Periodicity of the Clifford Algebras

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Abstract

In this paper we study the structure of Clifford Algebras $Cl_{p,q}$ associated with a non degenerate symmetric bilinear form of signature $(p, q)$, where $p, q$ are positive integer. Also we present a description of these algebras as matrix algebras, and then we will discuss the periodicity of these algebras completely. As a consequence, we create the related algebra matrix tables for these algebras, when $0 \leq p \leq 8$ and $8 \leq q \leq 13$. We also present an isomorphism between $Cl_{0,q,p}^0$ and $Cl_{0,p,q}^0$.

Keywords: Tensor algebra, Exterior algebra, Clifford algebra, Quadratic form, Bilinear form.

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1 Introduction

Given any vector space, $V$, over a field, $K$, there is a special $K$-algebra, $T(V)$, together with a linear map, $i : V \rightarrow T(V)$, following the universal mapping property [1]. The algebra, $T(V)$, is the tensor algebra of $V$. It may be constructed as the direct sum $T(V) = \bigoplus_{i \geq 0} V^\otimes i$, Where $V^0 = K$, and $V^\otimes i$ is the $i$-fold tensor product of $V$ with itself.

For every $i \geq 0$, there is a natural injection $\iota_n : V^\otimes n \rightarrow T(V)$ and in particular, an injection $\iota_0 : K \rightarrow T(V)$. The multiplicative unit, $1$, of $T(V)$ is the image, $\iota_0(1)$, in $T(V)$ of the unit, $1$, of the field $K$. Since every $v \in T(V)$ can be expressed as a finite

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sum \( v = v_1 + v_2 + \ldots + v_k \), where \( v_i \in V^{\otimes n_i} \) and \( n_i \) the are natural numbers with \( n_i \neq n_j \) if \( i \neq j \), to define multiplication in \( T(V) \), using bilinearity \([1]\), it is enough to define the multiplication \( V^{\otimes m} \times V^{\otimes n} \rightarrow V^{\otimes (m+n)} \). Of course, this is defined by:

\[
(v_1 \otimes \ldots \otimes v_m) \cdot (w_1 \otimes \ldots \otimes w_n) = v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_n.
\]

It is important to note that multiplication in \( T(V) \) is not commutative. Also, the unit, \( 1 \), of \( T(V) \) is not equal to 1, the unit of the field \( K \). However, in view of the injection \( \iota_0 : K \rightarrow T(V) \), for the sake of notational simplicity, we will denote 1 by 1.

More generally, in view of the injections \( \iota_n : V^{\otimes n} \rightarrow T(V) \), we identify elements of \( V^{\otimes n} \) with their images in \( T(V) \).

Most algebras of interest arise as well-chosen quotients of the tensor algebra \( T(V) \). This is true for the exterior algebra, \( \Lambda^* V \) (also called Grassmann algebra), where we take the quotient of \( T(V) \) modulo the ideal generated by all elements of the form \( v \otimes v \), where \( v \in V \), and for the symmetric algebra, \( \text{Sym} V \), where we take the quotient of \( T(V) \) modulo the ideal generated by all elements of the form \( v \otimes w - w \otimes v \), where \( v, w \in V \). A Clifford algebra may be viewed as a refinement of the exterior algebra, in which we take the quotient of \( T(V) \) modulo the ideal generated by all elements of the form \( v \otimes v - \Phi(v) \cdot 1 \), where \( \Phi \) is the quadratic form associated with a symmetric bilinear form, \( \varphi : V \times V \rightarrow K \), and \( \cdot : K \times T(V) \rightarrow T(V) \) denotes the scalar product of the algebra \( T(V) \). For simplicity, let us assume that we are now dealing with real algebras.

2 Preliminaries

Definition 2.1 Let \( V \) be a real finite-dimensional vector space. A quadratic form on \( V \) is a mapping \( \Phi : V \rightarrow \mathbb{R} \) such that

1. \( \Phi(\lambda v) = \lambda^2 \Phi(v) \) for all \( \lambda \in \mathbb{R} \), \( v \in V \).
2. the mapping \((x, y) \rightarrow (\Phi(x + y) - \Phi(x) - \Phi(y)) = \varphi(x, y)\) of \(V \times V\) into \(\mathbb{R}\) is bilinear.

