Periodicity of the Clifford Algebras

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Abstract

In this paper we study the structure of Clifford Algebras $Cl_{p,q}$ associated with a non degenerate symmetric bilinear form of signature $(p, q)$, where $p, q$ are positive integer. Also we present a description of these algebras as matrix algebras, and then we will discuss the periodicity of these algebras completely. As a consequence, We create the related algebra matrix tables for these algebras, when $0 \leq p \leq 8$ and $8 \leq q \leq 13$. We also present an isomorphism between $Cl^0_{q,p}$ and $Cl^0_{p,q}$.

Keywords: Tensor algebra, Exterior algebra, Clifford algebra, Quadratic form, Bilinear form.

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1 Introduction

Given any vector space, $V$, over a field, $K$, there is a special $K$-algebra, $T(V)$, together with a linear map, $i: V \rightarrow T(V)$, following the universal mapping property [1]. The algebra, $T(V)$, is the tensor algebra of $V$. It may be constructed as the direct sum $T(V) = \bigoplus_{i \geq 0} V^\otimes i$, Where $V^0 = K$, and $V^\otimes i$ is the $i$-fold tensor product of $V$ with itself. For every $i \geq 0$, there is a natural injection $\iota_n : V^\otimes n \rightarrow T(V)$ and in particular, an injection $\iota_0 : K \rightarrow T(V)$. The multiplicative unit, 1, of $T(V)$ is the image, $\iota_0(1)$, in $T(V)$ of the unit, 1, of the field $K$. Since every $v \in T(V)$ can be expressed as a finite

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sum $v = v_1 + v_2 + \ldots + v_k$, where $v_i \in V^{\otimes n_i}$ and $n_i$ the are natural numbers with $n_i \neq n_j$ if $i \neq j$, to define multiplication in $T(V)$, using bilinearity [1], it is enough to define the multiplication $V^{\otimes m} \times V^{\otimes n} \to V^{\otimes (m+n)}$. Of course, this is defined by:

$$(v_1 \otimes \ldots \otimes v_m).(w_1 \otimes \ldots \otimes w_n) = v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_n.$$  

It is important to note that multiplication in $T(V)$ is not commutative. Also, the unit, 1, of $T(V)$ is not equal to 1, the unit of the field $K$. However, in view of the injection $\iota_0 : K \to T(V)$, for the sake of notational simplicity, we will denote 1 by 1. More generally, in view of the injections $\iota_n : V^{\otimes n} \to T(V)$, we identify elements of $V^{\otimes n}$ with their images in $T(V)$.

Most algebras of interest arise as well-chosen quotients of the tensor algebra $T(V)$. This is true for the exterior algebra, $\Lambda^* V$ (also called Grassmann algebra), where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v$, where $v \in V$, and for the symmetric algebra, $\text{Sym} V$, where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes w - w \otimes v$, where $v, w \in V$. A Clifford algebra may be viewed as a refinement of the exterior algebra, in which we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v - \Phi(v).1$, where $\Phi$ is the quadratic form associated with a symmetric bilinear form, $\varphi : V \times V \to K$, and $\cdot : K \times T(V) \to T(V)$ denotes the scalar product of the algebra $T(V)$. For simplicity, let us assume that we are now dealing with real algebras.

2 Preliminaries

Definition 2.1 Let $V$ be a real finite-dimensional vector space. A quadratic form on $V$ is a mapping $\Phi : V \to \mathbb{R}$ such that

1. $\Phi(\lambda v) = \lambda^2 \Phi(v)$ for all $\lambda \in \mathbb{R}, \ v \in V$. 

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2. the mapping \((x, y) \rightarrow (\Phi(x + y) - \Phi(x) - \Phi(y)) = \varphi(x, y)\) of \(V \times V\) into \(\mathbb{R}\) is bilinear.

Then \(\varphi\) is called the bilinear form associated to \(\Phi\).

