the same line of research by introducing a new weighted version of the clique polynomial. Our main goal here is to show that how one can use the largest real root of this new graph polynomial to obtain a no-homomorphism criteria.

## 2. Weighted Clique Polynomials

Throughout the paper we will assume that  $G$  is a simple undirected graph. The graph terminology that we use is standard and generally follows from [4]. For a given graph  $G$ , we denoted by  $V(G)$  its set of vertices and by  $E(G)$  its set of edges. When  $S \subseteq V(G)$ , the *induced* subgraph  $G[S]$  consists of  $S$  and all edges whose endpoints are connected in  $S$ . The *neighborhood* of a vertex  $u$ , written  $N(u)$ , is the set of vertices adjacent to  $u$ . We write  $G - u$  for the subgraph of  $G$  obtained by deleting a vertex  $u$ . We also write  $G - uv$  for the subgraph obtained by deleting an edge  $uv \in E(G)$ . Here by an  *$i$ -clique*,  $i \geq 1$ , we mean a complete subgraph of  $G$  with  $i$  vertices. The *clique number* of a graph  $G$  denoted by  $\omega$  is the size of the largest clique in  $G$ . For simplicity of our arguments, we will assume that  $V(G) = [n] = \{1, 2, \dots, n\}$ .

For a given set  $A$ , it's *power set* denoted by  $\mathcal{P}(A)$ , is the set of all subsets of  $A$ . We will associate a real *weight* function  $\mathbf{w} : V(G) \mapsto \mathbb{R}$  with the given graph  $G$ , by associating an indeterminate  $w_i$  to each vertex  $i$  of  $G$  which can be viewed as the weight of the vertex  $i$ . More precisely, we

have

$$\begin{aligned} \mathbf{w} : V(G) &\mapsto \mathbb{R} \\ \mathbf{w}(i) &= w_i \quad \forall i \in V(G). \end{aligned}$$

The above mapping can be simply extended to the following multiplicative weight function (we will use the same symbol for the extended version)

$$\begin{aligned} \mathbf{w} : \mathcal{P}(V(G)) &\mapsto \mathbb{R} \\ \mathbf{w}(S) &= \prod_{i \in S} w_i, \quad \forall S \subseteq V(G). \end{aligned}$$

That is, the weight of any subset of vertices will be obtained as the product of the weights of those vertices. By convention, the empty product (corresponding to the set  $S = \emptyset$ ) is defined to be 1.

**Remark 2.1.** *It is worthy to note that the above weight function on vertices of a graph  $G$  can be considered as the multiplicative version of the following additive weight function:*

$$\begin{aligned} \mathbf{w} : \mathcal{P}(V(G)) &\mapsto \mathbb{R} \\ \mathbf{w}(S) &= \sum_{i \in S} w_i, \quad \forall S \subseteq V(G). \end{aligned}$$

Now, we are ready to give the definition of the weighted clique polynomial.

**Definition 2.2.** *Let  $G$  be a graph on  $n$  vertices associated with the multiplicative weight function  $\mathbf{w} : V(G) \mapsto \mathbb{R}$ . We define the weighted clique polynomial of  $G$  denoted by  $C(G, x; \mathbf{w})$ , as follows*

$$C(G, x; \mathbf{w}) = \sum_{i=0}^{\omega} c_i(G, \mathbf{w}) x^i$$

where  $\mathbf{w} = (w_1, \dots, w_n)$  is the weight vector of vertices of  $G$  and  $c_i(G, \mathbf{w})$ ,  $i \geq 1$ , the weighted-sum of all  $i$ -cliques in  $G$ , is defined by

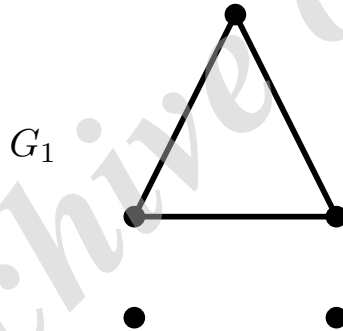
$$c_i(G, \mathbf{w}) = \sum_{S: S \subseteq V(G), G[S] \text{ is an } i\text{-clique}} \mathbf{w}(S).$$

By convention, we assume  $c_0(G, \mathbf{w}) = 1$  for any weight vector  $\mathbf{w}$  and any graph  $G$ . In particular, if all weights are equal to one then we obtain the clique polynomial of  $G$  [3].

