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Abstract

In this paper, we introduce a new technique in semi-analytic method for solving non-linear differential equations. At first, a non-linear differential equation will be converted to a linear differential equation (or some of them) by using homotopy method. Then, for finding parameter $\alpha$ in the linear differential equation we use Galerkin method, which is easier and less computations in comparison with similar methods, such as [5, 7]. Then, we will solve linear differential equation by using analytic method where it's evident that analytic method is better than the variational iteration method in [1, 3]. Also in [7] perturbation techniques are depend on small parameter but the above mentioned method is not depend on small parameter. The obtained results are shown the high accuracy.

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1 Introduction

Non-Linear differential equations are a class of important models in applied science and engineering. Analytic methods applied by engineers are perturbation techniques

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for solving non-linear equations. But perturbation methods depend on small parameter (see [7]) and choose unsuitable small parameter can be lead to wrong solution. Homotopy is an important part of topology [6], and it can transform any non-linear problem into a finite linear problems and it doesn’t depend on small parameter. For illustrating homotopy perturbation method corresponding with [1, 2], we have a general non-linear differential equation with boundary conditions

\[ A(u) - f(t) = 0, \quad t \in \Omega \]  

\[ B(u, \partial u/\partial n) = 0, \quad n \in \Gamma \]

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(t) \) is a known analytic function, \( \Gamma \) is the boundary of the domain \( \Omega \). The operator \( A \) can be divided into two operators \( L \) and \( N \), where \( L \) and \( N \) are linear and non-linear operators sequentially. So, we can write equation 1 as the following form:

\[ L(u) + N(u) - f(t) = 0. \]  

(2)

Homotopy perturbation \( H(\nu, p) \) may be proposed as follows:

\[ H : \Omega \times [0, 1] \to R, \]

\[ H(\nu, p) = L(\nu) - L(u_0) + pL(u_0) + p [N(\nu) - f(t)] = 0, \]  

(3)

where \( p \in [0, 1], t \in \Omega, \nu \) is an approximation of \( u \), \( u_0 \) is an initial approximation of \( u \) and \( p \) is an embedding parameter.

2 Application of Homotopy perturbation method

In this section, we convert a non-linear differential equation to linear differential equation by using homotopy perturbation method, Eq. (3). So we consider a non-linear
differential equation that is solved in [1, 4, 7],

\[
\frac{d^2u}{dt^2} + w^2u + (4q^2u^2)\frac{d^2u}{dt^2} + 4q^2u\left(\frac{du}{dt}\right)^2 = 0, \quad t \in \Omega
\]  

(4)

\[
u(0) = A,
\]

\[
u'(0) = 0,
\]

where \(w\) and \(q\) are known constants. By considering Eq.(2) we have

\[
L(u) = u'' + w^2u,
\]

(5)

\[
N(u) = u^2u'' + u(u')^2.
\]

Also, by using homotopy perturbation Eq.(3) and Eq.(5), we have

\[
H(\nu, p) = (\nu'' + w^2\nu) - (u_0^2 + w^2u_0) + p(u_0'' + w^2u_0) + p\left[4q^2\left(\nu'' + \nu(u')^2\right)\right] = 0.
\]

(6)

Assuming the approximate solution of Eq.(6) has the following form:

\[
\nu = \nu_0 + p\nu_1 + p^2\nu_2 + ...;
\]

(7)

\[
u = \lim(p \rightarrow 1)(\nu_0 + p\nu_1 + p^2\nu_2 + ...), \text{when } p \rightarrow 1.
\]

By substituting Eq.(7) into Eq.(6), we have

\[
(L(\nu_0) - L(u_0)) + p\left(L(\nu_1) + L(u_0) + 4q^2\nu_0^2\nu_0'' + 4q^2\nu_0\nu_0'(\nu_0')^2\right) + ... = 0.
\]

(8)

Now, let the coefficients of Eq.(8) equal by zero, we obtain

\[
L(\nu_0) = L(u_0),
\]

(9)

\[
L(\nu_1) + L(u_0) + 4q^2\nu_0^2\nu_0'' + 4q^2\nu_0\nu_0'(\nu_0')^2 = 0.
\]

(10)
From initial conditions of problem (4), and assuming \( u_0 \) as an initial approximation of \( u \) as follows:

\[
u_0(t) = a \cos \alpha wt + b \sin \alpha wt,
\]

\[
A = u_0(0) = a,
\]

\[
u_0'(0) = -\alpha wb_0 = 0.
\]