It is obvious from the definition that \(\varphi\) is symmetric:

\[
\varphi(x, y) = \varphi(y, x)
\]

and \(\varphi(x, x) = \Phi(x)\).

Two elements \(x, y\) of \(V\) such that \(\varphi(x, y) = 0\) are said to be orthogonal to each other.

**Definition 2.2** Let \(V\) be a real finite-dimensional vector space together with a symmetric bilinear form \(\varphi : V \times V \rightarrow \mathbb{R}\), and associated quadratic form, \(\Phi(x) = \varphi(x, x)\). A Clifford algebra associated with \(V\) and \(\Phi\) is a real algebra, \(Cl(V, \Phi)\), together with a linear map, \(i : V \rightarrow Cl(V, \Phi)\) satisfying the condition \((i(v))^2 = \Phi(v)\).1 for all \(v \in V\) and so that for every real algebra, \(A\), and every linear map, \(f : V \rightarrow A\), with

\[
(f(v))^2 = \Phi(v)\)1 for all \(v \in V\),

there is a unique algebra homomorphism, \(\bar{f} : Cl(V, \Phi) \rightarrow A\) so that

\[
f = \bar{f}oi,
\]

as in the diagram below:

\[
\begin{array}{c}
V \xrightarrow{i} Cl(V, \Phi) \\
\downarrow{f} \quad \downarrow{\bar{f}} \\
A
\end{array}
\]

We use the notation, \(\lambda u\), for the product of a scalar, \(\lambda \in \mathbb{R}\) and of an element, \(u\), in the algebra \(Cl(V, \Phi)\) and juxtaposition, \(uv\), for the multiplication of two elements, \(u, v \in Cl(V, \Phi)\).
By a familiar argument, any two Clifford algebras associated with $V$ and $\Phi$ are isomorphic.

To show the existence of $Cl(V, \Phi)$, observe that $T(V)/U$ does the job, where $U$ is the ideal of $T(V)$ generated by all elements of the form $v \otimes v - \Phi(v)$.1, where $v \in V$. The map $i : V \to Cl(V, \Phi)$ is the composition

$$V \xrightarrow{i} T(V) \xrightarrow{\pi} \frac{T(V)}{U}$$

where $\pi$ is the natural quotient map. We often denote the Clifford algebra $Cl(V, \Phi)$ simply by $Cl(\Phi)$.

Observe that when $\Phi \equiv 0$ is the quadratic form identically zero everywhere, then the Clifford algebra $Cl(V, 0)$ is just the exterior algebra, $\Lambda^*V$.

**Remark:** As in the case of the tensor algebra, the unit of the algebra $Cl(\Phi)$ and the unit of the field $\mathbb{R}$ are not equal.

Since

$$\Phi(u + v) - \Phi(u) - \Phi(v) = 2\varphi(u, v)$$

and

$$(i(u + v))^2 = i(u)^2 + i(v)^2 + i(u)i(v) + i(v)i(u),$$

using the fact that

$$(i(u))^2 = \Phi(u).1,$$ 

We get:

$$i(u)i(v) + i(v)i(u) = 2\varphi(u, v).1.$$ 

As a consequence, if $(u_1, \ldots, u_n)$ is an orthogonal basis w.r.t. $\varphi$(which means that $\varphi(u_j, u_k) = 0$ for all $j \neq k$ ), we have:

$$i(u_j)i(u_k) + i(u_k)i(u_j) = 0 \quad for \ all \ j \neq k.$$
Proposition 2.3 For every vector space, $V$, of finite dimension $n$, the map $i : V \to Cl(\Phi)$ is injective. Given a basis $(e_1, e_2, \ldots, e_n)$ of $V$ the $2^n - 1$ products

$$i(e_1)i(e_2) \cdots i(e_k), \quad 1 \leq i_1 < i_2 < \ldots < i_k \leq n,$$

and 1 form a basis of $Cl(\Phi)$. Thus, $Cl(\Phi)$ has dimension $2^n$.

Proof. See[4].