**Example 2.3.** Consider the graph  $G_1$  with  $\mathbf{w} = (1, 1, 1, 1, 1)$  as depicted in Fig.1, we obtain

$$C(G_1, x; \mathbf{w}) = 1 + 5x + 3x^2 + x^3.$$

The above polynomial has at least one real root, because of having odd degree. Moreover, since the quadratic polynomial  $\frac{d}{dx}C(G_1, x; \mathbf{w}) = 5 + 6x + 3x^2$  has the discriminant  $\Delta = 9 - 15 = -6 < 0$ , then by the first derivative criteria  $C(G_1, x; \mathbf{w})$  is an increasing function on its domain and hence the clique polynomial of  $G_1$  has only one real root.



**Figure 1.** The clique polynomial of the graph  $G_1$  has only one real root.

It seems that as in the case of the clique polynomials, we have always at least a real root for the weighted clique polynomials for any arbitrary choices of non-negative real weights. Next, we present the necessary tools for proving this interesting result.

The following counting lemma is key for proving the existence of a real root for the weighted clique polynomials with non-negative real weights. Here, for the given edge  $e = \{u, v\}$ , the notation  $N(e)$  stands

for  $N(e) = N(u) \cap N(v)$ . We also define the multiplicative weight function associated with the subgraph  $H$  of  $G$  simply by restricting the weight function associated with  $G$  to  $H$ . More precisely, we only need to choose the weight of a vertex which is not in  $V(H)$  to be zero.

**Lemma 2.4.** *Let  $G$  be a graph and  $u, v \in V(G)$  with non-negative real weights  $w_u$  and  $w_v$ . Then, we have*

$$i) \quad C(G, x; \mathbf{w}) = C(G - u, x; \mathbf{w}_1) + w_u x C(G[N(u)], x; \mathbf{w}_2), \quad (1)$$

$$ii) \quad C(G, x; \mathbf{w}) = C(G - e, x; \mathbf{w}_3) + w_u w_v x^2 C(G[N(e)], x; \mathbf{w}_4) \quad (2)$$

where  $e = uv \in E(G)$  and  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$  and  $\mathbf{w}_4$  are the weight vectors for the subgraphs  $G - u, G[N(u)], G - e$  and  $G[N(e)]$ , respectively.

**Proof.** We first note that the following recurrence relations

$$c_i(G, \mathbf{w}) = c_i(G - u, \mathbf{w}_1) + w_u c_{i-1}(G[N(u)], \mathbf{w}_2), \quad (i \geq 1)$$

$$c_i(G, \mathbf{w}) = c_i(G - e, \mathbf{w}_3) + w_u w_v c_{i-2}(G[N(e)], \mathbf{w}_4), \quad (i \geq 2),$$

can be easily proved using a simple counting and case analysis arguments. Now, by multiplying both sides of the above relations to  $x^i$  and summing over all  $i$ 's we obtain the desired results.  $\square$

The *join* of two simple graphs  $G$  and  $H$ , written  $G \vee H$ , is defined as a graph with the vertex set  $V(G) \cup V(H)$  and the edge set  $E(G) \cup E(H) \cup \{xy | x \in V(G) \wedge y \in V(H)\}$ . Considering the definition of the weighted clique polynomials and using Lemma 2.4, we get the following multiplicative property for the weighted clique polynomials.

**Proposition 2.5.** *Let  $G$  and  $H$  be arbitrary graphs with their weight vectors  $\mathbf{w}_1 = (w_1^{(g)}, \dots, w_m^{(g)})$  and  $\mathbf{w}_2 = (w_1^{(h)}, \dots, w_n^{(h)})$ . Then, we have*

$$C(G \vee H, x; \mathbf{w}) = C(G, x; \mathbf{w}_1)C(H, x; \mathbf{w}_2)$$

where

$$\mathbf{w} = (w_1^{(g)}, \dots, w_m^{(g)}, w_1^{(h)}, \dots, w_n^{(h)}).$$

**Definition 2.6.** Let  $G$  be a graph and  $\mathcal{Z}(G)$  be the set of all negative real roots of  $C(G, x; \mathbf{w})$ . We define  $\zeta_G$  by

$$\zeta_G = \begin{cases} \max \mathcal{Z}(G) & \text{if } \mathcal{Z}(G) \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases}$$