From the above relations, \( a = A \) and \( b = 0 \) may be obtained, also \( u_0(t) \) can be written the following

\[
u_0(t) = A \cos \alpha wt, \tag{11}
\]

by substituting Eq. (11) into Eq. (9 – 10), we have

\[
L(\nu_1) = (\alpha^2 + 2q^2A^2\alpha^2 - 1)w^2A\cos \alpha wt + 2q^2\alpha^2w^2A^3\cos 3\alpha wt. \tag{12}
\]

Also by considering Eq. (7) and initial conditions of problem (4), we have

\[
A = u(0) = \nu_0(0) + p\nu_1(0) + p^2\nu_2(0) + ..., \\
0 = \nu'(0) = \nu'(0) + p\nu'_1(0) + p^2\nu'_2(0) + ....
\]

So

\[
\nu_0(0) = A, \nu_0'(0) = 0 and \nu_i(0) = 0, \nu_i'(0) = 0, \forall i \geq 1. \tag{13}
\]

From Eqs. (12 – 14), we obtain a linear differential equation as follows:

\[
\nu''_1 + wn_1 = (\alpha^2 + 2q^2A^2\alpha^2 - 1)w^2A\cos \alpha wt + 2q^2\alpha^2w^2A^3\cos 3\alpha wt, \tag{13}
\]

\[
\nu_1(0) = 0, \nu'_1(0) = 0.
\]
In [1], problem (15) is solved by using variational iteration method. But for finding $\alpha$ and solution of problem (15) we use Galerkin method and analytic solution sequentially. From analytic solution we achieve an exact solution but variational iteration method prepares an approximate solution, so in this paper we employ analytic solution. Also we can obtain $\alpha$ by using Galerkin method with only two elements of basis.

3 Galerkin Method

In this section, we use Galerkin method for solving problem (15), thus suppose 
\[ \{ \phi_i(t) = \cos i\alpha t \}_{i=1}^{n} \] are basis elements, such that $\nu_1(t)$ is linear combination of them. In the other words:

\[ \nu_1(t) = \sum_{i=1}^{n} a_i \phi_i(t), \]  

also from initial conditions of problem (15), we have

\[ \sum_{i=1}^{n} a_i = 0. \]  

According to $L(\nu_1) = \nu''_1 + w^2 \nu_1$ and Galerkin method, we can write

\[ L \left( \sum_{i=1}^{n} a_i \phi_i(t) \right) = f(t), \]

where $f(t)$ is right hand side of problem (15). Since $L$ is linear operator, so

\[ \sum_{i=1}^{n} a_i L(\phi_i(t)) = f(t). \]  

By multiplying both sides of Eq.(18) by basis elements, so

\[ \sum_{i=1}^{n} a_i < L(\phi_i(t)), \phi_j(t) >= < f(t), \phi_j(t) >, j = 1, 2, 3, \ldots n. \]
where

\[
< L(\phi_i(t)), \phi_j(t) > = \int_0^{\pi} L(\phi_i(t))\phi_j(t)dt = \int_0^{\pi} \phi''_i(t) + w^2\phi_i(t)\phi_j(t)dt.
\]

Because orthonormality of basis elements, so coefficient matrix is diagonal and unknowns are obtained by the following form:

\[
a_i = \frac{< L(\phi_i(t)), \phi_i(t) >}{< f(t), \phi_i(t) >}, \\
i = 1, 2, ..., n.
\]

(20)

From Eq.(20) and Eq.(17), for \(n = 2\), we conclude

\[
a_1 = -\frac{A}{\alpha^2 - 1} \left[ -1 + \alpha^2 + 2q^2A^2\alpha^2 \right], \\
a_2 = 0, \\
a_1 + a_2 = 0.
\]

So

\[
\alpha = \frac{1}{\sqrt{1 + 2q^2A^2}}.
\]

(21)

Now, by considering Eq.(21), we solve problem(15) by using analytic method.