Then \(\varphi\) is called the bilinear form associated to \(\Phi\).

It is obvious from the definition that \(\varphi\) is symmetric:

\[\varphi(x, y) = \varphi(y, x)\]

and \(\varphi(x, x) = \Phi(x)\).

Two elements \(x, y\) of \(V\) such that \(\varphi(x, y) = 0\) are said to be orthogonal to each other.

**Definition 2.2** Let \(V\) be a real finite-dimensional vector space together with a symmetric bilinear form \(\varphi : V \times V \rightarrow \mathbb{R}\), and associated quadratic form, \(\Phi(x) = \varphi(x, x)\). A Clifford algebra associated with \(V\) and \(\Phi\) is a real algebra, \(Cl(V, \Phi)\), together with a linear map, \(i : V \rightarrow Cl(V, \Phi)\) satisfying the condition \((i(v))^2 = \Phi(v)\) for all \(v \in V\) and so that for every real algebra, \(A\), and every linear map, \(f : V \rightarrow A\), with

\[(f(v))^2 = \Phi(v)\] for all \(v \in V\),

there is a unique algebra homomorphism, \(\bar{f} : Cl(V, \Phi) \rightarrow A\) so that

\[f = \bar{f}oi,\]

as in the diagram below:

\[
\begin{array}{ccc}
V & \xrightarrow{i} & Cl(V, \Phi) \\
\downarrow{f} & & \downarrow{\bar{f}} \\
A
\end{array}
\]

We use the notation, \(\lambda u\), for the product of a scalar, \(\lambda \in \mathbb{R}\) and of an element, \(u\), in the algebra \(Cl(V, \Phi)\) and juxtaposition, \(uv\), for the multiplication of two elements, \(u, v \in Cl(V, \Phi)\).
By a familiar argument, any two Clifford algebras associated with $V$ and $\Phi$ are isomorphic.

To show the existence of $Cl(V, \Phi)$, observe that $T(V)/U$ does the job, where $U$ is the ideal of $T(V)$ generated by all elements of the form $v \otimes v - \Phi(v).1$, where $v \in V$.

The map $i : V \to Cl(V, \Phi)$ is the composition

$$V \xrightarrow{i_1} T(V) \xrightarrow{\pi} \frac{T(V)}{U}$$

where $\pi$ is the natural quotient map. We often denote the Clifford algebra $Cl(V, \Phi)$ simply by $Cl(\Phi)$.

Observe that when $\Phi \equiv 0$ is the quadratic form identically zero everywhere, then the Clifford algebra $Cl(V, 0)$ is just the exterior algebra, $\Lambda^*V$.

**Remark:** As in the case of the tensor algebra, the unit of the algebra $Cl(\Phi)$ and the unit of the field $\mathbb{R}$ are not equal.

Since

$$\Phi(u + v) - \Phi(u) - \Phi(v) = 2\varphi(u, v)$$

and

$$(i(u + v))^2 = i(u)^2 + i(v)^2 + i(u)i(v) + i(v)i(u),$$

using the fact that

$$(i(u))^2 = \Phi(u).1,$$

We get:

$$i(u)i(v) + i(v)i(u) = 2\varphi(u, v).1.$$

As a consequence, if $(u_1, \ldots, u_n)$ is an orthogonal basis w.r.t. $\varphi$ (which means that $\varphi(u_j, u_k) = 0$ for all $j \neq k$), we have:

$$i(u_j)i(u_k) + i(u_k)i(u_j) = 0 \text{ for all } j \neq k.$$
Proposition 2.3 For every vector space, \( V \), of finite dimension \( n \), the map \( i : V \to Cl(\Phi) \) is injective. Given a basis \((e_1, e_2, \ldots, e_n)\) of \( V \) the \( 2^n - 1 \) products
\[
i(e_1)i(e_2)\cdots i(e_k), \quad 1 \leq i_1 < i_2 < \ldots < i_k \leq n,
\]
and 1 form a basis of \( Cl(\Phi) \). Thus, \( Cl(\Phi) \) has dimension \( 2^n \).

