Effect of Polynomial Identity $x[x, y] = (x[x, y])^n$ in the Commutativity of Rings

Z. Tabatabaei
Islamic Azad University-Marvdasht Branch

Abstract. In this paper we study some sufficient conditions for commutativity of a ring according to Jacobson's idea. Jacobson proved that if $R$ is a ring satisfying $x^n = x$ ($n > 1$) for each $x \in R$, then $R$ is commutative. In this paper, we show that $R$ is commutative if for every $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $(x[x, y])^n = x[x, y]$.

AMS Subject Classification: 13PXX, 14A05.
Keywords and Phrases: Commutator, left(right) s-unital, left semisimple ring, Jacobson Radical, left Primitive ring, division ring, faithful simple left R-module.

1. Introduction

Throughout this paper $R$ denotes an associative ring with center $C$ and Jacobson radical $J(R)$.

In 1950, Jacobson proved that if for each $x$ in $R$ there exists a positive integer $n > 1$ such that $x^n = x$, then $R$ is commutative. After that in [1], Searcoid and MacHale proved that if for each $x, y$ in $R$, there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = xy$, then $R$ is commutative. In [3], Hirano and Yaqub studied the rings satisfying $(x - x^n)(y - y^n) = 0$ for every $x, y \in R$ and recently, Bell, Yaqub and Abu-khuzam in [5], [6], [7] have considered some conditions and periodicity conditions for rings to be commutative.

We will fix commutator $[x, y] = xy - yx$ in place of $y$ and obtain commutative results for rings.
A ring $R$ is called left (resp. right) s-unital ([2]) if for each $x \in R$ we have $x \in Rx$ (resp. $x \in xR$). A ring $R$ is called s-unital if for each $x$ in $R$, $x \in xR \cap Rx$.

If $R$ is a s-unital ring, then for any finite subset $K$ of $R$, there exists an element $e$ in $R$ such that $xe = ex = x$ for all $x \in K$ (see [2]). Such an element $e$ will be called a pseudo-identity of $K$.

A ring $R$ is called to be semisimple if its Jocobson radical $J(R)$ is zero.

A ring $R$ is called left primitive if there exists a simple faithful left $R$-module.

A ring $R$ is said to be a subdirect product of the family of rings $\{R_i| i \in I\}$ if $R$ is a subring of the direct product $\prod_{i \in I} R_i$ such that $\pi_k(R) = R_k$ for every $k \in I$, where $\pi_k : \prod_{i \in I} R_i \rightarrow R_k$ is the canonical epimorphism.

## 2. Preliminaries

First of all, we recall some concepts and prove some results that are used in the sequel.

**Theorem 2.1.** A nonzero ring $R$ is semisimple if and only if $R$ is isomorphic to a subdirect product of primitive rings.

**Proof.** See [8, pro. 3.2]. □

**Theorem 2.2.** If $R$ is a ring, then we have

(i) $J(R)$ is the intersection of all the left annihilators of simple left $R$-modules.

(ii) $J(R)$ is the intersection of all the regular maximal left ideals of $R$.

(iii) $J(R)$ is the intersection of all the left primitive ideals of $R$.

(iv) $J(R)$ is a left quasi-regular ideal which contains every left quasi-regular left ideal of $R$.

**Proof.** See [8, Theorem 2.3]. □

We say $R$ satisfies $(\ast)$ if, for each $x, y$ in $R$, there exists a positive integer $n = n(x, y) > 1$ such that $(x[y, x])^n = x[y, x]$. 
Lemma 2.3. Let $R$ and $S$ be rings such that $R$ satisfies $(\ast)$ and $\varphi: R \to S$ is a ring epimorphism. Then $S$ satisfies $(\ast)$.

**Proof.** Let $x, y \in S$. Since $\varphi$ is onto, there exist $s, t \in R$ such that $\varphi(t) = x$, $\varphi(s) = y$. But $R$ satisfies $(\ast)$. Therefore there exists a positive integer $n = n(t, s) > 1$ such that $(t[t, s])^n = t[t, s]$. On the other hand, we have

$$\varphi([t, s]) = \varphi(ts - st),$$

and since $\varphi$ is a ring homomorphism, we have

$$\varphi([t, s]) = \varphi(t)\varphi(s) - \varphi(s)\varphi(t),$$

$$= [\varphi(t), \varphi(s)],$$

$$= [x, y].$$

Thus

$$x[x, y] = \varphi(t)[\varphi(t), \varphi(s)],$$

$$= \varphi(t[t, s]),$$

$$= \varphi((t[t, s])^n),$$

$$= (\varphi(t)[\varphi(t), \varphi(s)])^n,$$

$$= (x[x, y])^n.$$ The result follows. \(\square\)

Lemma 2.4. If $R$ is a ring, then the quotient ring $\frac{R}{J(R)}$ is semisimple.

**Proof.** See [8. Theorem 2.14.]. \(\square\)

Lemma 2.5. Let $b \in R$ and $a \in J(R)$ such that $ab = b$. Then $b = 0$.

