On the Epsilon Hypercyclicity of a Pair of Operators

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Abstract. In this paper we prove that if a pair of operators is $\epsilon$-hypercyclic for all $\epsilon > 0$, then it is topologically transitive.

AMS Subject Classification: 47B37; 47B33.
Keywords and Phrases: hypercyclic vector, $\epsilon$-hypercyclicity, transitive operator.

1. Introduction

From now on, let $T_1$, $T_2$ be commutative bounded linear operators on an infinite dimensional Banach space $X$ and consider the pair $T = (T_1, T_2)$.

Definition 1.1. Put $F = \{T_1^mT_2^n : m, n \geq 0\}$. For $x \in X$, the orbit of $x$ under $T$ is the set $\text{Orb}(T, x) = \{Sx : S \in F\}$. The vector $x$ is called a hypercyclic vector for the pair $T$ if $\text{Orb}(T, x)$ is dense in $X$.

Definition 1.2. We say that the pair $T = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets $U$ and $V$ of $X$ there exists $S \in F$ such that $S(U) \cap V \neq \emptyset$.

Definition 1.3. Let $\epsilon \in (0, 1)$ and $x \in X$. If for every non-zero vector $y \in X$, there exist integers $m$, $n$ such that

$$\| T_1^mT_2^n x - y \| < \epsilon \| y \|,$$
then the vector $x$ is called $\epsilon$-hypercyclic for the pair $T = (T_1, T_2)$. A pair of operators is $\epsilon$-hypercyclic if it admits an $\epsilon$-hypercyclic vector.

For some sources on these topics see [1–18].

2. Main Results

In this section we prove that if a pair is $\epsilon$-hypercyclic for all $\epsilon > 0$, then it is topologically transitive and consequently it is hypercyclic. We will extend Theorem 1.4 in [1] for a pair of operators and we will use the idea of it’s proof. We will denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

**Theorem 2.1.** Let $X$ be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. If for every $\epsilon > 0$, $T$ is $\epsilon$-hypercyclic, then $T$ is topologically transitive.

**Proof.** Suppose that $U$ and $V$ are nonempty open subsets of $X$. Let $u \in U$ and $v \in V$ be two nonzero vectors, and consider

$$0 < \delta < \min\{\|u\|, \|v\|\}$$

small enough such that $B(u, \delta) \subset U$ and $B(v, \delta) \subset V$. Choose

$$\epsilon < \delta/(6\max\{\|u\|, \|v\|\}),$$

and let $x \in X$ be an $\epsilon$-hypercyclic vector for $T$. This implies that there exist nonnegative integers $m_0$ and $n_0$ such that

$$\|T_1^{m_0}T_2^{n_0}x - u\| < \epsilon \|u\| < \delta.$$

Hence

$$T_1^{m_0}T_2^{n_0}x \in B(u, \delta) \subset U.$$

We want to show that

$$V \cap \{T_1^mT_2^n x : m, n \in \mathbb{N}_0\}$$
contain infinitely many elements. Suppose on the contrary that it contains only the elements $T_1^{n_1}T_2^{n_2}x$ for $i = 1, ..., k$. As we saw earlier, for each $v' \in B(v, \frac{2\delta}{3})$ there exist integers $m(v')$ and $n(v')$ which satisfies

$$\| T_1^{m(v')}T_2^{n(v')}x - v' \| \leq \epsilon \| v' \| \leq 2\epsilon \| v \| < \frac{\delta}{3}.$$ 

Hence

$$T_1^{m(v')}T_2^{n(v')}x \in \{T_1^{n_1}T_2^{n_2}x : i = 1, ..., k\},$$

because

$$\| T_1^{m(v')}T_2^{n(v')}x - v \| \leq \| T_1^{m(v')}T_2^{n(v')}x - v' \| + \| v' - v \| < \delta.$$ 

Therefore

$$B(v, \frac{2\delta}{3}) \subset \bigcup_{i=1}^{k} B(T_1^{n_1}T_2^{n_2}x, \frac{\delta}{3}),$$

that is a contradiction since $X$ is infinite dimensional. Thus indeed the set

$$V \cap \{T_1^{m}T_2^{n}x : m, n \in \mathbb{N}_0\}$$

contain infinitely many elements and so the set

$$B(v, \delta) \cap \{T_1^{m}T_2^{n}x : m, n \in \mathbb{N}_0\}$$

has infinite elements. In particular, there exist $m, n \in \mathbb{N}_0$ satisfying $m \geq m_0$ and $n \geq n_0$ such that $T_1^{m}T_2^{n}x \in V$. Thus we get

$$T_1^{m-m_0}T_2^{n-n_0}T_1^{m_0}T_2^{n_0}x = T_1^{m-m_0}T_2^{n-n_0}x,$$

which belongs to

$$T_1^{m-m_0}T_2^{n-n_0}(U) \cap V.$$ 

This completes the proof. \hfill \Box

In the proof of the following lemma, we use a method of the proof of Theorem 1.2 in [5] to extend the results for tuples. We will use HC($T$) for the collection of hypercyclic vectors for the pair of operator $T$. 

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Lemma 2.2. Let $X$ be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. Then $T$ is topologically transitive if and only if $\text{HC}(T)$ is dense in $X$.

Proof. Fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of the open balls in $X$ with rational radii, and centers in a countable dense subset of $X$. By the continuity of the operators $T_1$ and $T_2$, the sets

$$G_k = \bigcup \{T_1^{-m}T_2^{-n}(B_k) : m, n \in \mathbb{N}_0\}$$

are open. Clearly $\text{HC}(T)$ is equal to

$$\bigcap \{G_k : k \in \mathbb{N}\}.$$ 

Now let $T$ be topologically transitive and let $W$ be an arbitrary nonempty open set in $X$. Then for all $k \in \mathbb{N}$, there exist $m(k)$ and $n(k)$ in $\mathbb{N}$ such that

$$T_1^{m(k)}T_2^{n(k)}W \cap B_k \neq \emptyset$$

which implies that $W \cap G_k \neq \emptyset$ for all $k$. Thus each $G_k$ is dense in $X$ and so by the Bair Category Theorem $\text{HC}(T)$ is also dense in $X$. Conversely, if $\text{HC}(T)$ is dense in $X$, then each set $G_k$ is so. This implies clearly that $T$ is topologically transitive.

Corollary 2.3. Let $X$ be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. If for every $\epsilon > 0$, $T$ is $\epsilon$-hypercyclic, then $T$ is hypercyclic.

Proof. If for every $\epsilon > 0$, $T$ is $\epsilon$-hypercyclic then by Theorem 2.1, $T$ is topologically transitive. Now, by the Lemma 2.2, $\text{HC}(T)$ is dense in $X$ and this implies clearly that the pair $T$ is hypercyclic. □
References


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