Existence of solutions for a class of functional integral equations of Volterra type in two variables via measure of noncompactness

A. Aghajani* and A. S. Haghighi

Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
E-mail: aghajani@iust.ac.ir

Abstract

This paper presents some results concerning the existence of solutions for a functional integral equation of Volterra type in two variables, via measure of noncompactness. Two examples are included to illustrate the main result.

Keywords: Measure of noncompactness; Darbo fixed point theorem; fixed point; Volterra integral equation

1. Introduction

Recently measures of noncompactness have been successfully applied to investigate the solvability and behavior of solutions of a large variety of integral equations (Aghajani et al., 2011; Banas et al, 2008; Benchohra, 2012; Darwish et al., 2011). Banas and Dhage (2008), Banas and Rzepka (2003), Hu and Yan (2006), Liu and Kang (2007) and Liu et al. (2012) studied the existence and behavior of solutions of integral equation of Volterra type on unbounded interval in one variable and Aghajani and Jalilian (2011) extended many of the above results by considering the following integral equation in general form

\[ x(t) = f(t, x(\alpha(t)), \int_0^\beta g(t, s, x(\eta(s))))ds \]

on \( BC(\mathbb{R}, \mathbb{C}) \). Moreover, many authors studied the existence of solutions for systems of integral equations of Volterra type in one variable on unbounded intervals (Aghajani et al., 2011; Olszowy, 2009). Li, Gao and Peng in (2012) studied the existence of mild solutions for a class of semilinear fractional differential equations with nonlocal conditions in Banach spaces. Benchahra and Seba (2012) studied the existence of solutions for an integral equation of fractional order with multiple time delays in Banach spaces, and M. Mursaleen and A. Mohiuddine in (2012) applied the technique of measures of noncompactness to the theory of infinite system of differential equations in the Banach sequence spaces \( \ell_p \) (1 \( \leq \) p \( \leq \) \( \infty \)).

In this paper, we study the existence of solutions for the following functional integral equation in two variables

\[ x(t,s) = f(t,s, x(\xi(t)), \xi_2(s)), \int_0^{\beta_1} \int_0^{\beta_2} g_1(t,s,v,w, x(\eta_1(v)), \eta_2(w)))dv dw, \]
\[ \int_0^{\beta_1} \int_0^{\beta_2} g_2(t,s,v, x(\xi_1(v)), \xi_2(s)))dv \]

where \( f, g_1, \xi_1, \eta_1, \xi_2 \) and \( \beta_1 \) satisfy some certain conditions, using the technic of measure of noncompactness.

The first measure of noncompactness was defined by Kuratowski (1930). For a bounded subset \( S \) of a metric space \( X \), the Kuratowski measure of noncompactness is defined as

\[ \alpha(S):= \inf \left \{ S = \bigcup_{i=1}^n S_i \text{for some } S_i \text{ with } \bigcup_{i=1}^n \text{diam}(S_i) \leq \delta \text{for } 1 \leq i \leq n < \infty \right \} \]

where \( \text{diam}(T) \) denotes the diameter of a set \( T \subset X \), namely

\[ \text{diam}(T) := \sup \{ \text{d}(x, y) | x, y \in T \} \]

Here, we recall some basic facts concerning measures of noncompactness from (Banas, 1980), which is defined axiomatically in terms of some natural conditions. Denote by \( \mathbb{R} \) the set of real numbers and put \( \mathbb{R}_+ = [0, \infty) \). Let \( (E, \| \|) \) be a Banach space. The symbol \( \overline{X}, ConvX \) will
denote the closure and closed convex hull of a subset \( X \) of \( E \), respectively. Moreover, let \( \mathfrak{M}_E \) and \( \mathfrak{N}_E \) indicate the family of all nonempty and bounded subsets of \( E \) and \( \mathfrak{N}_E \) indicate the family of all nonempty and relatively compact subsets.

**Definition 1.1.** A mapping \( \mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+ \) is said to be a measure of noncompactness in \( E \) if it satisfies the following conditions.

1° The family \( \text{Ker} \mu = \{ \mu(X) = 0 \} \) is nonempty and \( \mu \subseteq \mathfrak{M}_E \).

2° \( X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y) \).

3° \( \mu(\overline{X}) = \mu(X) \).