Proof. See [4].

Remark: Since \( i \) is injective, for simplicity of notation, from now on, we write \( u \) for \( i(u) \) Proposition 2.3 implies that if \((e_1, e_2, \ldots, e_n)\) is an orthogonal basis of \( V \), then \( Cl(\Phi) \) is the algebra presented by the generators \((e_1, e_2, \ldots, e_n)\) and the relations
\[
e_j^2 = \Phi(e_j)1, \quad 1 \leq j \leq n, \quad \text{and} \quad e_j e_k = -e_k e_j, \quad 1 \leq j, k \leq n, \quad j \neq k.
\]

In other words, Clifford algebra \( Cl(\Phi) \) consists of certain kinds of "polynomials," linear combinations of monomials of the form \( \sum_j \lambda_j e_J \), where \( J = \{i_1, i_2, \ldots, i_k\} \) is any subset (possibly empty) of \( \{1, \ldots, n\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \), and the monomial \( e_j \) is the "product" \( e_{i_1} e_{i_2} \cdots e_{i_k} \).

Definition 2.4 The even-graded elements (the elements of \( Cl^0(\Phi) \)) are those generated by 1 and the basis elements consisting of an even number of factors, \( e_{i_1} e_{i_2} \cdots e_{i_{2k}} \), and the odd-graded elements (the elements of \( Cl^1(\Phi) \)) are those generated by the basis elements consisting of an odd number of factors, \( e_{i_1} e_{i_2} \cdots e_{i_{2k+1}} \).

Remark: we assume that \( \Phi \) is the quadratic form on \( \mathbb{R}^n \) defined by
\[
\Phi(x_1, \ldots, x_n) = -(x_1^2 + \cdots + x_n^2)
\]

Let \( Cl_n \) denote the Clifford algebra \( Cl(\Phi) \).

Example 2.5 \( Cl_1 \) is spanned by the basis \((1, e_1)\). We have
\[
e_1^2 = -1.
\]
Under the bijection
\[ e_1 \mapsto i \]
\[ Cl_1 \] is isomorphic to the algebra of complex numbers, \( \mathbb{C} \).

**Example 2.6** Let \((e_1, e_2)\) be the canonical basis of \( \mathbb{R}^2 \), then \( Cl_2 \) is spanned by the basis by \((1, e_1, e_2, e_1e_2)\). Furthermore, we have:

\[ e_2e_1 = -e_1e_2, \quad e_1^2 = -1, \quad e_2^2 = -1, \quad (e_1e_2)^2 = -1. \]

Under the bijection
\[ e_1 \mapsto i, \quad e_2 \mapsto j, \quad e_1e_2 \mapsto k, \]
it is easily checked that the quaternion identities

\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

hold, and thus, \( Cl_2 \), is isomorphic to the algebra of quaternions, \( \mathbb{H} \).

**Definition 2.7** For every non degenerate quadratic form \( \Phi \) over \( \mathbb{R} \) there is an orthogonal basis with respect to which \( \Phi \) is given by

\[ \Phi(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2) \]

where \( p \) and \( q \) only depend on \( \Phi \). The quadratic form corresponding to \((p, q)\) is denoted \( \Phi_{p,q} \) and we call \((p, q)\) the signature of \( \Phi_{p,q} \). Let \( n = p+q \) We denote the Clifford algebra associated with \( \mathbb{R}^n \) and \( \Phi_{p,q} \) where has \( \Phi_{p,q} \) signature \((p, q)\) by \( Cl_{p,q} \). Note that with this new notation, \( Cl_n = Cl_{0,n} \).

