Third Term of the Lower Autocentral Series of Abelian Groups

M. Naghshineh
Islamic Azad University, Jahrom-Branch

M. R. R. Moghaddam
Islamic Azad University, Mashhad-Branch

F. Parvaneh
Islamic Azad University, Kermanshah-Branch

Abstract. Let $G$ be a group and $Aut(G)$ be the group of automorphisms of $G$. Then $[g, \alpha, \beta] = (g^{-1}g^{\alpha})^{-1}(g^{-1}g^{\alpha})^{\beta}$ is the autocommutator of the element $g \in G$ and $\alpha, \beta \in Aut(G)$ of weight 3. Also, we define $K_2(G) = \langle [g, \alpha, \beta] : g \in G, \alpha, \beta \in Aut(G) \rangle$ to be the third term of the lower autocentral series of subgroups of $G$. In this paper, it is shown that every finite abelian group is isomorphic to the third term of the autocentral series of some finite abelian group.

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1. Introduction

For every element $g$ of a group $G$ and any automorphism $\alpha \in Aut(G)$,

$$[g, \alpha] = g^{-1}g^{\alpha}$$

is the autocommutator of the element $g$ and the automorphism $\alpha$. Clearly, if $\alpha$ is taken to be an inner automorphism then we obtain the commutator of two elements of $G$. Now the autocommutator subgroup of $G$ is
defined as follows:
\[ K_1(G) = \langle [g, \alpha] : g \in G, \alpha \in \text{Aut}(G) \rangle, \]
which is a characteristic subgroup of \( G \).

There are some results on the autocommutator subgroup of a finite group \( G \) (see [2, 3, 4]). Recently, C. Chis, M. Chis and G. Silberberg ([1]) in 2008 showed that, every finite abelian group is the autocommutator subgroup of some finite abelian group. In the present paper, it is shown that a similar result holds for the autocommutator of weight three.

The following definition is vital in our investigation.

**Definition 1.1.** Let \( g \) be an element of a given group \( G \) and \( \alpha, \beta \in \text{Aut}(G) \). Then we define
\[ [g, \alpha, \beta] = [g, \alpha]^{-1}[g, \alpha]^\beta \]
to be the autocommutator of the element \( g \) and the automorphisms \( \alpha \) and \( \beta \) of weight 3. Clearly, when \( \alpha \) and \( \beta \) are taken to be inner automorphisms of \( G \), one obtains the usual commutator of weight 3.

Now we define the third term of the lower autocentral series of subgroups as follows:
\[ K_2(G) = \langle [g, \alpha, \beta] : g \in G, \alpha, \beta \in \text{Aut}(G) \rangle. \]

Note that one may define the lower autocentral series of subgroups of a given group, which are all characteristic subgroups. The main objective of this paper is to prove the following result.

**Main Theorem.** Every finite abelian group is the third term of the lower autocentral series of some finite abelian group.

**2. Preparatory Results**

To prove our main theorem we first establish some preparatory results. All groups, which are considered in this paper are finite.
Lemma 2.1. Let $A$ and $B$ be characteristic subgroups of a given group $G$ such that $G = A \times B$. Then $K_2(G) = K_2(A) \times K_2(B)$.

Proof. Let $\varphi : Aut(G) \rightarrow Aut(A) \times Aut(B)$, given by $\alpha \mapsto (\alpha|_A, \alpha|_B)$ and $\theta : Aut(A) \times Aut(B) \rightarrow Aut(G)$, given by $(\mu, \eta) \mapsto \bar{\mu} \bar{\eta}$, where $\bar{\mu} \bar{\eta}$ is an automorphism of $G$, defined as follows:

\[
\bar{\mu}(ab) = a\mu b,
\]

\[
\bar{\eta}(ab) = ab^n,
\]

for all $a \in A$ and $b \in B$. Since $A$ and $B$ are characteristic subgroups, it follows that $\varphi$ and $\theta$ are inverse isomorphisms and hence we may identify $Aut(G)$ with $Aut(A) \times Aut(B)$.

Now, for any $g = ab = ba \in G = A \times B$ and $\alpha, \beta \in Aut(G)$ we have

\[
[g, \alpha, \beta] = [ab, \alpha, \beta] = [ab, \alpha]^{-1}[ab, \alpha]^{\beta}
\]

\[
= (ab)^{-\alpha}(ab)(ab)^{-\beta}(ab)^{\alpha \beta}
\]

\[
= b^{-\alpha}a^{-\alpha}ab.a^{-\beta}b^{-\beta}.a^{\alpha \beta}b^{\alpha \beta}
\]

\[
= a^{-\alpha}a^{-\beta}a^{\alpha \beta}b^{-\alpha}b^{-\beta}b^{\alpha \beta}
\]

\[
= [a, \alpha|_A, \beta|_B][b, \alpha|_A, \beta|_B].
\]

Thus $K_2(G) \subseteq K_2(A) \times K_2(B)$.

Now, for any $a \in A$ and $\mu, \mu' \in Aut(A)$,

\[
[a, \mu, \mu'] = [a, \bar{\mu}, \bar{\mu}'] \in K_2(G).
\]

Hence $K_2(A)$ is contained in $K_2(G)$. Similarly $K_2(B) \subseteq K_2(G)$ and so $K_2(G) = K_2(A) \times K_2(B)$. □

Lemma 2.2. Let $G$ be a finite cyclic group. Then $K_2(G) = G^4$.

