Some kinds of \((e, e \vee q_{\delta})\)-fuzzy ideals of ternary semigroups

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Abstract

Generalizing the concepts of \((e, e \vee q)\)-fuzzy (left, right, lateral) ideals, \((e, e \vee q)\)-fuzzy quasi-ideals and \((e, e \vee q)\)-fuzzy bi (generalized bi-) ideals in ternary semigroups, the notions of \((e, e \vee q_{\delta})\)-fuzzy (left, right, lateral) ideals, \((e, e \vee q_{\delta})\)-fuzzy quasi-ideals and \((e, e \vee q_{\delta})\)-fuzzy bi (generalized bi-) in ternary semigroups are introduced and several related properties are investigated. Some new results are obtained.

Keywords: Ternary semigroups; \((e, e \vee q_{\delta})\)-fuzzy ideals; \((e, e \vee q_{\delta})\)-fuzzy quasi-ideals; \((e, e \vee q_{\delta})\)-fuzzy bi-ideals

1. Introduction

Lehmer in 1932 introduced the notion of ternary semigroup in his pioneering paper [1]. To develop the theory of ternary semigroups, ideal theory plays an important role. Sioson in [2], studied ternary semigroups with a special reference of ideals and radicals and characterized regular ternary semigroups by the properties of quasi-ideals. Dixit and Dewan studied quasi-ideals and bi-ideals of ternary semigroups [3]. Zadeh introduced the concept of fuzzy subset of a set in his seminal paper [4]. There has been rapid growth worldwide in the interest of fuzzy set theory and its applications from the past two decades. An extensive account of applications of fuzzy set theory have been found in different fields such as expert system, coding theory, computer science, artificial intelligence, control engineering, information sciences, operation research, robotics etc. After the introduction of fuzzy set by Zadeh, reconsideration of the concepts of classical mathematics began. On the other hand, because of the importance of group theory in mathematics as well as its applications in many areas, the notion of fuzzy subgroup was introduced by Rosenfeld [5]. Kuroki is responsible for much of the fuzzy ideal theory of semigroups (see [6-10]). The monograph by Mordeson et al. concentrates on the theory of fuzzy semigroups and its applications in fuzzy coding theory, fuzzy finite state machines and fuzzy languages (see [11]). Pu and Liu in [12], gave the idea of quasi-coincidence of a fuzzy point with a fuzzy set. Murali [13], introduced the notion of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. Bhakat in [14, 15], and Bhakat and Das [16, 17], introduced the concept of \((\alpha, \beta)\)-fuzzy subgroup by using the combined notions of `belongingness' and `quasi-coincidence' between a fuzzy point and a fuzzy subset and introduced the concept of \((e \vee q)\)-level subset, \((e, e \vee q)\)-fuzzy normal, quasinormal, maximal subgroup and \((e, e \vee q)\)-fuzzy subgroup.

In fact, \((e, e \vee q)\)-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems with other algebraic structures. Davvaz defined \((e, e \vee q)\)-fuzzy subnear rings and ideals of a near ring in his paper [18]. Kazanci and Yamak in [19], discussed \((e, e \vee q)\)-fuzzy bi-ideals of a semigroup. In [20], Shabir et al. characterized...
regular semigroups by \((e_i \in \mathbb{R})\)-fuzzy ideals. Dudek et al. in [21] and Ma and Zhan in [22] defined \((\alpha, \beta)\)-fuzzy ideals in hemirings, and investigated some related properties of hemirings. Akram in [23], studied \((e_i \in \mathbb{R})\)-fuzzy \(K\)-algebras. Zhan and Yin considered \((e_i \in \mathbb{R})\)-fuzzy subnear-rings and ideals of a near-ring in [24]. Ma et al. in [25], introduced the concept of \((e_i \in \mathbb{R})\)-fuzzy ideals of BCI-algebras. Recently Shabir and Ali characterized semigroups by the properties of their \((e_i \in \mathbb{R})\)-fuzzy ideals [26].

In the current paper we initiate the study of \((e_i \in \mathbb{R})\)-fuzzy ternary subsemigroup, \((e_i \in \mathbb{R})\)-fuzzy left (right, lateral) ideals, \((e_i \in \mathbb{R})\)-fuzzy quasi-ideals, \((e_i \in \mathbb{R})\)-fuzzy bi (generalized bi-) ideals of ternary semigroups and investigated several related properties.