4° \( \mu(\text{ConvX}) = \mu(X) \).

5° \( \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y) \)

for \( \lambda \in [0,1] \).

6° If \( \{X_n\} \) is a sequence of closed sets from \( \mathfrak{M}_E \) such that \( X_{n+1} \subseteq X_n \) for \( n = 1, 2, \ldots \) and if

\[
\lim_{n \to \infty} \mu(X_n) = 0,
\]

then \( \bigcap_{n=1}^{\infty} X_n \neq \emptyset \).

In 1955, Darbo published the following fixed point theorem, using the concept of measures of noncompactness, which guarantees the existence of fixed point for condensing operators.

**Theorem 1.1.** (Darbo, 1955) Let \( \Omega \) be a nonempty, bounded, closed, and convex subset of a Banach space \( E \) and let \( F : \Omega \rightarrow \Omega \) be a continuous mapping such that there exists a constant \( k \in [0,1) \) with the property

\[
\mu(F(X)) \leq k \mu(X)
\]

for any \( X \subseteq \Omega \). Then \( F \) has a fixed point in the set \( \Omega \).

Now let \( BC(R \times R_+) \) be the Banach space of all bounded and continuous functions on \( R \times R_+ \) equipped with the standard norm

\[
\|x\| = \sup \{x(t,s) : t, s \geq 0\}.
\]

For any nonempty bounded subset \( X \) of \( BC(R \times R_+) \), \( x \in X \), \( L > 0 \), and \( \varepsilon \geq 0 \), let

\[
\omega^L(x, \varepsilon) = \sup \left\{ \frac{|f(t,s) - x(u,v)|}{t - u, s - v \in [0,L], |t - u| \leq \varepsilon, |s - v| \leq \varepsilon} \right\},
\]

\[
\omega^L(X, \varepsilon) = \sup \{ \omega^L(x, \varepsilon) : x \in X \},
\]

\[
\omega^L_0(X) = \lim_{\varepsilon \to 0} \omega^L(X, \varepsilon),
\]

\[
\omega^L_0(X) = \lim_{L \to \infty} \omega^L_0(X)
\]

\[
X(t,s) = \{x(t,s) : x \in X\},
\]

and

\[
\mu(X) = \omega^L_0(X) + \limsup_{t,s \to \infty} \text{diam} X(t,s),
\]

(3)

where

\[
\limsup_{t,s \to \infty} \text{diam} X(t,s) := \inf_{t,s > 0} \{\text{diam} X(t,s)\}
\]

Similar to (Banas et al., 2003) (cf. also (Banas et al., 2009)), it can be shown that the function \( \mu \) is a measure of noncompactness in the space \( BC(R \times R_+) \) (in the sense of Definition 1.1).

The rest of the paper is organized as follows: In Section 2, we present an extension of Darbo fixed point theorem and state our main results concerning the existence of solutions of the integral equation (1). In Section 3, we provide two examples to show the usefulness and applicability of main results.

**2. Main results**

In this section, we study the functional integral equation (1) with the following conditions:

i. \( \xi_i, \eta_i, \beta_i, \zeta_i : R \to R_+ \) (\( i = 1, 2, 3 \)) are continuous and \( \xi_i(t) \to \infty \) as \( t \to \infty \) (\( i = 1, 2 \)).

ii. \( f : R \times R \times R \times R \times R \to R \) is continuous.

Moreover, there exist a constant \( k \in [0,1) \) and nondecreasing continuous functions \( \Phi_i, \Phi_2 : R \to R \) with \( \Phi_i(0) = 0 \) (\( i = 1, 2 \)) such that

\[
|f(t,s,x,y,v) - f(t,s,u,z,w)| \leq k |x - u| + \Phi_1(m_1(t,s)|y - z|)
\]

\[
+ \Phi_2(m_2(t,s)|v - w|)
\]

(4)

where \( m_1 : R \times R \to R_+ \) (\( i = 1, 2 \)) are continuous functions.

iii. \( M := \sup \{ |f(t,s,0,0,0)| : t, s \in R_+ \} < \infty \)

iv. \( g_1 : R_+ \times R_+ \times R_+ \times R_+ \to R \) is continuous and

\[
D_i := \sup \left\{ m_i(t,s) \left| \int_0^{\infty} \int_0^{\infty} g_i(t,s,v,u, w, z) \right| dv dw \leq \infty \right\}
\]

(5)
Further,
\[
\lim_{t,s \to \infty} m_1(t,s)\left| \int_0^{\beta(t)} \int_0^{\beta(s)} \frac{g_1(t,s,v,x(\eta_1(v),\eta_1(w)))}{dvdw} \right| = 0
\]
uniformly with respect to \( x, y \in BC(R_+ \times R_+) \).