Remark: Since $i$ is injective, for simplicity of notation, from now on, we write $u$ for $i(u)$ Proposition 2.3 implies that if $(e_1, e_2, \ldots, e_n)$ is an orthogonal basis of $V$, then $Cl(\Phi)$ is the algebra presented by the generators $(e_1, e_2, \ldots, e_n)$ and the relations

$$e_j^2 = \Phi(e_j).1, \quad 1 \leq j \leq n, \quad \text{and} \quad e_j e_k = -e_k e_j, \quad 1 \leq j, k \leq n, \quad j \neq k.$$

In other words, Clifford algebra $Cl(\Phi)$ consists of certain kinds of "polynomials," linear combinations of monomials of the form $\sum J \lambda e_J$, where $J = \{i_1, i_2, \ldots, i_k\}$ is any subset (possibly empty) of $\{1, \ldots, n\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and the monomial $e_J$ is the "product" $e_{i_1}e_{i_2}\ldots e_{i_k}$.

Definition 2.4 The even-graded elements (the elements of $Cl^0(\Phi)$ ) are those generated by 1 and the basis elements consisting of an even number of factors, $e_{i_1}e_{i_2}\ldots e_{i_{2k}}$, and the odd-graded elements (the elements of $Cl^1(\Phi)$) are those generated by the basis elements consisting of an odd number of factors, $e_{i_1}e_{i_2}\ldots e_{i_{2k+1}}$.

Remark: we assume that $\Phi$ is the quadratic form on $\mathbb{R}^n$ defined by

$$\Phi(x_1, \ldots, x_n) = -(x_1^2 + \ldots + x_n^2)$$

Let $Cl_n$ denote the Clifford algebra $Cl(\Phi)$.

Example 2.5 $Cl_1$ is spanned by the basis $(1, e_1)$. We have

$$e_1^2 = -1.$$
Under the bijection

\[ e_1 \mapsto i \]

\( Cl_1 \) is isomorphic to the algebra of complex numbers, \( \mathbb{C} \).

**Example 2.6** Let \((e_1, e_2)\) be the canonical basis of \( \mathbb{R}^2 \), then \( Cl_2 \) is spanned by the basis by \((1, e_1, e_2, e_1e_2)\). Furthermore, we have:

\[ e_2e_1 = -e_1e_2, \quad e_1^2 = -1, \quad e_2^2 = -1, \quad (e_1e_2)^2 = -1. \]

Under the bijection

\[ e_1 \mapsto i, \quad e_2 \mapsto j, \quad \varepsilon_1e_2 \mapsto k, \]

it is easily checked that the quaternion identities

\[ i^2 = j^2 = k^2 = -1, \quad \varepsilon_1 = -j\varepsilon_1 = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

hold, and thus, \( Cl_2 \), is isomorphic to the algebra of quaternions, \( \mathbb{H} \).

**Definition 2.7** For every non-degenerate quadratic form \( \Phi \) over \( \mathbb{R} \) there is an orthogonal basis with respect to which \( \Phi \) is given by

\[ \Phi(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2) \]

where \( p \) and \( q \) only depend on \( \Phi \). The quadratic form corresponding to \((p, q)\) is denoted \( \Phi_{p,q} \) and we call \((p, q)\) the signature of \( \Phi_{p,q} \). Let \( n = p+q \) We denote the Clifford algebra associated with \( \mathbb{R}^n \) and \( \Phi_{p,q} \) where has \( \Phi_{p,q} \) signature \((p, q)\) by \( Cl_{p,q} \). Note that with this new notation, \( Cl_n = Cl_{0,n} \).