2. Preliminaries

A ternary semigroup is an algebraic structure \((T, [\cdot, \cdot, \cdot])\) such that \(T\) is a non-empty set and \([\cdot, \cdot, \cdot] : T \times T \times T \to T\) a ternary operation satisfying the associative law: \(\langle x, y, z \rangle = \langle y, z, u \rangle = \langle x, y, \langle z, u \rangle \rangle\) for all \(x, y, z, u \in T\). For the sake of simplicity we write \([xyz]\) as \(x\cdot y\cdot z\) and consider the ternary operation as multiplication. A non-empty subset \(A\) of a ternary semigroup \(T\) is called a ternary subsemigroup of \(T\) if \(AAA \subseteq A\). By a left (right, lateral) ideal of a ternary semigroup \(T\) we mean a non-empty subset \(A\) of \(T\) such that \(TTA \subseteq A\) \((ATT \subseteq A, TAT \subseteq A\). If a non-empty subset \(A\) of \(T\) is a left and right ideal of \(T\), then it is called an ideal of \(T\). A non-empty subset \(A\) of a ternary semigroup \(T\) is called a quasi-ideal of \(T\) if \(ATT \cap TAT \subseteq A\) and \(ATT \cap TTA \subseteq A\). A non-empty subset \(A\) of a ternary semigroup \(T\) is called a generalized bi-ideal of \(T\) if \(ATT \cap TTA \subseteq A\) and \(ATT \cap TAT \subseteq A\). A fuzzy subset \(f\) of a universe \(X\) of the form

\[
\mathcal{F}(X) = \{e_i \in \mathbb{R} \mid f(x) \geq t\}
\]

is said to be a fuzzy point with support \(x\) and value \(t\) and is denoted by \(x_t\). For a fuzzy point \(x_t\), a fuzzy set \(f\) in a set \(X\), Pu and Liu [12] gave meaning to the symbol \(x_{t, \alpha f}\), where \(\alpha \in \{e_i \in \mathbb{R} \mid f(x) \geq t\} \subseteq \mathbb{R}\). A fuzzy point \(x_t\) is said to belong to (resp. be quasi-coincident with) a fuzzy set \(f\) written \(x \in f\) \((\text{resp. } x_{t, \alpha f})\) if \(f(x) \geq t\) \((\text{resp. } f(x) + t > 1)\), and in this case, \(x_t \in \mathbb{R}\) \((\text{resp. } x_t \in \mathbb{R}\) means that \(x_t \in f\) \((\text{resp. } x_t \in f)\). By \(x_{t, \alpha f}\), we mean that \(x_t \in f\) does not hold.

For any two fuzzy subsets \(f\) and \(g\) of \(T\), \(f \leq g\) means that, for all \(x \in T\), \(f(x) \leq g(x)\). The symbols \(f \wedge g\) and \(f \vee g\) will mean the following fuzzy subsets of \(T\):

\[
\begin{align*}
(f \wedge g)(x) & = f(x) \wedge g(x) \\
(f \vee g)(x) & = f(x) \vee g(x)
\end{align*}
\]

Let \(f, g, h\) be three fuzzy subsets of a ternary semigroup \(T\). The product \(f \circ g \circ h\) is a fuzzy subset of \(T\) defined by:

\[
(f \circ g \circ h)(x) = \begin{cases} 
(f \circ g \circ h)(x) & \text{if there exist } x, y, z \in T \text{ such that } \\
0 & \text{otherwise.}
\end{cases}
\]

Let \(f\) be a fuzzy subset of a ternary semigroup \(T\). Then the set

\[
U(f; t) = \{x \in T \mid f(x) \geq t\},
\]

is called a quasi-ideal of \(T\) if \(ATT \cap TTA \subseteq A\). A ternary subsemigroup of a ternary semigroup \(T\) is called bi-ideal if \(A\) is a generalized bi-ideal of \(T\). It is clear that every left (right, lateral) ideal of \(T\) is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal of \(T\).

A fuzzy subset \(f\) of a universe \(X\) is a function from \(X\) into the unit closed interval \([0,1]\), that is, \(f : X \to [0,1]\).

A fuzzy subset \(f\) of a universe \(X\) of the form

\[
f(y) = \begin{cases} 
t \in (0,1] & \text{if } y = x \\
0 & \text{if } y \neq x
\end{cases}
\]

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where $t \in [0,1]$, is called a level subset of $f$.

**Definition 1.** A fuzzy subset $f$ of a ternary semigroup $T$ is a fuzzy ternary subsemigroup of $T$ if $f(xyz) \geq f(x) \land f(y) \land f(z)$ for all $x,y,z \in T$.

**Definition 2.** A fuzzy subset $f$ of a ternary semigroup $T$ is a fuzzy left (right, lateral) ideal of $T$ if $f(xyz) \geq f(x) \land f(y) \land f(z)$ for all $x,y,z,u,v \in T$, and is called a fuzzy bi-ideal of $T$ if it is both a fuzzy ternary subsemigroup and a fuzzy generalized bi-ideal of $T$.