**Theorem 2.7.** Let  $G$  be a graph and  $H$  be the induced subgraph of  $G$ . Then  $\zeta_H \leq \zeta_G$ .

**Proof.** We proceed by strong induction on  $|V(G)| = n$ . If  $n = 1, 2$ , then the assertion is easily followed. Moreover, if  $H = G$  then we are also done. Therefore, we will assume that  $H$  is a proper induced subgraph of  $G$ . Since any proper induced subgraph  $H$  of  $G$  can be obtained from it by a sequence of vertex-deletion operations and the binary relation  $\leq$  is transitive, it is sufficient to prove the assertion for  $H = G - u$ . If  $\mathcal{Z}(G - u) = \emptyset$ , by the definition of  $\zeta_{G-u}$ , we are done. Otherwise, plugging  $\alpha = \zeta_{G-u}$  ( $< 0$ ) into both sides of (1), we get

$$C(G, \alpha; \mathbf{w}) = w_u \alpha C(G[N(u)], \alpha; \mathbf{w}_2).$$

Now, we distinguish between two cases.

**Case 1.** If  $C(G[N(u)], \alpha; \mathbf{w}_2) \geq 0$ , then by positivity of the weights we conclude that  $C(G, \alpha; \mathbf{w}) \leq 0$ . This implies that, using intermediate value theorem and  $C(G, 0; \mathbf{w}) = 1$ , the polynomial  $C(G, x; \mathbf{w})$  has a real root in the interval  $[\alpha, 0)$ . Hence, we immediately obtain  $\zeta_{G-u} = \alpha \leq \zeta_G$ .

**Case 2.** Otherwise,  $C(G[N(u)], \alpha; \mathbf{w}_2) < 0$ . Now, we claim that this case is impossible. To do this, note that the last inequality implies that  $C(G[N(u)], x; \mathbf{w}_2)$  has a real root in the interval  $[\alpha, 0)$  (let say  $\beta$ ), once again by applying intermediate value theorem. Hence we conclude that

$$\zeta_{G-u} = \alpha \leq \beta \leq \zeta_{G[N(u)]}.$$

But this last inequality is impossible by the induction hypothesis, since  $G[N(u)]$  is an induced subgraph of  $G - u$  and we obtain the inequality

$$\zeta_{G[N(u)]} \leq \zeta_{G-u}.$$

Thus, the only possibility is Case 1 which implies the desired inequality  $\zeta_{G-u} \leq \zeta_G$ . This completes the proof by mathematical induction.  $\square$

**Corollary 2.8.** *For any graph  $G$ , let  $w_u$  be the weight of the vertex  $u$  which has the maximum weights among all vertices. Then,  $-\frac{1}{w_u} \leq \zeta_G < 0$ .*

**Proof.** First of all, by definition of  $\zeta_G$  it is clear that  $\zeta_G < 0$ . Next, let  $u$  be the vertex of  $G$  with the maximum weight  $w_u$  and let  $H$  be the induced subgraph  $G[u]$ . Then, clearly  $C(H, x; w_u) = 1 + w_u x$ . Hence,  $\zeta_H = -\frac{1}{w_u}$ . Now applying Theorem 2.7, we get  $-\frac{1}{w_u} \leq \zeta_G$ .  $\square$

**Remark 2.9.** *It is worth to note that the above corollary shows that the weighted clique polynomial has always a real root, provided that the weights are non-negative real numbers and the weight vector is not identically zero.*

As we already saw, when  $H$  is an induced subgraph of  $G$  we obtain  $\zeta_H \leq \zeta_G$ . Next, we show that for a spanning subgraph  $H$  of  $G$  we have the reverse inequality; that is,  $\zeta_H \geq \zeta_G$ . Recall that a *spanning* subgraph  $H$  of a given graph  $G$  is the one with the same vertex-set as  $G$ ; that is,  $V(H) = V(G)$ .