**Example 2.8** Let \( Cl_{p,q} = Cl(\mathbb{R}^{p+q}, \Phi_{p,q}) \), where \( \Phi \) has signature \((p, q)\), and orthonormal basis is written as \( \{e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_q\} \) where \( e_1^2 = \cdots = e_p^2 = 1, \varepsilon_1^2 = \cdots = e_q^2 = -1 \).
\[ \ldots = \varepsilon^2_q = -1. \text{ Thus, we have:} \]

\[ Cl_{1,0} = \mathbb{R} \oplus \mathbb{R} \quad \text{with} \quad e_1 = \pm 1; \]

\[ Cl_{0,1} = \mathcal{C}; \quad \text{with} \quad \varepsilon_1 = i; \]

\[ Cl_{2,0} = M_2(\mathbb{R}), \quad \text{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \]

\[ Cl_{0,2} = \mathbb{H}, \quad \text{with} \quad \varepsilon_1 = i \quad \varepsilon_2 = j, \quad \varepsilon_1 \varepsilon_2 = k; \]

\[ Cl_{1,1} = M_2(\mathbb{R}), \quad \text{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1 \varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

### 3 Main Results

It turns out that the real algebras $Cl_{p,q}$ can be built up as tensor products of the basic algebras $\mathbb{R}$, $\mathcal{C}$ and $\mathbb{H}$. According to [6], the description of the real algebras $Cl_{p,q}$ as matrix algebras and the 8-periodicity was first discovered by Elie Cartan in 1908. Of course, Cartan used a very different notation. These facts were rediscovered independently by [2] in the 1960's (see Raoul Bott's comments in Volume 2 of his Collected papers.).

As mentioned in Example 2.3, we have:

\[ Cl_{1,0} = \mathcal{C}, \quad Cl_{0,2} = \mathbb{H}, \quad Cl_{1,0} = \mathbb{R} \oplus \mathbb{R}, \quad Cl_{2,0} = M_2(\mathbb{R}), \]

And

\[ Cl_{1,1} = M_2(\mathbb{R}). \]

The key to the classification is the following lemma:

**Lemma 3.1** We have the isomorphisms

\[ Cl_{0,n+2} \approx Cl_{n,0} \otimes Cl_{0,2} \]

\[ Cl_{n+2,0} \approx Cl_{0,n} \otimes Cl_{2,0} \]

\[ Cl_{p+1,q+1} \approx Cl_{p,q} \otimes Cl_{1,1} \]

for all $n, p, q \geq 0$. 
Proof. Let $\Phi_{0,n+2}(x) = -\|x\|^2$, where $\|x\|$ is the standard Euclidean norm on $\mathbb{R}^{n+2}$, and let $(e_1, \ldots, e_{n+2})$ be an orthonormal basis for $\mathbb{R}^{n+2}$ under the standard Euclidean inner product. We also let $(e_1', \ldots, e_n')$ be a set of generators for $Cl_{0,0}$ and $(e_1'', e_2'')$ be a set of generators for $Cl_{0,2}$. We can define a linear map $f : \mathbb{R}^{n+2} \to Cl_{0,0} \otimes Cl_{0,2}$ by its action on the basis $(e_1, \ldots, e_{n+2})$ as follows:

$$f(e_i) = \begin{cases} e_i' \oplus e_i'' & 1 \leq i \leq n \\ 1 \oplus e_{i-n}' & n + 1 \leq i \leq n + 2 \end{cases}$$

Observe that for $1 \leq i, j \leq n$ we have

$$f(e_i) f(e_j) + f(e_j) f(e_i) = (e_i' e_j' + e_j' e_i') \otimes (e_i'' e_2'') = -2\delta_{ij} 1 \otimes 1,$$

Since $(e_2'')^2 = (e_1'')^2 = -1$, $e_1' e_2'' = -e_2' e_1''$ and $e_i' e_j' = -e_j' e_i'$, for all $i \neq j$, and $(e_i')^2 = 1$, for all $i$ with $1 \leq i \leq n$. Also for $n + 1 \leq i, j \leq n + 2$ we have

$$f(e_i) f(e_j) + f(e_j) f(e_i) = 1 \otimes (e_i'' e_j'' - e_j'' e_i'') = -2\delta_{ij} 1 \otimes 1,$$