**Example 2.8** Let \( Cl_{p,q} = Cl(\mathbb{R}^{p+q}, \Phi_{p,q}) \), where \( \Phi \) has signature \((p, q)\), and orthonormal basis is written as \( \{e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_q\} \) where \( e_1^2 = \cdots = e_p^2 = 1, \varepsilon_1^2 = \cdots = \varepsilon_q^2 = -1 \).
\[ ... = \varepsilon_q^2 = -1. \] Thus, we have:

\[ Cl_{1,0} = \mathbb{R} \oplus \mathbb{R} \quad \text{with} \quad e_1 = \pm 1; \]

\[ Cl_{0,1} = \mathbb{C}; \quad \text{with} \quad \varepsilon_1 = i; \]

\[ Cl_{2,0} = M_2(\mathbb{R}), \quad \text{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \]

\[ Cl_{0,2} = \mathbb{H}, \quad \text{with} \quad \varepsilon_1 = i \quad \varepsilon_2 = j, \quad \varepsilon_1 \varepsilon_2 = k; \]

\[ Cl_{1,1} = M_2(\mathbb{R}), \quad \text{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1 e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

### 3 Main Results

It turns out that the real algebras \( Cl_{p,q} \) can be build up as tensor products of the basic algebras \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{H} \). According to [6], the description of the real algebras \( Cl_{p,q} \) as matrix algebras and the 8-periodicity was first discovered by Elie Cartan in 1908. Of course, Cartan used a very different notation. These facts were rediscovered independently by [2] in the 1960's (see Raoul Bott's comments in Volume 2 of his Collected papers.).

As mentioned in Example 2.3, we have:

\[ Cl_{0,1} = \mathbb{C}, \quad Cl_{0,2} = \mathbb{H}, \quad Cl_{1,0} = \mathbb{R} \oplus \mathbb{R}, \quad Cl_{2,0} = M_2(\mathbb{R}), \]

And

\[ Cl_{1,1} = M_2(\mathbb{R}). \]

The key to the classification is the following lemma:

**Lemma 3.1** We have the isomorphisms

\[ Cl_{0,n+2} \approx Cl_{n,0} \otimes Cl_{0,2} \]

\[ Cl_{n+2,0} \approx Cl_{0,n} \otimes Cl_{2,0} \]

\[ Cl_{p+1,q+1} \approx Cl_{p,q} \otimes Cl_{1,1} \]

for all \( n, p, q \geq 0 \).
Proof. Let $\Phi_{0,n+2}(x) = -\|x\|^2$, where $\|x\|$ is the standard Euclidean norm on $\mathbb{R}^{n+2}$, and let $(e_1, \ldots, e_{n+2})$ be an orthonormal basis for $\mathbb{R}^{n+2}$ under the standard Euclidean inner product. We also let $(e'_1, \ldots, e'_n)$ be a set of generators for $Cl_{0,0}$ and $(e''_1, e''_2)$ be a set of generators for $Cl_{0,2}$. We can define a linear map $f : \mathbb{R}^{n+2} \to Cl_{0,0} \otimes Cl_{0,2}$ by its action on the basis $(e_1, \ldots, e_{n+2})$ as follows:

$$f(e_i) = \begin{cases} e'_i \oplus e''_2 & 1 \leq i \leq n \\ 1 \oplus e''_{i-n} & n + 1 \leq i \leq n + 2 \end{cases}$$

Observe that for $1 \leq i, j \leq n$ we have

$$f(e_i) f(e_j) + f(e_j) f(e_i) = (e'_i e'_j + e'_j e'_i) \otimes (e''_1 e''_2)^2 = -2\delta_{ij} 1 \otimes 1,$$

Since $(e''_2)^2 = (e''_1)^2 = -1, e'_1 e'_2 = -e''_2 e''_1$ and $e'_i e'_j = -\delta_{ij} e'_i$, for all $i \neq j$, and $(e'_i)^2 = 1$, for all $i$ with $1 \leq i \leq n$. Also for $n + 1 \leq i, j \leq n + 2$ we have

$$f(e_i) f(e_j) + f(e_j) f(e_i) = 1 \otimes (e''_{i-n} e''_{j-n} + e''_{j-n} e''_{i-n}) = -2\delta_{ij} 1 \otimes 1,$$

and

$$f(e_i) f(e_k) + f(e_k) f(e_i) = 2e'_i \otimes (e''_1 e''_{k-n} + e''_{k-n} e''_1) = 0,$$

for all $1 \leq i, j \leq n$ and $n + 1 \leq k \leq n + 2$ (since $e''_{k-n} = e''_1$ or $e''_{k-n} = e''_2$). Thus, we have:

$$f(x)^2 = -\|x\|^2.1 \otimes 1 \quad for \ all \ x \in \mathbb{R}^{n+2},$$

and by the universal mapping property of $Cl_{0,n+2}$, we get an algebra map:

$$\tilde{f} : Cl_{0,n+2} \to Cl_{0,0} \otimes Cl_{0,2}.$$

Since $\tilde{f}$ maps onto a set of generators, it is surjective. However,

$$\dim(Cl_{0,n+2}) = 2^{n+2} = 2^n.2 = \dim(Cl_{0,0})\dim(Cl_{0,2}) = \dim(Cl_{0,0} \otimes Cl_{0,2})$$

and $\tilde{f}$ is an isomorphism.

The proof of the second identity is analogous. For the third identity, we have:

$$\Phi_{p,q}(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),$$

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And let \((e_1, \ldots, e_{p+1}, \varepsilon_1, \ldots, \varepsilon_{q+1})\) be an orthogonal basis for \(\mathbb{R}^{p+q+2}\) so that \(\Phi_{p+1,q+1}(e_i) = +1\) and \(\Phi_{p+1,q+1}(\varepsilon_j) = -1\) for \(i = 1, \ldots, p+1\) and \(j = 1, \ldots, q+1\). Also, let \((e'_1, \ldots, e'_p, \varepsilon'_1, \ldots, \varepsilon'_q)\) be a set of generators for \(\mathbb{C}l_{p,q}\) and \((e''_1, \varepsilon''_1)\) be a set of generators for \(\mathbb{C}l_{1,1}\). We define a linear map \(f : \mathbb{R}^{p+q+2} \rightarrow \mathbb{C}l_{p,q} \otimes \mathbb{C}l_{1,1}\) by its action on the basis as follows:

\[
f(e_i) = \begin{cases} 
e'_i \otimes \varepsilon'_1 \varepsilon''_1 & 1 \leq i \leq p \\
1 \otimes \varepsilon''_1 & i = p + 1
\end{cases} \quad \text{and} \quad f(\varepsilon_j) = \begin{cases} 
e''_j \otimes e'_1 \varepsilon''_1 & 1 \leq j \leq q \\
1 \otimes \varepsilon''_1 & j = q + 1
\end{cases}
\]

We can check that

\[f(x)^2 = \Phi_{p+1,q+1}(x) \otimes 1 \quad \text{for all} \quad x \in \mathbb{R}^{p+q+2},\]

and we finish the proof as in the first case.

To apply this lemma, we need some further isomorphisms among various matrix algebras.

**Proposition 3.2** The following isomorphisms hold:

\[
\begin{align*}
M_m(\mathbb{R}) \otimes M_n(\mathbb{R}) & \cong M_{mn}(\mathbb{R}) \quad \text{for all} \quad m, n \geq 0 \\
M_n(\mathbb{R}) \otimes R_k & \cong M_n(k) \quad \text{for all} \quad K = \mathbb{C} \text{ or } K = \mathbb{H} \text{ and all} \quad n \geq 0 \\
\mathbb{C} \otimes \mathbb{C} & \cong \mathbb{C} \oplus \mathbb{C} \\
\mathbb{C} \otimes \mathbb{H} & \cong M_4(\mathbb{C})
\end{align*}
\]

**Proof.** See[5].

**Proposition 3.3** (Cartan/Bott) For all \(n \geq 0\) we have the following isomorphisms:

\[
\begin{align*}
\mathbb{C}l_{0,n+8} & \cong \mathbb{C}l_{0,n} \otimes \mathbb{C}l_{0,8} \\
\mathbb{C}l_{n+8,0} & \cong \mathbb{C}l_{n,0} \otimes \mathbb{C}l_{8,0}
\end{align*}
\]