If $A \subseteq T$, then the characteristic function of $A$ is a function $C_A$ of $T$ onto $\{0,1\}$ defined by:

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For a fuzzy point $x_i$ and a fuzzy subset $f$ of a ternary semigroup $T$, we say that:

1. $x_i \in_\gamma f$ if $f(x) \geq t > \gamma$
2. $x_i \in_\delta f$ if $f(x) + t > 2\delta$
3. $x_i \notin_\gamma \land_\delta f$ if $x_i \in_\gamma f \lor x_i \notin_\delta f$
4. $x_i \notin_\gamma \lor_\delta f$ if $x_i \in_\gamma f \land x_i \notin_\delta f$
5. $x_i \notin_\alpha f$ means that $x_i \in_\gamma f$ does not hold.

3. $(\alpha, \beta)$-fuzzy ideals

Throughout this paper $\gamma, \delta \in [0,1]$, where $\gamma < \delta$, $\alpha, \beta \in \{\in_\gamma, \land_\beta, \lor_\beta, \lor_\gamma, \land_\delta\}$ and $\alpha \neq \in_\gamma \land_\delta$.

We omit the case of $\alpha = \in_\gamma \land_\delta$, because if $f$ is a fuzzy subset of a ternary semigroup $T$ such that $f(x) \leq \delta$ for all $x \in T$ and $t \in [0,1]$ be such that $x_i \in_\gamma \land_\delta f$, then $f(x) \geq t > \gamma$ and $f(x) + t > 2\delta$. This implies that $2\delta < f(x) + t \leq f(x) + f(x) = 2f(x)$.

$2\delta < f(x) + t \leq f(x) + f(x) = 2f(x)$ which implies $f(x) > \delta$. Thus $\{x_i : x_i \in_\gamma \land_\delta f\} = \emptyset$.

**Definition 4.** A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\alpha, \beta)$-fuzzy ternary subsemigroup of $T$ if

$(T1)\quad x_\alpha f$, $y_\alpha f$ and $z_\alpha f$ implies $(xyz)_{\min[t,r,s]} \beta f$ for all $x,y,z \in T$ and $t,r,s \in (\gamma,1]$.

**Definition 5.** A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\alpha, \beta)$-fuzzy left (right, lateral) ideal of $T$, if it satisfies:

$(T2)\quad z_\alpha f$ implies $(xyz)_{\min[t,r,s]} \beta f$ for all $x,y,z \in T$ and $t \in (\gamma,1]$.

A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\alpha, \beta)$-fuzzy two sided ideal of $T$ if it is an $(\alpha, \beta)$-fuzzy left ideal and an $(\alpha, \beta)$-fuzzy right ideal of $T$. A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\alpha, \beta)$-fuzzy ideal of $T$ if it is an $(\alpha, \beta)$-fuzzy left ideal, $(\alpha, \beta)$-fuzzy right ideal and an $(\alpha, \beta)$-fuzzy lateral ideal of $T$.

**Definition 6.** A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\alpha, \beta)$-fuzzy generalized bi-ideal of $T$, if it satisfies:

$(T3)\quad x_\alpha f$, $y_\alpha f$ and $z_\alpha f$ implies $(xyz)_{\min[t,r,s]} \beta f$ for all $x,y,z,u,v \in T$ and $t,r,s \in (\gamma,1]$.

A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\alpha, \beta)$-fuzzy bi-ideal of $T$, if it satisfies $(T1)$ and $(T3)$. 

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Theorem 1. Let \( 2\delta = 1 + \gamma \) and \( f \) be an \((\alpha, \beta)\)-fuzzy ternary subsemigroup of a ternary semigroup \( T \). Then the set \( f_r = \{ x \in T : f(x) > \gamma \} \) is a ternary subsemigroup of \( T \).

Proof: Suppose \( x, y, z \in f_r \). Then \( f(x) > \gamma \) and \( f(y) > \gamma \). Assume that \( f(xyz) \leq \gamma \). If \( \alpha \in \{ e_\gamma, e_\gamma \land q_\delta \} \), then
\[
x_f(x)f, y_f(y)f, z_f(z)f
\]
but
\[
f(xyz) \leq \gamma < \min \{ f(x), f(y), f(z) \} \quad \text{and} \quad f(xyz)+\min \{ f(x), f(y), f(z) \} \leq \gamma + 1 = 2\delta.
\]
This implies that \( (xyz)_{\min(f(x),f(y),f(z))} \overline{\beta}f \) for every \( \beta \in \{ e_\gamma, q_\delta, e_\gamma \land q_\delta \} \), which is a contradiction. Hence \( f(xyz) > \gamma \), that is, \( xyz \in f_r \). Also, \( f(x) + \gamma > \gamma + \gamma = 2\delta \), \( f(y) + 1 + \gamma = 2\delta \) and \( f(z) + 1 + \gamma = 2\delta \). This implies that \( x, q_\delta f, y, q_\delta f, z, q_\delta f \), but
\[
f(xyz) \leq \gamma \quad \text{so} \quad (xyz)_{\overline{\epsilon}f} \quad \text{and} \quad f(xyz) + 1 \gamma + 1 = 2\delta, \quad \text{so} \quad (xyz)_{q_\delta f}, \quad \text{a contradiction. Hence} \quad f(xyz) > \gamma, \quad \text{that is,} \quad xyz \in f \). Therefore, \( f_r \) is a ternary subsemigroup of \( T \).