**Theorem 2.10.** *Let  $G$  be a graph and  $H$  be the spanning subgraph of  $G$ . Then  $\zeta_H \geq \zeta_G$ .*

**Proof.** We proceed by strong induction on the number of edges. It is sufficient to prove the assertion for the case  $H = G - e$ , where  $e = uv$  is an edge of  $G$ . Now by substituting  $\zeta_G$  in both sides of (2), we get

$$C(G - uv, \zeta_G; \mathbf{w}_3) = -w_u w_v \zeta_G^2 C(G[N(e)], \zeta_G; \mathbf{w}_4). \quad (3)$$

Since  $G[N(e)]$  is an induced subgraph of  $G$ , then by Theorem 2.7 the right-hand side of (3) is negative which implies that  $C(G - uv, \zeta_G; \mathbf{w}_3)$  is also negative. Considering the fact that  $C(G - uv, 0; \mathbf{w}_3) = 1$  and applying the intermediate value theorem, we get the desired result.  $\square$

**Definition 2.11.** *An independent set in a graph is a set of pairwise nonadjacent vertices. The independence number of a graph  $G$ , written  $\alpha(G)$ , is the maximum size of an independent set of vertices.*

**Proposition 2.12.** *Let  $G$  be a graph with  $n$  vertices and  $\alpha(G)$  its independence number. Let  $\mathbf{w} = (w_1, \dots, w_n)$  be the weight vector of  $G$  with  $w = \min_{1 \leq i \leq n} w_i$ . Then, we have  $\alpha(G) \leq -\frac{1}{w\zeta_G}$ .*

**Proof.** Assume that  $S = \{i_1, i_2, \dots, i_k\}$  is an independent set of size  $\alpha(G) = k$  in  $G$  and  $H$  is the induced subgraph  $G[S]$ . Since  $H$  has no edges, we obtain

$$C(H, x; \mathbf{w}) = 1 + (w_{i_1} + w_{i_2} + \dots + w_{i_k})x.$$

Now, set  $w = \min_{1 \leq i \leq n} w_i$ . Then  $\xi_H = -\frac{1}{w_{i_1} + \dots + w_{i_k}} \geq -\frac{1}{\alpha(G)w}$ , and since  $\zeta_H \leq \zeta_G$  by Theorem 2.7, we finally get

$$\alpha(G) \leq -\frac{1}{w\zeta_G}. \quad \square$$

### 3. Weighted Clique Polynomials and Homomorphisms

In this section we will discuss about one of the applications of the weighted clique polynomials for obtaining a no-homomorphism criteria. We first review some basics of graph homomorphism. The reader may consult the reference [5].

**Definition 3.1.** *Let  $G$  and  $H$  be two simple graphs. A homomorphism of  $G$  to  $H$ , written as  $f : G \rightarrow H$  is a mapping  $f : V(G) \rightarrow V(H)$  such that  $f(u)f(v) \in E(H)$  whenever  $uv \in E(G)$ . A homomorphism of  $G$  to  $H$  is also called an  $H$ -coloring of  $G$ . we shall call a homomorphism  $f : G \rightarrow H$  surjective, if the mapping  $f : V(G) \rightarrow V(H)$  is surjective.*

**Definition 3.2.** *Let  $G$  and  $H$  be two simple graphs and  $f : G \rightarrow H$  a homomorphism. We associate a partition  $\theta_f$  with  $f$  consisting of the preimages of  $f$ , i.e., the set  $f^{-1}(x)$ ,  $x \in V(H)$ . Clearly the set  $S_x = f^{-1}(x)$  is an independent set, if there is no loop at vertex  $x \in V(H)$ . Thus, the mapping  $\theta_f$  partitions the vertex set  $V(G)$  into independent sets.*

**Remark 3.3.** *It is not hard to see that every weighted clique polynomial with non-negative integer weights can be viewed as the clique polynomial*



with clusters of vertices. To do this, We need the following definition of blow-up graphs.

**Definition 3.4.** For a given graph  $G = (V, E)$  with the non-negative integer weight function  $\mathbf{w} : V(G) \mapsto \mathbb{R}$  and the vertex set  $V = \{1, 2, \dots, n\}$ , the blow-up graph  $G_b$  of  $G$  is defined as a graph  $G_b = (V_b, E_b)$  such that  $V_b = \{A_1, \dots, A_n\}$  where  $A_i$  is the set of  $w_i$  vertices with no edges among them, that we will call it a cluster of vertices ( $|A_i| = w_i$ ). There is an edge  $A_i A_j \in E_b$  if there is an edge between vertices  $i, j \in V$ .