Proof. Let $G = \langle x : x^n = 1 \rangle$ be the cyclic group of order $n$. Clearly $\varphi$ is an automorphism of $G$ if and only if

\[
\varphi : x \mapsto x^i, \quad 1 \leq i \leq n,
\]

where $(i, n) = 1$, (see [5]). Assume $n$ is an odd number, then since the group $G$ is abelian, the map $\alpha$ given by $x \mapsto x^{-1}$ is an automorphism of $G$. Hence, for all $g \in G$,

\[
g^4 = [g, \alpha, \alpha] \in K_2(G).
\]
Thus $G^4 \subseteq K_2(G)$. By the assumption $(4, n) = 1$, then there exist some integers $s, r \in \mathbb{Z}$ such that $4s + nr = 1$. Thus $g = g^{4s+nr} = (g^s)^4 \in G^4$. This shows that $G$ and hence $K_2(G)$ is contained in $G^4$. Therefore in this case, $K_2(G) = G^4$.

Now, we assume $n$ is even. Hence for a non-trivial automorphism of $G$ given by $x \mapsto x^i$, the integer $i$ must be odd and greater than 2. Therefore for any $\alpha, \beta \in \text{Aut}(G)$, with $\alpha(x) = x^i$ and $\beta(x) = x^j$,

$$[x, \alpha, \beta] = [x^{i-1}, \beta] = x^{1-i}x^{(i-1)j} = x^{(i-1)(j-1)} \in G^4.$$

Thus $K_2(G) = G^4$. \[\square\]

**Lemma 2.3.** Let $G$ be an abelian group of odd order and $Z_2$ the cyclic group of order 2. Then $K_2(G)$ and $K_2(G \times Z_2)$ are both isomorphic with $G$.

**Proof.** Since $G$ is abelian of odd order, say, we conclude that $(4, n) = 1$. Lemma 2.2. implies that $K_2(G) = G$. On the other hand, $Z_2$ and $G$ are characteristic subgroups in $G \times Z_2$. Thus Lemma 2.1. gives the assertion. \[\square\]

**Lemma 2.4.** Let $G$ be a cyclic group of order $2^m$ and $H$ an abelian $2$-group of exponent $2^n$, with $n < m$. Then $K_2(G \times H) = G^4 \times H^2$.

**Proof.** Let $G = \langle x : x^{2^m} = 1 \rangle$, then for every element $h \in H$, we may define a unique automorphism $\alpha_h \in \text{Aut}(G \times H)$ in the following way:

$$\alpha_h : (x, h_1) \longrightarrow (x, h^{-1}h_1^{-1}).$$

Thus $h^2 = [x, \alpha_h, \alpha_{e_H}] \in K_2(G \times H)$, and so

$$H^2 \subseteq K_2(G \times H).$$

Using Lemma 2.2. we have $K_2(G) = G^4$. Hence

$$G^4 = K_2(G) \subseteq K_2(G \times H).$$

So $G^4 \times H^2 \subseteq K_2(G \times H)$. On the other hand, for all $\alpha, \beta \in \text{Aut}(G \times H)$ and noting the structures of the groups $G$ and $H$ we have

$$[g, \alpha, \beta] \in G^4 \times H^2, \text{ for all } g \in G \times H.$$
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Hence the result holds. □

Using the above lemma, we obtain the following.

Proposition 2.5. For natural numbers \( m > n_1 \geq n_2 \geq \ldots \geq n_r \), such that \( m \geq 2 \), we have

\[
K_2(Z_{2^m} \times Z_{2^{n_1}} \times \ldots \times Z_{2^{n_r}}) = Z_{2^{m-2}} \times Z_{2^{n_1-1}} \times \ldots \times Z_{2^{n_r-1}}.
\]

3. Proof of the Main Theorem

Let \( G \) be a finite abelian group. Then \( G \) can be written as a direct product of its Sylow \( p \)-subgroups.

If \( (4, |G|) = 1 \), then \( G \) is of odd order and has no Sylow 2-subgroups. Since \( G \) is abelian, all of its Sylow \( p \)-subgroups are characteristic and of odd order, which by Lemma 2.3. implies that \( K_2(G) = G \). Otherwise, \( G \) has a Sylow 2-subgroup, say \( P \), and it can be written as follows:

\[
P = Z_{2^m} \times Z_{2^{n_1}} \times \ldots \times Z_{2^{n_r}}.
\]

Now, we construct the abelian group

\[
H = Z_{2^{m+2}} \times Z_{2^{n_1+1}} \times \ldots \times Z_{2^{n_r+1}} \times P_1 \times \ldots \times P_k,
\]

where \( P_i \)'s are all Sylow \( p_i \)-subgroups of \( G \), except \( P \). Using Proposition 2.5, we obtain

\[
K_2(H) \cong G,
\]

which proves the theorem. □

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References


Mohammad Naghshineh
Department of Mathematics
Islamic Azad University, Jahrom-Branch
Jahrom, Iran.
E-mail: naghshineh@jia.ac.ir

Mohammad Reza R. Moghaddam
Department of Mathematics
Islamic Azad University, Mashhad-Branch
Mashhad, Iran.
E-mail: mrrm5@yahoo.ca

Foroud Parvaneh
Department of Mathematics
Islamic Azad University, Kermanshah-Branch
Kermanshah, Iran.
E-mail: foroudparvane@kiau.ac.ir