**Proof.** Let $a \in J(R)$. By theorem 2.2, there exists $r \in R$ such that $r + a - ra = 0$ and therefore

$$0 = 0b = (r + a - ra)b = rb + ab - rab = b. \square$$
Theorem 2.6. Let $K$ be a division ring. If for any $x, y \in K$ there exists a positive integer $n = n(x, y) > 1$ such that $[x, y]^n = [x, y]$, then $K$ is commutative.

Proof. see [9. Theorem 12.10]. □

Theorem 2.7. Let $K$ be a division ring which satisfies $(\ast)$. Then $K$ is commutative.

Proof. Let $x, y$ be arbitrary elements in $K$. If $x = 0$, then $[x, y] = 0$ and $xy = yx$. If $x \neq 0$, we put $z = x^{-1}y$. Hence by $(\ast)$, there exists a positive integer $n = n(x, z) > 1$ such that

$$(x[x, z])^n = x[x, z],$$

$$([x, xz])^n = [x, xz],$$

$$[x, y]^n = [x, y].$$

Hence for every $x, y \in K$, there exists a positive integer $n = n(x, y) > 1$ such that $[x, y]^n = [x, y]$. Thus, by theorem 2.6, $K$ is commutative. □

Theorem 2.8. Let $R$ be a left primitive ring which satisfies $(\ast)$. Then $R$ is commutative.

Proof. Let $R$ be a left primitive ring. By Structure Theorem for Left primitive ring ([9]), there exists a division ring $K = \text{End}_R(V)$ ( $V$ is a faithful simple left $R$-module ) such that we have one of the following statements:

i) There exists a positive integer $m$ such that $R \cong M_m(K)$.

ii) For any integer $m > 1$, there exists a subring $R_m$ of $R$ which admits a ring homomorphism onto $M_m(K)$.

But for any $m \geq 2$, the division ring $M_m(K)$ doesn’t satisfy $(\ast)$. For example, if $x = E_{11}$, $y = E_{12}$, then $x[x, y] = y$ and therefore $(x[x, y])^n = y^n = 0$. Thus $m = 1$ and $R \cong K$. By theorem 2.6, $R$ is commutative. □
EFFECT OF POLYNOMIAL IDENTITY $x[x, y] = (x[x, y])^n$...

**Theorem 2.9.** Let $R$ be a semisimple ring which satisfies $(*)$. Then $R$ is commutative.

**Proof.** By theorem 2.1, $R$ is isomorphic to a subdirect product of primitive rings. By Lemma 2.3, every $R_i$ satisfies $(*)$. Therefore, by theorem 2.8, each $R_i$ is commutative and so $R$ is commutative. □

### 3. Main Results

**Theorem 3.1.** Let $R$ be a ring which satisfies $(*)$. Then $x[x, y] = 0$, for every $x, y \in R$.

**Proof.** The semisimple ring $R = \frac{R}{J(R)}$ satisfies $(*)$ and so by theorem 2.9, $R$ is commutative. Therefore, for each $x, y$ in $R$, $[x, y] \in J(R)$ and there exist a positive integer $n = n(x, y) > 1$ such that

$$([x, xy])^n = (x[x, y])^n = x[x, y] = [x, xy],$$

and so

$$[x, xy]^{n-1}[x, xy] = [x, xy]$$

by lemma 2.5., $x[x, y] = 0$. □

**Corollary 1.** Let $R$ be a left s-unital ring satisfies $(*)$. Then $R$ is commutative.

**Proof.** Since $R$ is left s-unital, so for every $x$ in $R$, there exists $e \in R$ that $x = ex$. Now we show that $R$ is right s-unital.

If $x \neq xe$, then by $(*)$ there exists a positive integer $n > 1$ such that

$$e[e, xe - x] = (e[e, xe - x])^n.$$ But we have

$$(e[e, xe - x])^2 = (xe - x - xe^2 + xe)^2 = 0.$$ Therefore,

$$e[e, xe - x] = 0$$
and so
\[ xe - x - xe^2 + xe = 0. \]
and hence, \( x = x(2e - e^2). \) If \( e = 2e - e^2, \) for every \( x \in R, \) there exists \( e \in R \) such that \( x = xe \) and thus \( R \) is s-unital.

For every \( x, y \) in \( R, \) there exists \( e \in R \) such that \( xe = ex = x \) and \( ye = ey = y \) (see[2]) and, by theorem 3.1, we have
\[
0 = [x + e, (x + e)y],
= [x + e, xy + y],
= [x, xy + y] + [e, xy + y],
\]

since \([e, xy + y] = 0, so\)
\[
0 = [x, xy + y],
= [x, xy] + [x, y].
\]

By theorem 3.1., \([x, xy] = 0, thus [x, y] = 0. Therefore, R is commutative. □\]

**Corollary 2.** In corollary 1, being s-unital is necessary, for example the following noncommutative ring isn’t s-unital:

\[
A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \text{ are any real numbers} \right\}.
\]

But for every \( x, y, z \) in \( A, xyz = 0 \) and so \( A \) satisfies (*), and \( A \) is noncommutative ring.

**References**


EFFECT OF POLYNOMIAL IDENTITY $x[x, y] = (x[x, y])^n$

Archive of SID


Zohre Tabatabaei
Department of Mathematics
Islamic Azad University-Marvdasht Branch
Shiraz, Iran.
E-mail: Parivash.tabatabae@yahoo.com

www.SID.ir