Theorem 2. Let \( 2\delta = 1 + \gamma \) and \( f \) be an \((\alpha, \beta)\)-fuzzy left (right, lateral, two sided) ideal of a ternary semigroup \( T \). Then the set \( f_r = \{ x \in T : f(x) > \gamma \} \), is a left (right, lateral, two sided) ideal of \( T \).

Proof: Let \( f \) be an \((\alpha, \beta)\)-fuzzy left ideal of \( T \) and \( z \in f_r \). Suppose there exist \( x, y \in T \) such that \( f(xyz) \leq \gamma \). Now, if \( \alpha \in \{ e_\gamma, e_\gamma \land q_\delta \} \), then
\[
z_{f(z)}f \quad \text{but} \quad f(xyz) \leq \gamma < f(z).
\]
This implies that \( (xyz)_{f(z)} \overline{\epsilon}f \). Also,
\[
f(xyz) + f(z) \leq \gamma + f(z) \leq \gamma + 1 = 2\delta, \quad \text{so} \quad (xyz)_{f(z)}q_\delta f. \quad \text{This implies that} \quad (xyz)_{f(z)} \overline{\beta}f
\]
for every \( \beta \in \{ e_\gamma, q_\delta, e_\gamma \land q_\delta \} \), which is a contradiction. Hence \( f(xyz) > \gamma \). This implies that \( xyz \in f_r \). If \( \alpha = q_\delta \), then \( z, q_\delta f \).
But \( f(xyz) \leq \gamma \), so \( (xyz)_{\overline{\epsilon}f} \) and \( f(xyz) + 1 \gamma + 1 = 2\delta \). So \( (xyz)_{q_\delta f} \). Thus \( (xyz)_{\overline{\beta}f} \) for every \( \beta \in \{ e_\gamma, q_\delta, e_\gamma \land q_\delta \} \), which is a contradiction. Hence \( f(xyz) > \gamma \).
This implies that \( xyz \in f_r \). Therefore, \( f_r \) is a left ideal of \( T \).

Theorem 3. (1) Let \( 2\delta = 1 + \gamma \) and \( f \) be an \((\alpha, \beta)\)-fuzzy generalized bi-ideal of a ternary semigroup \( T \). Then the set \( f_r = \{ x \in T : f(x) > \gamma \} \) is a generalized bi-ideal of \( T \).

(2) Let \( 2\delta = 1 + \gamma \) and \( f \) be an \((\alpha, \beta)\)-fuzzy bi-ideal of \( T \). Then the set \( f_r = \{ x \in T : f(x) > \gamma \} \) is a bi-ideal of \( T \).

Proof: The proof is similar to the proof of Theorem 1.

Theorem 4. Let \( 2\delta = 1 + \gamma \) and \( A \) be a non-empty subset of a ternary semigroup \( T \). Then \( A \) is a ternary subsemigroup of \( T \) if and only if the fuzzy subset \( f \) of \( T \) defined by:
\[
f(x) = \begin{cases} 
\geq \delta \text{ if } x \in A \\
\leq \gamma \text{ if } x \notin A,
\end{cases}
\]
is an \((\alpha, e_\gamma \land q_\delta)\)-fuzzy ternary subsemigroup of \( T \), where \( \alpha \in \{ e_\gamma, q_\delta, e_\gamma \land q_\delta \} \).

Proof: Let \( A \) be a ternary subsemigroup of \( T \).

(1) For \( \alpha = e_\gamma \). Let \( x, y, z \in T \) and \( t, r, s \in (\gamma, 1] \) be such that \( x, e_\gamma f \), \( y, e_\gamma f \) and \( z, e_\gamma f \). Then \( f(x) \geq t \gamma > \delta \), \( f(y) \geq r \gamma > \delta \) and \( f(z) \geq s \gamma > \delta \). Thus \( x, y, z \in A \). Since \( A \) is a ternary subsemigroup of \( T \), we have \( xyz \in A \), that is, \( f(xyz) \geq \delta \).
If \( \min \{t, r, s\} \leq \delta \), then \( f(xyz) \geq \delta \geq \min \{t, r, s\} \geq \gamma \), which implies that \( \langle xyz \rangle_{\min \{t, r, s\}} \in_{\gamma} f \).

If \( \min \{t, r, s\} > \delta \), then \( f(xyz) + \min \{t, r, s\} > \delta + \gamma = 2\delta \), which implies that \( \langle xyz \rangle_{\min \{t, r, s\}} \in_{\gamma} f \). Hence \( f \) is an \( \langle \epsilon_{\gamma}, \epsilon_{\gamma} \vee q_{\delta} \rangle \)-fuzzy ternary subsemigroup of \( T \).