\( \mathbf{v.} \) \( g_2 : R_+ \times R_+ \times R_+ \times R^2 \to R \) is continuous and
\[
D_2 := \sup_{t,s \in R_+} \left\{ m_2(t,s) \left| \int_0^{\beta(t)} \int_0^{\beta(s)} x(\gamma_1(v),\gamma_2(s))(dv) \right| \right\} < \infty.
\]
Moreover,
\[
\lim_{t,s \to \infty} m_2(t,s)\left| \int_0^{\beta(t)} \int_0^{\beta(s)} -g_2(t,s,v,x(\gamma_1(v),\gamma_2(s)))(dv) \right| = 0
\]
uniformly with respect to \( x, y \in BC(R_+ \times R_+) \).

Before we discuss the existence of solutions for the functional integral equation (1) and prove the main theorem, let us provide some auxiliary lemmas in this respect.

**Lemma 2.1.** Let \( C \) be a nonempty, bounded, closed and convex subset of a Banach space \( E \) and let \( F : C \to E \) be an operator such that
\[
\| F(x) - F(y) \| \leq k\|x - y\|
\]
for \( i = 1, 2, ..., n \). Let us fix arbitrarily \( 1 \leq k \leq m \). Then by (8) and properties of \( \Phi_i \) we obtain
\[
diam(T(S_k)) \leq diam(F(S_k)) + \sum_{i=1}^{m} \Phi_i(diam(G_i(S_k))) \leq \alpha(F(X)) + \varepsilon + \sum_{i=1}^{m} \Phi_i(\varepsilon),
\]
and since \( \varepsilon \) is an arbitrarily positive number and \( \Phi_i \) are nondecreasing continuous functions, it concludes that
\[
\alpha(T(X)) \leq \alpha(F(X)).
\]

Now, we show that \( T \) satisfies (2). To do this, fix arbitrary \( x, y \in X \) then we have
\[
\| F(x) - F(y) \| \leq k\|x - y\| \leq k\ diam X
\]
So
\[
diam(F(X)) \leq k\ diam X,
\]
which gives
\[
\alpha(F(X)) \leq k\alpha(X).
\]
From (7) and (10) we deduce that
\[
\alpha(T(X)) \leq k\alpha(X)
\]
Also, from (8), \( T \) is a continuous operator and the application of Theorem 1.1 completes the proof.

**Lemma 2.2.** Assume that \( g_1 \) satisfies the hypothesis iv, then \( g_1 : BC(R_+ \times R_+) \to BC(R_+ \times R_+) \) defined by
\[
g_i(t,s) = m_i(t,s) \int_0^{\beta(t)} \int_0^{\beta(s)} g_1(t,s,v,w,x(\eta_1(v),\eta_1(w)))(dv)dw
\]
is a compact and continuous operator.

**Proof:** Obviously, \( g_i(t,s) \) for any \( x \in BC(R_+ \times R_+) \) is continuous on \( R_+ \times R_+ \) and by (5), \( g_i \) is a self operator on
Now we show that $G_1$ is continuous. To verify this, take $x \in BC(R_+ \times R_+)$ and $\varepsilon > 0$ arbitrarily. Moreover, take $y \in BC(R_+ \times R_+)$ with $\|x - y\| < \varepsilon$. Then we have