Furthermore,

\[
\mathbb{C}l_{0,8} = \mathbb{C}l_{8,0} = M_{16}(\mathbb{R}).
\]
Proof. By Lemma 3.1 we have the isomorphisms:

\[ Cl_{0,n+2} \approx Cl_{n,0} \otimes Cl_{0,2}, \quad Cl_{n+2,0} \approx Cl_{0,n} \otimes Cl_{2,0}, \]

and thus,

\[ Cl_{0,n+8} \approx Cl_{n+6,0} \otimes Cl_{0,2} \approx Cl_{0,n+4} \otimes Cl_{2,0} \otimes Cl_{0,2} \approx \cdots \approx Cl_{0,n} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{0,2}. \]

Since \( Cl_{0,2} = H \) and \( Cl_{2,0} = M_2(\mathbb{R}) \), by Proposition 3.1, we get:

\[ Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \approx H \otimes H \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \approx M_4(\mathbb{R}) \otimes M_4(\mathbb{R}) \approx M_{16}(\mathbb{R}). \]

The second isomorphism is proved in a similar fashion.

**Lemma 3.4** \( Cl_{p+4,q} \approx Cl_{p,q} \otimes M_2(\mathbb{H}) \approx Cl_{p,q+4} \).

**Proof.** We will prove the first isomorphism. Take \( A = Cl_{p,q} \otimes M_2(\mathbb{H}) \), define

\[
\begin{align*}
    f(e_r) &= e_r' \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad r = 1, \ldots, p, \\
    f(\varepsilon_s) &= \varepsilon_s' \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad s = 1, \ldots, q,
\end{align*}
\]

and on the remaining four basic vectors, define

\[
\begin{align*}
    f(e_{p+1}) &= 1 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
    f(e_{p+2}) &= 1 \otimes \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \\
    f(e_{p+3}) &= 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
    f(e_{p+4}) &= 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

From all this, we can deduce the following Theorem:

**Theorem 3.5** For \( 0 \leq p \leq 8 \) and \( 8 \leq q \leq 13 \) matrix representations of the Clifford
algebras $Cl_{p,q}$ are exhibited in the following table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$p$</th>
<th>$M_{16}(\mathbb{R})$</th>
<th>$M_{16}(\mathbb{C})$</th>
<th>$M_{16}(\mathbb{H})$</th>
<th>$M_{16}(\mathbb{H}) \oplus M_{16}(\mathbb{H})$</th>
<th>$M_{32}(\mathbb{H})$</th>
<th>$M_{64}(\mathbb{H})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$M_{16}(\mathbb{R}) \oplus M_{16}(\mathbb{R})$</td>
<td>$M_{32}(\mathbb{R})$</td>
<td>$M_{32}(\mathbb{C})$</td>
<td>$M_{32}(\mathbb{H})$</td>
<td>$M_{32}(\mathbb{H}) \oplus M_{32}(\mathbb{H})$</td>
<td>$M_{64}(\mathbb{H})$</td>
<td>$M_{128}(\mathbb{H})$</td>
</tr>
<tr>
<td></td>
<td>$M_{32}(\mathbb{R}) \oplus M_{32}(\mathbb{R})$</td>
<td>$M_{64}(\mathbb{R})$</td>
<td>$M_{64}(\mathbb{C})$</td>
<td>$M_{64}(\mathbb{H}) \oplus M_{64}(\mathbb{H})$</td>
<td>$M_{64}(\mathbb{H}) \oplus M_{64}(\mathbb{H})$</td>
<td>$M_{128}(\mathbb{H}) \oplus M_{128}(\mathbb{H})$</td>
<td>$M_{512}(\mathbb{H})$</td>
</tr>
<tr>
<td></td>
<td>$M_{64}(\mathbb{R}) \oplus M_{64}(\mathbb{R})$</td>
<td>$M_{128}(\mathbb{R})$</td>
<td>$M_{128}(\mathbb{C})$</td>
<td>$M_{128}(\mathbb{H}) \oplus M_{128}(\mathbb{H})$</td>
<td>$M_{128}(\mathbb{H}) \oplus M_{128}(\mathbb{H})$</td>
<td>$M_{256}(\mathbb{H}) \oplus M_{256}(\mathbb{H})$</td>
<td>$M_{1024}(\mathbb{H})$</td>
</tr>
<tr>
<td></td>
<td>$M_{128}(\mathbb{R}) \oplus M_{128}(\mathbb{R})$</td>
<td>$M_{256}(\mathbb{R})$</td>
<td>$M_{256}(\mathbb{C})$</td>
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<td>$M_{512}(\mathbb{H}) \oplus M_{512}(\mathbb{H})$</td>
<td>$M_{1024}(\mathbb{C})$</td>
<td>$M_{1024}(\mathbb{C})$</td>
</tr>
</tbody>
</table>

Remark: A table of the Clifford algebras $Cl_{p,q}$ for $0 \leq p, q \leq 7$ can be found in [7].