and

$$f(e_i) f(e_k) + f(e_k) f(e_i) = 2e_i' \otimes (e_1'' e_2'' e_k - e_k'' e_1'' e_2') = 0,$$

for all $1 \leq i, j \leq n$ and $n + 1 \leq k \leq n + 2$ (since $e_k'' e_1'' = e_1''$ or $e_k'' e_2'' = e_2''$). Thus, we have:

$$f(x)^2 = -\|x\|^2.1 \otimes 1 \quad \text{for all} \quad x \in \mathbb{R}^{n+2},$$

and by the universal mapping property of $Cl_{0,n+2}$, we get an algebra map:

$$\tilde{f} : Cl_{0,n+2} \to Cl_{0,0} \otimes Cl_{0,2}.$$ 

Since $\tilde{f}$ maps onto a set of generators, it is surjective. However,

$$\dim(\text{dim}(Cl_{0,n+2}) = 2^{n+2} = 2^n 2 = \dim(\text{dim}(Cl_{0,0}) \dim(Cl_{0,2}) = \dim(Cl_{0,0} \otimes Cl_{0,2})$$

and $\tilde{f}$ is an isomorphism.

The proof of the second identity is analogous. For the third identity, we have:

$$\Phi_{p,q}(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),$$
And let \((e_1, \ldots, e_{p+1}, \varepsilon_1, \ldots, \varepsilon_{q+1})\) be an orthogonal basis for \(\mathbb{R}^{p+q+2}\) so that \(\Phi_{p+1,q+1}(e_i) = +1\) and \(\Phi_{p+1,q+1}(\varepsilon_j) = -1\) for \(i = 1, \ldots, p + 1\) and \(j = 1, \ldots, q + 1\). Also, let \((e'_1, \ldots, e'_p, \varepsilon'_1, \ldots, \varepsilon'_{q})\) be a set of generators for \(Cl_{p,q}\) and \((e''_1, \varepsilon''_1)\) be a set of generators for \(Cl_{1,1}\). We define a linear map \(f : \mathbb{R}^{p+q+2} \rightarrow Cl_{p,q} \otimes Cl_{1,1}\) by its action on the basis as follows:

\[
f(e_i) = \begin{cases} 
    e'_i \otimes e''_1 & 1 \leq i \leq p \\
    1 \otimes e''_1 & i = p + 1
  \end{cases}, \quad f(\varepsilon_j) = \begin{cases} 
    \varepsilon'_j \otimes e''_1 & 1 \leq j \leq q \\
    1 \otimes \varepsilon''_1 & j = q + 1
  \end{cases}
\]

We can check that

\[
f(x)^2 = \Phi_{p+1,q+1}(x) \cdot 1 \otimes 1 \quad \text{for all} \quad x \in \mathbb{R}^{p+q+2},
\]

and we finish the proof as in the first case.

To apply this lemma, we need some further isomorphisms among various matrix algebras.

**Proposition 3.2** The following isomorphisms hold:

\[
M_m(\mathbb{R}) \otimes M_n(\mathbb{R}) \cong M_{mn}(\mathbb{R}) \quad \text{for all} \quad m, n \geq 0
\]
\[
M_n(\mathbb{R}) \otimes \mathbb{R}^k \cong M_n(k) \quad \text{for all} \quad K = \mathbb{C} \text{ or } K = \mathbb{H} \text{ and all} \quad n \geq 0
\]
\[
\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}
\]
\[
\mathbb{C} \otimes \mathbb{H} \cong M_4(\mathbb{C})
\]

**Proof.** See [5].

**Proposition 3.3** *(Cartan/Bott)* For all \(n \geq 0\) we have the following isomorphisms:

\[
Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8}
\]
\[
Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0}
\]