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq m_{i,T} \beta_T^2 \partial_T(\varepsilon), \]

where

\[ \beta_T = \sup \{ \beta_i(t) : t \in [0,T], 1 \leq i \leq 3 \}, \]
\[ m_{i,T} = \sup \{ m_i(t,s) : t,s \in [0,T] \}, \]
\[ \partial_T(\varepsilon) = \sup \{ g(t,s,v,w,x) - g(t,s,v,w,y) : t,s \in [0,T], v,w \in [0,\beta_T], x,y \in [-b,b], |x-y| \leq \varepsilon \}. \]

with $b = \|x\| + \varepsilon$. By using the continuity of $g_1$ on the compact set $[0,T] \times [0,T] \times [0,\beta_T] \times [-b,b]$, we have $\partial_T(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $G_1$ is a continuous function on $BC(R_+ \times R_+)$. To complete the proof we need to verify that $G_1$ is a compact operator. Let $X$ be a nonempty and complete the proof we need to verify that $1$

This result together condition (iii) imply that there exists $T > 0$ such that for $t,s > T$ we have

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq \varepsilon, \]

and if $t,s \in [0,T]$, then the inequality in (12) follows that

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq m_{i,T} \beta_T^2 \partial_T(\varepsilon), \]

where

\[ \beta_T = \sup \{ \beta_i(t) : t \in [0,T], 1 \leq i \leq 3 \}, \]
\[ m_{i,T} = \sup \{ m_i(t,s) : t,s \in [0,T] \}, \]
\[ \partial_T(\varepsilon) = \sup \{ g(t,s,v,w,x) - g(t,s,v,w,y) : t,s \in [0,T], v,w \in [0,\beta_T], x,y \in [-b,b], |x-y| \leq \varepsilon \}. \]

with $b = \|x\| + \varepsilon$. By using the continuity of $g_1$ on the compact set $[0,T] \times [0,T] \times [0,\beta_T] \times [-b,b]$, we have $\partial_T(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $G_1$ is a continuous function on $BC(R_+ \times R_+)$. To complete the proof we need to verify that $G_1$ is a compact operator. Let $X$ be a nonempty and complete the proof we need to verify that $1$

This result together condition (iii) imply that there exists $T > 0$ such that for $t,s > T$ we have

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq \varepsilon, \]

and if $t,s \in [0,T]$, then the inequality in (12) follows that

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq m_{i,T} \beta_T^2 \partial_T(\varepsilon), \]

where

\[ \beta_T = \sup \{ \beta_i(t) : t \in [0,T], 1 \leq i \leq 3 \}, \]
\[ m_{i,T} = \sup \{ m_i(t,s) : t,s \in [0,T] \}, \]
\[ \partial_T(\varepsilon) = \sup \{ g(t,s,v,w,x) - g(t,s,v,w,y) : t,s \in [0,T], v,w \in [0,\beta_T], x,y \in [-b,b], |x-y| \leq \varepsilon \}. \]

with $b = \|x\| + \varepsilon$. By using the continuity of $g_1$ on the compact set $[0,T] \times [0,T] \times [0,\beta_T] \times [-b,b]$, we have $\partial_T(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $G_1$ is a continuous function on $BC(R_+ \times R_+)$. To complete the proof we need to verify that $G_1$ is a compact operator. Let $X$ be a nonempty and complete the proof we need to verify that $1$

This result together condition (iii) imply that there exists $T > 0$ such that for $t,s > T$ we have

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq \varepsilon, \]

and if $t,s \in [0,T]$, then the inequality in (12) follows that

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq m_{i,T} \beta_T^2 \partial_T(\varepsilon), \]

where

\[ \beta_T = \sup \{ \beta_i(t) : t \in [0,T], 1 \leq i \leq 3 \}, \]
\[ m_{i,T} = \sup \{ m_i(t,s) : t,s \in [0,T] \}, \]
\[ \partial_T(\varepsilon) = \sup \{ g(t,s,v,w,x) - g(t,s,v,w,y) : t,s \in [0,T], v,w \in [0,\beta_T], x,y \in [-b,b], |x-y| \leq \varepsilon \}. \]

with $b = \|x\| + \varepsilon$. By using the continuity of $g_1$ on the compact set $[0,T] \times [0,T] \times [0,\beta_T] \times [-b,b]$, we have $\partial_T(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $G_1$ is a continuous function on $BC(R_+ \times R_+)$. To complete the proof we need to verify that $G_1$ is a compact operator. Let $X$ be a nonempty and complete the proof we need to verify that $1$

This result together condition (iii) imply that there exists $T > 0$ such that for $t,s > T$ we have