4 Analytic method

We use characteristic equation of homogeneous linear differential equation according to problem(15), as follows:

\[
m^2 + w^2 = 0 \rightarrow m = \pm iw, \\
\bar{v}_1(t) = C\cos wt + D\sin wt.
\]

(22)
Also, private solution of problem (15) is

\[ \nu_1^*(t) = M \cos \omega t + N \cos 3\omega t. \]  

(23)

By substituting Eq.(23) into problem (15), we have

\[ M = \frac{A \left[ -1 + \alpha^2 + 2q^2 A^2 \alpha^2 \right]}{1 - \alpha^2}, \quad N = \frac{2q^2 \alpha^2 A^3}{1 - 9\alpha^2}. \]  

(24)

From Eqs.(22 - 24) the following results derived

\[ \nu_1(t) = \bar{\nu}_1(t) + \nu_1^*(t) = C \cos \omega t + D \sin \omega t + \] 

\[ \frac{A \left[ -1 + \alpha^2 + 2q^2 A^2 \alpha^2 \right]}{1 - \alpha^2} \cos \omega t + \frac{2q^2 \alpha^2 A^3}{1 - 9\alpha^2} \cos 3\omega t. \]  

(25)

We obtain C and D in Eq.(25), by using initial conditions of problem (15),

\[ \nu_1(0) = 0 \rightarrow C = -\frac{A \left[ -1 + \alpha^2 + 2q^2 A^2 \alpha^2 \right]}{(1 - \alpha^2)} - \frac{2q^2 \alpha^2 A^3}{1 - 9\alpha^2}, \] 

\[ \nu_1'(0) = 0 \rightarrow D = 0. \]

So

\[ \nu_1(t) = \left( -\frac{A \left[ -1 + \alpha^2 + 2q^2 A^2 \alpha^2 \right]}{1 - \alpha^2} - \frac{2q^2 \alpha^2 A^3}{1 - 9\alpha^2} \right) \cos \omega t \] 

\[ + \frac{\left[ -1 + \alpha^2 + 2q^2 A^2 \alpha^2 \right]}{(1 - \alpha^2)} \cos \omega t + \frac{2q^2 \alpha^2 A^3}{1 - 9\alpha^2} \cos 3\omega t. \]  

(26)

By considering equations(7, 12, 21, 26), solution of problem (4) is given by

\[ u(t) \equiv \nu_0(t) + \nu_1(t) = A \cos \omega t + \frac{2q^2 \alpha^2 A^3}{1 - 9\alpha^2} \left[ \cos 3\omega t - \cos \omega t \right]. \]  

(27)
5 Discussion on parameter $\alpha$

Problem (4) has a periodic solution such that the exact period $T$, was given in [7] by

$$\frac{T}{T_0} = \left(\frac{2}{\pi}\right) \int_0^{\frac{\pi}{2}} \sqrt{1 + 4q^2A^2\cos^2\phi} \, d\phi,$$

(28)

where $T_0 = \frac{2\pi}{w}$ and $w$ is known constant.

In [7], by using perturbation techniques (first and second order) approximation of $\frac{T}{T_0}$ was given as follows:

- first order: $\frac{T_1}{T_0} = 1 + q^2A^2$,
  \hspace{1cm} (29)

- second order: $\frac{T_2}{T_0} = 1 + q^2A^2 - q^4A^4$,
  \hspace{1cm} (30)

the perturbation formulas Eq.(29) and Eq.(30), are valid only in the case $(qA)^2 \leq 0.1$, but Eq.(21) in the above mentioned method is valid for any values of $(qA)^2$, (see Figure 1),

where values of $(qA)^2$ is shown on horizontal axis and $\frac{T_{\text{exact}}}{T_{\text{approximate}}}$ is shown on vertical axis.

Also, even in the case of $(qA)^2$ tends to infinity, we have

$$\text{Limit}_{|qA| \to \infty} \frac{T_{\text{exact}}}{T_{\text{Eq.(21)}}} = \text{limit} \frac{\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 + 4q^2A^2\cos^2\phi} \, d\phi}{\sqrt{1 + 2q^2A^2}} \approx 0.90,$$

(31)
and, for any value of $(qA)^2$, relative error is the following form:

$$0 \leq \frac{T_{\text{exact}} - T_{\text{Eq.(21)}}}{T_{\text{Eq.(21)}}} \leq 0.1.$$ (32)

But, even for $(qA)^2 = 1$, the perturbation formulas Eq.(29) and Eq.(30) have a relative error about 0.16 and 0.67, respectively. Of course in[1, 4] results are equal to our method but with more complexity computations.

6 Conclusion

In this paper, we introduced a method for solving non-linear differential equations, which is a combination of homotopy, projection and analytic methods. We get a formula for approximation of $\alpha$ by using projection method Eq.(21), it is valid for any values of $(qA)^2$ even in the case of $(qA)^2$ tend to infinity. But the perturbation formulas in [7] are valid only in the case $(qA)^2 \leq 0.1$, also they have relative error greater than our method.

References