Corollary 1. Let \( 2\delta = 1 + \gamma \) and \( \alpha \in \{\epsilon_{\gamma}, q_{\delta} \} \). Then a non-empty subset \( T \) of a ternary semigroup \( T \) is a ternary subsemigroup of \( T \) if and only if the characteristic function \( C_{\alpha} \) of \( T \) is an \( \langle \alpha_{\gamma}, \epsilon_{\gamma} \vee q_{\delta} \rangle \)-fuzzy ternary subsemigroup of \( T \).

In a similar manner we can prove the following:

Theorem 5. Let \( 2\delta = 1 + \gamma \) and \( \alpha \in \{\epsilon_{\gamma}, q_{\delta} \} \), and \( A \) be a non-empty subset of \( T \). Define a fuzzy subset \( f \) of \( T \) by:

\[
    f(x) = \begin{cases} 
    \geq \delta & \text{if } x \in A \\
    \leq \gamma & \text{if } x \notin A.
\end{cases}
\]

Then

(1) \( f \) is an \( \langle \alpha_{\gamma}, \epsilon_{\gamma} \vee q_{\delta} \rangle \)-fuzzy left (right, lateral, two sided) ideal of \( T \) if and only if \( A \) is a left (right, lateral, two sided) ideal of \( T \).

(2) \( f \) is an \( \langle \alpha_{\gamma}, \epsilon_{\gamma} \vee q_{\delta} \rangle \)-fuzzy bi-ideal (generalized bi-ideal) of \( T \) if and only if \( A \) is a bi-ideal (generalized bi-ideal) of \( T \).

Corollary 2. Let \( 2\delta = 1 + \gamma \) and \( \alpha \in \{\epsilon_{\gamma}, q_{\delta} \} \). If \( A \) is a non-empty subset of \( T \), then

(1) \( A \) is a left (right, lateral) ideal of \( T \) if and only if the characteristic function \( C_{\alpha} \) of \( A \) is a left (right, lateral) ideal of \( T \).

(2) \( A \) is a bi-ideal (generalized bi-ideal) of \( T \) if and only if the characteristic function \( C_{\alpha} \) of \( A \) is a bi-ideal (generalized bi-ideal) of \( T \).

Theorem 6. (1) Every \( \langle q_{\delta}, \epsilon_{\gamma} \vee q_{\delta} \rangle \)-fuzzy ternary subsemigroup of \( T \) is an \( \langle \epsilon_{\gamma}, \epsilon_{\gamma} \vee q_{\delta} \rangle \)-fuzzy ternary subsemigroup of \( T \).
(2) Every \((q_δ, ε_γ ∨ q_δ)\)-fuzzy left (right, lateral, two sided) ideal of \(T\) is an \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy left (right, lateral, two sided) ideal of \(T\).

(3) Every \((q_δ, ε_γ ∨ q_δ)\)-fuzzy bi-ideal (generalized bi-ideal) of \(T\) is an \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy bi-ideal (generalized bi-ideal) of \(T\).

Proof: We prove (1). The proofs of (2) and (3) are similar to (1).

(1) Let \(f\) be a \((q_δ, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroup of \(T\). Let \(x, y, z ∈ T\) and \(t, r, s ∈ (γ, 1]\) be such that \(x_i ∈ f\), \(y_i ∈ f\) and \(z_i ∈ f\).

Then \(f(x) ≥ t > γ\), \(f(y) ≥ r > γ\) and \(f(z) ≥ s > γ\). Suppose \((xyz)_{min(t, r, s)} ∈ ε_γ ∨ q_δ\). Then \(f(xyz) < min\{t, r, s\}\) and \(f(xyz) + min\{t, r, s\} ≤ 2δ\). This implies that \(f(xyz) + f(xyz) < f(xyz) + min\{t, r, s\} ≤ 2δ\).

This implies that \(f(xyz) < δ\). Now, max \(\{f(xyz), γ\} < min\{f(x), f(y), f(z), δ\}\).

Choose \(t_i ∈ (γ, 1]\) such that \(2δ − max \{f(xyz), γ\} ≥ t_i > 2δ − min\{f(x), f(y), f(z), δ\}\).

This implies that \(t_i > 2δ − f(x), t_i > 2δ − f(y), t_i > 2δ − f(z)\).

This implies that \(f(x) + t_i > 2δ, f(y) + t_i > 2δ, f(z) + t_i > 2δ\)

and \(2δ − f(xyz) > t_i\). This implies that \(f(xyz) + t_i < 2δ\) and \(f(xyz) < δ < t_i\). Thus \(x_i q_δ f, y_i q_δ f\) and \(z_i q_δ f\) but \((xyz)_{t_i} ∈ ε_γ ∨ q_δ f\), which is a contradiction.

Hence \(f\) is an \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroup of \(S\).