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq \varepsilon, \]

and if $t,s \in [0,T]$, then the inequality in (12) follows that

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq m_{i,T} \beta_T^2 \partial_T(\varepsilon), \]

where

\[ \beta_T = \sup \{ \beta_i(t) : t \in [0,T], 1 \leq i \leq 3 \}, \]
\[ m_{i,T} = \sup \{ m_i(t,s) : t,s \in [0,T] \}, \]
\[ \partial_T(\varepsilon) = \sup \{ g(t,s,v,w,x) - g(t,s,v,w,y) : t,s \in [0,T], v,w \in [0,\beta_T], x,y \in [-b,b], |x-y| \leq \varepsilon \}. \]

with $b = \|x\| + \varepsilon$. By using the continuity of $g_1$ on the compact set $[0,T] \times [0,T] \times [0,\beta_T] \times [-b,b]$, we have $\partial_T(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $G_1$ is a continuous function on $BC(R_+ \times R_+)$. To complete the proof we need to verify that $G_1$ is a compact operator. Let $X$ be a nonempty and complete the proof we need to verify that $1$

This result together condition (iii) imply that there exists $T > 0$ such that for $t,s > T$ we have

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq \varepsilon, \]

and if $t,s \in [0,T]$, then the inequality in (12) follows that

\[ |G_1(x)(t,s) - G_1(y)(t,s)| \leq m_{i,T} \beta_T^2 \partial_T(\varepsilon), \]

where

\[ \beta_T = \sup \{ \beta_i(t) : t \in [0,T], 1 \leq i \leq 3 \}, \]
\[ m_{i,T} = \sup \{ m_i(t,s) : t,s \in [0,T] \}, \]
\[ \partial_T(\varepsilon) = \sup \{ g(t,s,v,w,x) - g(t,s,v,w,y) : t,s \in [0,T], v,w \in [0,\beta_T], x,y \in [-b,b], |x-y| \leq \varepsilon \}. \]

with $b = \|x\| + \varepsilon$. By using the continuity of $g_1$ on the compact set $[0,T] \times [0,T] \times [0,\beta_T] \times [-b,b]$, we have $\partial_T(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $G_1$ is a continuous function on $BC(R_+ \times R_+)$. To complete the proof we need to verify that $G_1$ is a compact operator. Let $X$ be a nonempty and complete the proof we need to verify that $1$
On the other hand, by the uniform continuity of $g_i$ and $\beta_i$ on the compact sets $[0, T] \times [0, T] \times [0, \beta_T] \times [-r, r]$ and $[0, T]$, respectively, we have $\omega^T_i (g, \epsilon) \to 0$ and $\omega^T_i (\beta, \epsilon) \to 0$ as $\epsilon \to 0$. Therefore, we obtain $\omega^T_0 (G_1(X), \epsilon) = 0$, which gives

$$\omega_0 (G_1(X)) = 0.$$  \hspace{1cm} (14)

Moreover, for arbitrary $x, y \in X$ and $t, s \in R_+$, we have the following estimate

$$|G_1(x)(t,s) - G_1(y)(t,s)| \leq m_{1,T} \theta(t,s),$$

where

$$\theta(t,s) = \sup \left\{ \int_0^{\beta_1} \left| g_1(t,s,v,w,x(\eta(v),\eta(w))) - g_1(t,s,v,w,y(\eta(v),\eta(w))) \right| dv dw \mid x, y \in BC (R_+ \times R_+) \right\}.$$  \hspace{1cm} (15)

Thus, we obtain

$$\text{diam} \ G_1(X)(t,s) \leq m_{1,T} \theta(t,s).$$  \hspace{1cm} (16)

Taking $t, s \to \infty$ in the inequality (15), then using (iv) we arrive at

$$\limsup_{t, s \to \infty} \text{diam} \ G_1(X)(t,s) = 0.$$  \hspace{1cm} (17)

or, equivalently

$$\mu(G_1(X)) = 0.$$  \hspace{1cm} (18)

So, it is a $G_1$ compact operator and the proof is complete.

**Lemma 2.3.** Assume that $g_2$ satisfies the hypothesis (v), then $G_2 : BC(R_+ \times R_+) \to BC(R_+ \times R_+)$ defined by

$$G_2(x)(t,s) = m_2(T) \int_0^{r_{(i)}} g_2(t,s,v,x(\zeta(v),\zeta(s)))dv$$  \hspace{1cm} (19)

is a compact and continuous operator.