Furthermore,

\[
Cl_{0,8} = Cl_{8,0} = M_{16}(\mathbb{R}).
\]
Proof. By Lemma 3.1 we have the isomorphisms:

\[ Cl_{0,n+2} \approx Cl_{n,0} \otimes Cl_{0,2}, \quad Cl_{n+2,0} \approx Cl_{0,n} \otimes Cl_{2,0}, \]

and thus,

\[ Cl_{0,n+8} \approx Cl_{n+6,0} \otimes Cl_{0,2} \approx Cl_{0,n+4} \otimes Cl_{2,0} \otimes Cl_{0,2} \approx \cdots \approx Cl_{0,n} \otimes Cl_{2,0} \otimes Cl_{2,0} \otimes Cl_{0,2}. \]

Since \( Cl_{0,2} = H \) and \( Cl_{2,0} = M_2(\mathbb{R}) \), by Proposition 3.1, we get:

\[ Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \approx H \otimes H \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \approx M_4(\mathbb{R}) \otimes M_4(\mathbb{R}) \approx M_{16}(\mathbb{R}). \]

The second isomorphism is proved in a similar fashion.

Lemma 3.4 \( Cl_{p+4,q} \approx Cl_{p,q} \otimes M_2(H) \approx Cl_{p,q+4} \).

Proof. We will prove the first isomorphism. Take \( A = Cl_{p,q} \otimes M_2(H) \), define

\[ f : \mathbb{R}^{p+4} \rightarrow A \]

\[ f(e_r) = e'_r \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad r = 1, \ldots, p, \]

\[ f(\varepsilon_s) = \varepsilon'_s \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad s = 1, \ldots, q, \]

and on the remaining four basic vectors, define

\[ f(e_{p+1}) = 1 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad f(e_{p+2}) = 1 \otimes \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \]

\[ f(e_{p+3}) = 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_{p+4}) = 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

From all this, we can deduce the following Theorem:

Theorem 3.5 For \( 0 \leq p \leq 8 \) and \( 8 \leq q \leq 13 \) matrix representations of the Clifford
The Clifford algebras $\text{Cl}_{p,q}$ are exhibited in the following table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$p$</th>
<th>$\text{M}_{16}(\mathbb{R})$</th>
<th>$\text{M}_{16}(\mathbb{C})$</th>
<th>$\text{M}_{16}(\mathbb{H})$</th>
<th>$\text{M}<em>{16}(\mathbb{H}) \oplus \text{M}</em>{16}(\mathbb{H})$</th>
<th>$\text{M}_{32}(\mathbb{R})$</th>
<th>$\text{M}_{64}(\mathbb{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\downarrow$</td>
<td>$\text{M}<em>{32}(\mathbb{R}) \oplus \text{M}</em>{32}(\mathbb{R})$</td>
<td>$\text{M}_{64}(\mathbb{R})$</td>
<td>$\text{M}<em>{64}(\mathbb{R}) \oplus \text{M}</em>{64}(\mathbb{R})$</td>
<td>$\text{M}_{128}(\mathbb{R})$</td>
<td>$\text{M}<em>{128}(\mathbb{R}) \oplus \text{M}</em>{128}(\mathbb{R})$</td>
<td>$\text{M}_{256}(\mathbb{R})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{M}<em>{64}(\mathbb{R}) \oplus \text{M}</em>{64}(\mathbb{H})$</td>
<td>$\text{M}_{128}(\mathbb{H})$</td>
<td>$\text{M}<em>{128}(\mathbb{H}) \oplus \text{M}</em>{128}(\mathbb{H})$</td>
<td>$\text{M}_{256}(\mathbb{H})$</td>
<td>$\text{M}<em>{256}(\mathbb{H}) \oplus \text{M}</em>{256}(\mathbb{H})$</td>
<td>$\text{M}_{512}(\mathbb{H})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{M}_{256}(\mathbb{R})$</td>
<td>$\text{M}_{256}(\mathbb{C})$</td>
<td>$\text{M}_{256}(\mathbb{H})$</td>
<td>$\text{M}_{512}(\mathbb{C})$</td>
<td>$\text{M}_{1024}(\mathbb{R})$</td>
<td>$\text{M}_{1024}(\mathbb{C})$</td>
</tr>
</tbody>
</table>

Remark: A table of the Clifford algebras $\text{Cl}_{p,q}$ for $0 \leq p, q \leq 7$ can be found in [7].