Theorem 7. (1) Every \((ε_γ ∨ q_δ, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroup of a ternary semigroup \(T\) is an \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroup of \(T\).

(2) Every \((ε_γ ∨ q_δ, ε_γ ∨ q_δ)\)-fuzzy left (right, lateral, two sided) ideal of a ternary semigroup \(T\) is an \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy left (right, lateral, two sided) ideal of \(T\).

(3) Every \((ε_γ ∨ q_δ, ε_γ ∨ q_δ)\)-fuzzy bi-ideal (generalized bi-ideal) of a ternary semigroup \(T\) is an \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy bi-ideal (generalized bi-ideal) of \(T\).

Proof: The proof follows from the fact that if \(x_i ∈ f\), then \(x_i ∈ ε_γ ∨ q_δ f\).

The above discussion shows that every \((α, β)\)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of a ternary semigroup \(T\) is an \((α, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of \(T\). Also, every \((α, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of \(T\) is an \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of \(T\). Thus in the theory of \((α, β)\)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of \(T\) plays a central role.

4. \((ε_γ ∨ q_δ, ε_γ ∨ q_δ)\)-fuzzy ideals

In this section we introduce the concepts of \((ε_γ ∨ q_δ, ε_γ ∨ q_δ)\)-fuzzy ternary subsemigroups, \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy left (right, lateral) ideals, \((ε_γ, ε_γ ∨ q_δ)\)-fuzzy bi-ideal (generalized bi-ideal)
of ternary semigroups and investigate some new results.

**Definition 7.** A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy ternary subsemigroup if $x_i, y_i, z_i \in f$ implies $(xyz)_{\min(t,r,s)} \in \varepsilon \vee \delta f$ for all $x, y, z \in T$ and $t, r, s \in (\gamma, 1]$.

**Definition 8.** A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy left (right, lateral) ideal of $T$ if it satisfies:

$$z_i \in f \implies (xyz)_{\min(t,r,s)} \in \varepsilon \vee \delta f$$

for all $x, y, z \in T$ and $t, r, s \in (\gamma, 1]$.

A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy generalized bi-ideal of $T$ if it satisfies:

$$z_i \in f \implies (xuvz)_{\min(t,r,s)} \in \varepsilon \vee \delta f$$

for all $x, y, z, u, v \in T$ and $t, r, s \in (\gamma, 1]$.

**Definition 9.** A fuzzy subset $f$ of a ternary semigroup $T$ is called an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy generalized bi-ideal of $T$ if it satisfies:

$$z_i \in f \implies (xuvz)_{\min(t,r,s)} \in \varepsilon \vee \delta f$$

for all $x, y, z, u, v \in T$ and $t, r, s \in (\gamma, 1]$.

**Theorem 8.** A fuzzy subset $f$ of a ternary semigroup $T$ is an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy ternary subsemigroup of $T$ if and only if

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\}.$$

**Proof:** Suppose $f$ is an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy ternary subsemigroup of a ternary semigroup $T$. Assume that

$$\max\{f(xy), \gamma\} < \min\{f(x), f(y), f(z), \delta\}.$$

Choose $t \in (\gamma, 1]$ such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), f(z), \delta\}.$$

Then $x_i \in f$, $y_i \in f$, and $z_i \in f$, but

$$(xyz)_{\min(t,r,s)} \in \varepsilon \vee \delta f$$

and

$$f(xyz) + t < \delta + \delta = 2\delta.$$

This implies that $(xyz)_{\min(t,r,s)} \in \varepsilon \vee \delta f$, which is a contradiction. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\}.$$

Conversely, assume that

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\}.$$

Let $x_i \in f$, $y_i \in f$, and $z_i \in f$. This implies that $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$ and $f(z) \geq t > \gamma$. Since

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\} \geq \min\{t, t, t, \delta\} = \min\{t, \delta\}.$$

If $t \leq \delta$, then $\max\{f(xy), \gamma\} \geq t$. But

$t > \gamma$. So $f(xy) \geq t > \gamma$, Thus $(xyz)_{\min(t,r,s)} \in \varepsilon$. If $t > \delta$, then $\max\{f(xy), \gamma\} \geq \delta$. But

$\gamma < \delta$. So $f(xy) \geq \delta$. This implies that

$f(xy) + t > \delta + \delta = 2\delta$. Thus $(xyz)_{\min(t,r,s)} \in \varepsilon \vee \delta f$.

Therefore, $f$ is an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy ternary subsemigroup of $T$.

If we put $\gamma = 0$ and $\delta = 0.5$ in Theorem 8, we get the following corollary.

**Corollary 3.** [27] A fuzzy subset $f$ of a ternary semigroup $T$ is an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy ternary subsemigroup of $T$ if and only if

$$f(xy) \geq \min\{f(x), f(y), f(z), 0.5\}$$

for all $x, y, z \in T$.