**Proof:** Obviously, for any $x \in BC(R_+ \times R_+)$, $G_2(x)(t,s)$ is a continuous function and by (6), $G_2$ is a self operator on $BC(R_+ \times R_+)$. Similar to the proof of Lemma 2.2 we deduce that $G_2$ is continuous,

$$\omega^T_i (G_2(X), \epsilon) \leq m_{2,T} \left( \beta_i \omega^T_i (g, \epsilon) + \epsilon_i \omega^T_i (\beta, \epsilon) \right).$$  \hspace{1cm} (20)

and

$$\text{diam} \ G_2(X)(t,s) \leq m_{2,T} \varphi(t,s)$$

where

$$m_{2,T} = \sup \left\{ m_2(t,s) : t, s \in [0, T] \right\},$$

and $r_i \omega^T_i (\beta_i, \epsilon)$ are as defined in the proof of Lemma 2.2. So by the uniform continuity of $g_2$ and $\beta_i$ on the compact sets $[0, T] \times [0, T] \times [0, \beta_T] \times [-r, r]$ and $[0, T]$, respectively, we obtain $\omega^T_0 (g_2, \epsilon) \to 0$ and $\omega^T_i (\beta, \epsilon) \to 0$ as $\epsilon \to 0$, gives

$$\omega_0 (G_2(X)) = 0.$$  \hspace{1cm} (21)

Also, taking $t, s \to \infty$ in the inequality (19), then using (v) we arrive at

$$\limsup_{t, s \to \infty} \text{diam} \ G_2(X)(t,s) = 0.$$  \hspace{1cm} (22)

Thus,

$$\mu(G_2(X)) = 0.$$  \hspace{1cm} (23)
So, it is a $G_2$ compact operator and the proof is complete.

Now we are in a position to present the main result of this paper.

**Theorem 2.4.** Under the assumptions (i)-(v), Eq. (1) has at least one solution in $BC(R_+ \times R_+)$. 

**Proof:** Define the operators $F, T : BC(R_+ \times R_+) \to BC(R_+ \times R_+)$ by the formulas 

$$F(x)(t,s) = x(t,s)$$

and

$$T(x)(t,s) = f\left( t, s, x(\varphi(t), \varphi(s)), \int_0^\beta \int_0^{\beta_2} g_1(t, s, v, w, x(\eta(t), \eta_2)(w)) dv dw, \int_0^\beta_1 g_2(t, s, v, x(\zeta(t), \zeta_2)(v)) dv \right)$$

(21)

Using conditions (i)-(iv), for arbitrarily fixed $t \in R_+$ we have

$$|T(x)(t,s)| \leq k |x(t,s)| + \Phi_1 \left( \int_0^{\beta(t)} \int_0^{\beta_1} g_1(t, s, v, w, x(\eta(t), \eta_2)(w)) dv dw \right) + \Phi_2 \left( \int_0^{\beta_1} g_2(t, s, v, x(\zeta(t), \zeta_2)(v)) dv \right)$$

where

$$A = \int_0^{\beta(t)} \int_0^{\beta_1} g_1(t, s, v, w, x(\eta(t), \eta_2)(w)) dv dw$$

$$B = \int_0^{\beta_1} g_2(t, s, v, x(\zeta(t), \zeta_2)(v)) dv$$

Thus,

$$\|F(x)\| \leq k \|x\| + M + \Phi_1(D_1) + \Phi_2(D_2).$$

(22)

and $T(x) \in BC(R_+ \times R_+)$ for any $x \in BC(R_+ \times R_+)$. 

Inequality (22) yields that $T$ transforms the ball $B_{R_+}$ into itself where

$$M + \Phi_1(D_1) + \Phi_2(D_2).$$

Also, applying (4) and taking into account the definitions of $G_1, G_2, F$ and $T$ we obtain

$$|T(x)(t,s) - T(y)(t,s)| \leq |F(x)(t,s) - F(y)(t,s)|$$

$$+ \sum_{i=1}^2 \Phi_i \left( |G_i(x)(t,s) - G_i(y)(t,s)| \right)$$

Thus, $T$ satisfies (8) and by Lemma 2.1, $T$ has a fixed point.