**Lemma 3.6** We have the isomorphisms

$$Cl_{p,q} \cong Cl_{p+q+1}^0$$

$$Cl_{p+1,q}^0 \cong Cl_{q,p}$$

$$Cl_{p+1,q}^0 \cong Cl_{q+1,p}$$

for all $p, q \geq 0$.

**Proof.** Let $(e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_q)$ be an orthonormal basis for $\mathbb{R}^{p+q}$, We also let $(e'_1, \ldots, e'_p, \varepsilon'_1, \ldots, \varepsilon'_{q+1})$ be a set of generators for $Cl_{p,q+1}$. We can define a linear map $f : \mathbb{R}^{p+q} \to Cl_{p,q+1}^0$ by its action on the basis $(e_1, \ldots, e_n, \varepsilon_1, \ldots, \varepsilon_q)$ as follows:

$$f(e_i) = e'_i e'_{q+1} \quad i = 1, \ldots, p,$$

$$f(\varepsilon_j) = \varepsilon'_j e'_{q+1} \quad j = 1, \ldots, q.$$ 

We have

$$f(e_i) f(e_j) + f(e_j) f(e_i) = e'_i e'_{q+1} e'_j e'_{q+1} + e'_j e'_{q+1} e'_i e'_{q+1} = e'_i e'_j + e'_j e'_i = 2\delta_{ij},$$

And

$$f(\varepsilon_i) f(\varepsilon_j) + f(\varepsilon_j) f(\varepsilon_i) = e'_i e'_{q+1} e'_j e'_{q+1} + e'_j e'_{q+1} e'_i e'_{q+1} = e'_i e'_j + e'_j e'_i = -2\delta_{ij},$$
And also
\[
f(e_i)f(\varepsilon_j) + f(e_i)f(\varepsilon_j) = e'_i\varepsilon'_{q+1}\varepsilon'_j + \varepsilon'_j e'_q e'_{q+1} = e'_i\varepsilon'_j + \varepsilon'_j e'_i = 0.\]

Thus, by the universal mapping property of $\text{Cl}_{p,q}$, we get an algebra map:
\[
\tilde{f} : \text{Cl}_{p,q} \rightarrow \text{Cl}_{p,q+1}^0.
\]

Since $\tilde{f}$ maps onto a set of generators, it is surjective. However,
\[
dim(\text{Cl}_{p,q+1}^0) = \frac{2p+q+1}{2} = 2p+q = \dim(\text{Cl}_{p,q})
\]
and $\tilde{f}$ is an isomorphism.

For the second identity we define $f : \mathbb{R}^{q+p} \rightarrow \text{Cl}_{p+1,q}^0$ on basic vectors by:
\[
f(e_r) = e'_r e'_{p+1}, \quad r = 1, \ldots, q,
\]
\[
f(\varepsilon_s) = \varepsilon'_s e'_{p+1}, \quad s = 1, \ldots, p.
\]

Then
\[
f(e_r)^2 = e'_r e'_{p+1} e'_r e'_{p+1} = -e'^2_r e'^2_{p+1} = -1,
\]
\[
f(\varepsilon_s)^2 = \varepsilon'_s e'_{p+1} \varepsilon'_s e'_{p+1} = -\varepsilon'^2_s e'^2_{p+1} = -1,
\]

The rest of the proof is like the previous part. For the third identity, according to the previous parts, we have:
\[
\text{Cl}_{p+1,q} \approx \text{Cl}_{p+1,q+1}^0 \approx \text{Cl}_{q+1,p}.
\]

**Corollary:** $\text{Cl}_{p,q}^0 \approx \text{Cl}_{q,p}^0$.

**References**