**Remark 1.** Every fuzzy ternary subsemigroup of a ternary semigroup $T$ is an $(\varepsilon, \varepsilon, \varepsilon \vee \delta, \delta)$-fuzzy ternary subsemigroup of $T$ but the converse is not true in general.

**Example 1.** Consider $T = \{-i, 0, i\}$, where $T$ is a ternary semigroup under the usual multiplication of complex numbers. Define a fuzzy subset $f$ of $T$ by:

$$f(-i) = 0.43, \quad f(i) = 0.35, \quad f(0) = 0.6.$$
Then routine calculations show that \( f \) is an \((\epsilon_{0.5,\epsilon_{0.5}} \lor q_{0.5})\)-fuzzy ternary subsemigroup of \( T \) but not a fuzzy ternary subsemigroup of \( T \).

**Theorem 9.** A fuzzy subset \( f \) of a ternary semigroup \( T \) is an \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy left (right, lateral) ideal of \( T \) if and only if

\[
\max \{ f(xyz), \gamma \} \geq \min \{ f(z), \delta \} \{ \min \{ f(x), \delta \}, \min \{ f(y), \delta \} \}.
\]

**Proof:** The proof is similar to the proof of Theorem 8.

If we put \( \gamma = 0 \) and \( \delta = 0.5 \) in Theorem 9, we get the following corollary.

**Corollary 4.** [27] A fuzzy subset \( f \) of a ternary semigroup \( T \) is an \((\epsilon, \epsilon \lor q)\)-fuzzy left (right, lateral) ideal of \( T \) if and only if

\[
f(xyz) \geq f(z) \lor 0.5(f(x) \lor 0.5, f(y) \lor 0.5)
\]

for all \( x, y, z \in T \).

**Example 2.** Let \( T \) be a ternary semigroup as defined in Example 1. Define a fuzzy subset \( f \) of \( T \) by:

\[
f(-i) = 0.40, \quad f(i) = 0.52, \quad f(0) = 0.31.
\]

It is now simple to verify that \( f \) is an \((\epsilon_{0.6,\epsilon_{0.6}} \lor q_{0.1})\)-fuzzy ideal of \( T \).

**Lemma 1.** The intersection of any number of \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy left (right, lateral) ideals of a ternary semigroup \( T \) is an \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy left (right, lateral) ideal of \( T \).

**Proof:** Let \( \{ f_i \}_{i=1} \) be a family of \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy left ideals of \( T \) and \( x, y, z \in T \). Then

\[
(\wedge f_i)(xyz) \lor \gamma \geq (\wedge f_i)(z) \lor \delta.
\]

This implies that \( (\wedge f_i)(xyz) \lor \gamma \geq (\wedge f_i)(z) \lor \delta \).

Hence \( (\wedge f_i) \) is an \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy left ideal of \( T \).

In a similar manner we can prove the following lemma.

**Lemma 2.** The union of any number of \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy left (right, lateral) ideals of a ternary semigroup \( T \) is an \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy left (right, lateral) ideal of \( T \).

**Definition 10.** Let \( f \) be a fuzzy subset of a ternary semigroup \( T \). We define:

\[
f_i = \{ x \in T : x \in f \} = \{ x \in T : f(x) \geq t \lor \gamma \} = U \{ f; t \},
\]

\[
f_i^\delta = \{ x \in T : x \in f_i \} = \{ x \in T : f(x) \geq t \lor 2\delta \},
\]

\[
[(f_i^\delta)^\gamma = \{ x \in T : x \in f_i \} \lor q_{\delta}, f \}.
\]

Clearly, \( [f_i^\delta] \equiv f_i \lor f_i^\delta \).

The next theorem provides the relationship between \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy ternary subsemigroup and the crisp ternary subsemigroup of \( T \).

**Theorem 10.** Let \( f \) be a fuzzy subset of a ternary semigroup \( T \). Then \( f \) is an \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy ternary subsemigroup of \( T \) if and only if \( f_i (\neq \phi) \) is a ternary subsemigroup of \( T \) for all \( t \in (\gamma, \delta] \).

**Proof:** Let \( f \) be an \((\epsilon_{\gamma},\epsilon_{\gamma} \lor q_{\delta})\)-fuzzy ternary subsemigroup of \( T \) and \( x, y, z \in f_i \) for some \( t \in (\gamma, \delta] \). Then

\[
f(x) \geq t \lor \gamma, \quad f(y) \geq t \lor \gamma, \quad f(z) \geq t \lor \gamma.
\]

Now, by hypothesis

\[
\max \{ f(xyz), \gamma \} \geq \min \{ f(x), \gamma \}, \min \{ f(y), \delta \} \geq \min \{ f(z), \delta \} = t.
\]

This implies that \( \max \{ f(xyz), \gamma \} \geq t \lor \gamma \). But \( t \lor \gamma \), so \( f(xyz) \geq t \lor \gamma \). This implies that \( xyz \in f_i \). Thus \( f_i \) is a ternary subsemigroup of \( T \).