By a similar reasoning, one can derive the following consequences of Lemmas 2.1 and 2.2.

**Theorem 2.5.** Assume that the following conditions are satisfied:

i. $\zeta_i, \eta_i, \beta_i, \zeta_i : R_+ \to R_+$ (i = 1, 2) are continuous and $\zeta_i(t) \to \infty$ as $t \to \infty$ for $i = 1, 2$.

ii. $f : R_+ \times R_+ \times R_+ \times R_+ \times R_+ \to R$ is continuous. Moreover, there exist a constant $k \in [0,1)$ and nondecreasing continuous functions $\Phi_1, \Phi_2 : R_+ \to R$ with $\Phi_i(0) = 0$ for $i = 1, 2$ such that

$$|f(t, s, x, y, v) - f(t, s, u, z, w)| \leq k |x - u|$$

$$+ \Phi_1(m_1(t, s) |y - z|) + \Phi_2(m_2(t, s) |v - w|)$$

where $m_i : R_+ \times R_+ \to R_+$ (i = 1, 2) is a continuous function.

iii. $M := \sup \{|f(t, s, 0, 0, 0)| : t, s \in R_+ \} < \infty$.

iv. $g_i : R_+ \times R_+ \times R_+ \times R_+ \times R_+ \to R$ (i = 1, 2) are continuous and

$$D \geq \sup \left\{ \int_0^{\beta(t)} \int_0^{\beta_1} g_i(t, s, v, w, x(\eta(t), \eta_2)(w)) \ dv \ dw \right\}$$

where

$$m_i(t, s) = \sup \left\{ \int_0^{\beta(t)} \int_0^{\beta_1} g_i(t, s, v, w, x(\eta(t), \eta_2)(w)) \ dv \ dw \right\}$$

Moreover,

$$\lim_{t, s \to \infty} \left\{ \int_0^{\beta(t)} \int_0^{\beta_1} g_i(t, s, v, w, x(\eta(t), \eta_2)(w)) \ dv \ dw \right\} = 0$$

uniformly with respect to $x, y \in BC(R_+ \times R_+)$. Then the functional integral
\begin{align*}
x(t,s) &= (t,s,x(\xi(t),\theta(t)), \\
\int_0^{\beta(t)} \int_0^{\eta(t)} g_1(t,s,v,w), \\
\int_0^{\beta(t)} \int_0^{\eta(t)} g_2(t,s,v, \\
\int_0^{\beta(t)} \int_0^{\eta(t)} x(\eta(t),\eta(t)))dvdw, \\
\int_0^{\beta(t)} \int_0^{\eta(t)} x(\eta(t),\eta(t)))dvdw,
\end{align*}

has at least one solution in the space \( BC(\mathbb{R}_+ \times \mathbb{R}_+) \).

Proof: Similar to the proof of Theorem 2.4, consider \( F,G_1,G_2,T : BC(\mathbb{R}_+ \times \mathbb{R}_+) \rightarrow BC(\mathbb{R}_+ \times \mathbb{R}_+) \) by the formulas

\[
F(x)(t,s) = x(t,s),
G_1(x)(t,s) = m(t,s)\int_0^{\beta(t)} \int_0^{\eta(t)} g_1(t,s,v,w, \\
G_2(x)(t,s) = m(t,s)\int_0^{\beta(t)} \int_0^{\eta(t)} g_2(t,s,v, \\
T(x)(t,s) = \left\{
\begin{array}{ll}
\int_0^{\beta(t)} \int_0^{\eta(t)} g_1(t,s,v,w, \\
\int_0^{\beta(t)} \int_0^{\eta(t)} g_2(t,s,v, \\
\int_0^{\beta(t)} \int_0^{\eta(t)} x(\eta(t),\eta(t)))dvdw,
\end{array}
\right.
\end{align*}

Then by applying Lemma 2.1, we see that (23) has at least one solution in the space \( BC(\mathbb{R}_+ \times \mathbb{R}_+) \).

3. Examples

In this section, we provide two examples to show the efficiency of the main results.