Lemma 3.6 We have the isomorphisms

\[
\text{Cl}_{p,q} \cong \text{Cl}_{p,q+1}^0
\]

\[
\text{Cl}_{p+1,q}^0 \cong \text{Cl}_{q,p}
\]

\[
\text{Cl}_{p+1,q}^1 \cong \text{Cl}_{q+1,p}
\]

for all $p, q \geq 0$.

Proof. Let $(e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_q)$ be an orthonormal basis for $\mathbb{H}^{p+q}$, We also let $(e'_1, \ldots, e'_p, e'_{p+1}, \ldots, e'_{p+q})$ be a set of generators for $\text{Cl}_{p,q+1}$. We can define a linear map $f : \mathbb{H}^{p+q} \rightarrow \text{Cl}_{p,q+1}^0$ by its action on the basis $(e_1, \ldots, e_n, \varepsilon_1, \ldots, \varepsilon_q)$ as follows:

\[
f(e_i) = e'_i e'_{q+1} \quad i = 1, \ldots, p,
\]

\[
f(\varepsilon_j) = e'_j e'_{q+1} \quad j = 1, \ldots, q.
\]

We have

\[
f(e_i) \ f(e_j) + f(e_j) \ f(e_i) = e'_i e'_{q+1} e'_j e'_{q+1} + e'_j e'_{q+1} e'_i e'_{q+1} = e'_i e'_{q+1} + e'_j e'_{q+1} = 2\delta_{ij},
\]

And

\[
f(e_i) \ f(\varepsilon_j) + f(\varepsilon_j) \ f(e_i) = e'_i e'_{q+1} e'_j e_{q+1} + e'_j e'_{q+1} e'_i e_{q+1} = e'_i e'_{q+1} + e'_j e_{q+1} = -2\delta_{ij},
\]
And also
\[ f(e_i) f(\varepsilon_j) + f(\varepsilon_j) f(e_i) = e'_i \varepsilon'_{q+1} \varepsilon'_j + \varepsilon'_j e'_i e'_{q+1} = e'_i \varepsilon'_j + \varepsilon'_j e'_i = 0. \]

Thus, by the universal mapping property of \( Cl_{p,q} \), we get an algebra map:
\[ \tilde{f} : Cl_{p,q} \rightarrow Cl_{p,q+1}^0. \]

Since \( \tilde{f} \) maps onto a set of generators, it is surjective. However,
\[ \dim(Cl_{p,q}^0) = \frac{2^{p+q+1}}{2} = 2^{p+q} = \dim(Cl_{p,q}) \]
and \( \tilde{f} \) is an isomorphism.

For the second identity we define \( f : \mathbb{R}^{q+p} \rightarrow Cl_{p,q+1}^0 \) on basic vectors by:
\[
\begin{align*}
  f(e_r) &= e'_r e'_{p+1} & r &= 1, \ldots, q, \\
  f(\varepsilon_s) &= \varepsilon'_s e'_{p+1} & s &= 1, \ldots, p.
\end{align*}
\]

Then
\[
\begin{align*}
  f(e_r)^2 &= e'_r e'_{p+1} e'_r e'_{p+1} = -e'_r e'_{p+1} = -e'_r = -1, \\
  f(\varepsilon_s)^2 &= \varepsilon'_s e'_{p+1} \varepsilon'_s e'_{p+1} = -\varepsilon'_s e'_{p+1} = -\varepsilon'_s = +1,
\end{align*}
\]

The rest of the proof is like the previous part. For the third identity, according to the previous parts, we have:
\[ Cl_{p+1,q} \approx Cl_{p+1,q+1}^0 \approx Cl_{q+1,p}. \]

**Corollary:** \( Cl_{p,q}^0 \approx Cl_{q,p}^0. \)

**References**