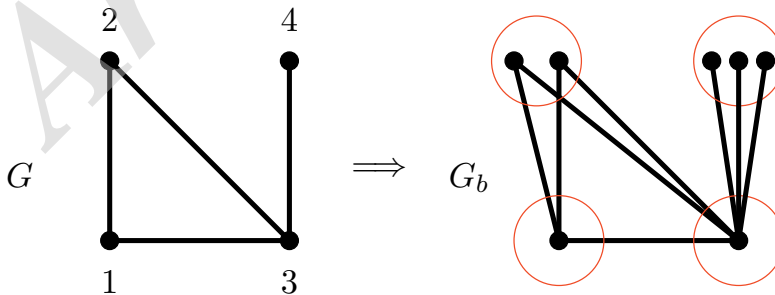
**Remark 3.5.** Note that to obtain a blow-up graph  $G_b$  from a graph  $G = (V, E)$ , we replace each vertex  $i \in V$  with a cluster of vertices of size  $w_i$  (blowing-up process) and then we replace each edge  $e = ij \in E$  in  $G$  with a complete bipartite graph  $K_{w_i, w_j}$  with bipartition  $(A_i, A_j)$  between two clusters of vertices of sizes  $w_i$  and  $w_j$  (see Fig 2). Now, using a simple counting argument based on inclusion - exclusion principle, one can show that

$$C(G, x; \mathbf{w}) = C(G_b, x), \tag{4}$$

provided that the weights are non-negative integers.

**Example 3.6.** In the following picture we depicted a graph  $G$  with the weight vector  $\mathbf{w} = (1, 2, 1, 3)$ , and it's blow-up graph  $G_b$ . Note that  $G_b$  has four clusters of vertices of sizes 1, 2, 1 and 3. It is not hard to see that

$$C(G, x; \mathbf{w}) = 1 + 7x + 8x^2 + 2x^3 = C(G_b, x).$$



**Figure 2.** A graph  $G$  and it's blow-up graph  $G_b$ .

Now we are at position to state the *main* result of this section.

**Theorem 3.7.** *Let  $G$  and  $H$  be two simple graphs and  $f : G \longrightarrow H$  be a surjective homomorphism. Then, we have*

$$\zeta_G \geq \zeta_H.$$

**Proof.** Let  $V(H) = \{1, 2, \dots, m\}$ . Set

$$\mathbf{w} = (|f^{-1}(1)|, |f^{-1}(2)|, \dots, |f^{-1}(m)|).$$

Since  $f : G \longrightarrow H$  is a homomorphism, the partition function  $\theta_f$  partitions the vertex set  $V(G)$  into independent sets  $A_i, i = 1, \dots, m$ , with  $A_i = f^{-1}(i)$ . Now the blow-up graph  $G_b$  of the graph  $G$  with clusters of vertices  $A_i$ 's has the clique polynomial  $C(G_b, x)$ . By surjectivity of  $f$ , its clear that the blow-up graph  $H_b$  of the graph  $H$  is an inducted subgraph of  $G_b$ . Therefore, using Theorem 2.7, we get

$$\zeta_{H_b} \leq \zeta_{G_b},$$

which is equivalent to  $\zeta_H \leq \zeta_G$ , applying the identity (4).  $\square$

**Corollary 2.8.** *Let  $G$  and  $H$  be two simple graphs such that  $\zeta_G < \zeta_H$ . Then, there is no surjective homomorphism from  $G$  to  $H$ .*

## References

- [1] D. C. Fisher, The number of words of length  $n$  in a free "semi abelian" monoid, *Amer. Math. Monthly*, 96 (1989), 610-614.
- [2] D. C. Fisher and A. E. Solow, Dependence polynomials, *Discrete Mathematics*, 82 (1990), 251-258.
- [3] H. Hajiabolhassan and M. L. Mehrabadi, On clique polynomials, *Australasian Journal of Combinatorics*, 18 (1998), 313-316.
- [4] D. B. West, *Intorduction to Graph Theory*, (Second edition) Prentice-Hall, (2001).

- [5] P. Hell and J. Nešetřil, *Graphs and Homomorphism*, Oxford University Press, (2004).

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