Example 3.1. Consider the following functional integral equation

\[
x(t,s) = \frac{t^2 s^2 + 1 + x(t,s)}{t^2 s^2 + 1} \frac{\sin(ts)}{t^2 s^2 + 1} + \arctan \left( \int_0^{\beta(t)} e^x \sin(x'(w,v))dv \right)
\]

for any \( x,y \in BC(\mathbb{R}_+ \times \mathbb{R}_+) \), which implies that condition (iv) is satisfied. Next we estimate

\[
M := \sup \{ |f(t,s,0,0,0)| : t,s \in \mathbb{R}_+ \}
\]

and condition (iii) of Theorem 2.4 is valid. Then by Theorem 2.7, the integral equation (24) has at least one solution in \( BC(\mathbb{R}_+ \times \mathbb{R}_+) \).

Example 3.2. Consider the following functional integral equation

\[
x(t,s) = \frac{3x(t,s)}{4s^2 + 1} + \int_0^{\beta(t)} u^3 \cos(u^2,s)) + e^x (2 + \sin(x'(u^2,s)))du
\]

for any \( x \in BC(\mathbb{R}_+ \times \mathbb{R}_+) \), which implies that condition (iv) is satisfied. Next we estimate

\[
M := \sup \{ |f(t,s,0,0,0)| : t,s \in \mathbb{R}_+ \}
\]

and condition (iii) of Theorem 2.4 is valid. Then by Theorem 2.4, the integral equation (24) has at least one solution in \( BC(\mathbb{R}_+ \times \mathbb{R}_+) \).

From the definitions of \( \xi_i, \eta_i, \beta_i \) and \( g_2 \), it is easy to see that conditions (i) and (v) of Theorem 2.4 are valid and taking \( m(t,s) = \frac{\sin(ts)}{e^x} \), \( k = \frac{1}{2} \) and \( \Phi(t) = t \), we can find that \( f, m \) and \( \Phi \) satisfy condition (ii) of Theorem 2.4. Also, \( g_1 \) is continuous on \( \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \) and

\[
g_2(t,s,v,w,x) = 0
\]
\( \xi_1(t) = \xi_2(t) = \xi_3(t) = t, \quad \beta_1(t) = t^2, \quad \xi_4(t) = t^t \)

\[ f(t, s, x, y, z) = \frac{3ts}{4s + 1} x + y + z \]

\[ g_1(t, s, v, w, x) = 0 \]

\[ g_2(t, s, u, x) = \frac{v^3 \cos(ux) + e^v (2 + \sin(x^4))}{e^{v^2} (2 + \sin(x^4))} \]

Moreover, we have bounded so condition (iii) of Theorem 2.4 is valid.

Then by Theorem 2.4, the integral equation (25) has at least one solution in \((R^\| R^\|)\)

and thus condition (v) of Theorem 2.4 is satisfied.

Now we check all conditions of Theorem 2.4. By the definitions of \( \xi_1, \xi_2, \beta_3 \) and \( g_1 \), it is easily seen that conditions (i), (ii) and (iv) are satisfied with \( k = \frac{3}{4} \) and also \( f(t, s, 0, 0, 0) = 0 \) is bounded so condition (iii) of Theorem 2.4 is valid. Moreover, we have

\[
\left| \frac{u^3 \cos(ux(\sqrt{u}, s)) + e^v (2 + s \sin(x^4(\sqrt{u}, s)))}{e^{v^2} (2 + \sin(x^4(\sqrt{u}, s)))} \right| 
\]

\[
\leq \frac{2u^3}{e^{v^2}} 
\]

Thus,

\[ D_2 = \sup_{t, s \in R, x, y \in BC(R \times R_+) \{ u^3 \cos(ux(\sqrt{u}, s)) + e^v (2 + s \sin(x^4(\sqrt{u}, s))) \} e^{v^2} (2 + \sin(x^4(\sqrt{u}, s))) } \int_0^x \left| \frac{u^3 \cos(ux(\sqrt{u}, s)) + e^v (2 + s \sin(x^4(\sqrt{u}, s)))}{e^{v^2} (2 + \sin(x^4(\sqrt{u}, s)))} \right| dt < x. \]

uniformly with respect to \( x, y \in BC(R_+ \times R_+) \) and thus condition (v) of Theorem 2.4 is satisfied. Then by Theorem 2.4, the integral equation (25) has at least one solution in \( BC(R_+ \times R_+) \).

References