Conversely, assume that \( \phi \neq f_i \) is a ternary subsemigroup of \( T \) for all \( t \in (\gamma, \delta] \). Suppose
there exist \( x, y, z \in T \) such that \( \max \{ f(xyz), \gamma \} < \min \{ f(x), f(y), f(z), \delta \} \). Choose \( t \in (\gamma, \delta) \) such that 
\[ \max \{ f(xyz), \gamma \} < t \leq \min \{ f(x), f(y), f(z), \delta \}. \]
Then \( x, y, z \in f_t \), but \( xyz \notin f_t \), which is a contradiction. Hence 
\[ \max \{ f(xyz), \gamma \} \geq \min \{ f(x), f(y), f(z), \delta \}. \]
Hence, \( f \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy ternary subsemigroup of \( T \).

In a similar manner we can prove the following theorems.

**Theorem 11.** A fuzzy subset \( f \) of a ternary semigroup \( T \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy left (right, lateral) ideal of \( T \) if and only if \( f_t(\neq \phi) \) is a left (right, lateral) ideal of \( T \) for all \( t \in (\gamma, \delta) \).

**Theorem 12.** A fuzzy subset \( f \) of a ternary semigroup \( T \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy bi-ideal (generalized bi-ideal) of \( T \) if and only if \( f_t(\neq \phi) \) is a bi-ideal (generalized bi-ideal) of \( T \) for all \( t \in (\gamma, \delta) \).

**Theorem 13.** Let \( 2\delta = 1 + \gamma \) and \( f \) be a fuzzy subset of a ternary semigroup \( T \). Then \( f \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy ternary subsemigroup of \( T \) if and only if \( f_t^\delta(\neq \phi) \) is a ternary subsemigroup of \( T \) for all \( t \in (\delta, 1] \).

**Proof:** Let \( f \) be an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy ternary subsemigroup of \( T \) and \( x, y, z \in f_t^\delta \). Then \( x, q_\delta f_t, y, q_\delta f_t, z, q_\delta f_t \). This implies that \( f(x) + t > 2\delta, f(y) + t > 2\delta, f(z) + t > 2\delta \). This implies that \( f(x) + t > 2\delta - t \geq 2\delta - 1 = \gamma \).

This implies that \( f(x) > \gamma \). Similarly, \( f(y) > \gamma \) and \( f(z) > \gamma \). Now, by hypothesis 
\[ \max \{ f(xyz), \gamma \} \geq \min \{ f(x), f(y), f(z), \delta \} \geq \min \{ 2\delta - t, 2\delta - t, 2\delta - t, 2\delta - t \}. \]
This implies that \( f(xyz) + t > 2\delta - t \). This implies that \( f(xyz) + t > 2\delta \). This implies that \( x, y, z \in f_t^\delta \). Hence \( f_t^\delta \) is a ternary subsemigroup of \( T \).

Conversely, assume that \( \phi \neq f_t^\delta \) is a ternary subsemigroup of \( T \) for all \( t \in (\delta, 1] \). Let \( x, y, z \in T \) be such that \( \max \{ f(xyz), \gamma \} < \min \{ f(x), f(y), f(z), \delta \} \). This implies that \( 2\delta - \min \{ f(x), f(y), f(z), \delta \} < 2\delta - \max \{ f(xyz), \gamma \} \).

Take \( r \in (\delta, 1] \) such that 
\[ \max \{ 2\delta - f(x), 2\delta - f(y), 2\delta - f(z), \delta \} < r \leq \min \{ 2\delta - f(xyz), 2\delta - \gamma \}. \]
Then \( 2\delta - f(x) < r, 2\delta - f(y) < r, 2\delta - f(z) < r, \) and \( r \leq 2\delta - f(xyz) \). This implies that \( f(x) + r > 2\delta, f(y) + r > 2\delta, f(z) + r > 2\delta \) and \( f(xyz) + r > 2\delta \). This implies that \( x, q_\delta f, y, q_\delta f, z, q_\delta f \) but \( (xyz), f_0 f \) which is a contradiction.

Thus \( f(xyz), \gamma \) \( \geq \min \{ f(x), f(y), f(z), \delta \} \).

Hence \( f \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy ternary subsemigroup of \( T \).

In a similar manner we can prove the following theorems.

**Theorem 14.** Let \( 2\delta = 1 + \gamma \) and \( f \) be a fuzzy subset of a ternary semigroup \( T \). Then \( f \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy left (right, lateral) ideal of \( T \) if and only if \( f_t^\delta(\neq \phi) \) is a left (right, lateral) ideal of \( T \) for all \( t \in (\delta, 1] \).

**Theorem 15.** Let \( 2\delta = 1 + \gamma \) and \( f \) be a fuzzy subset of a ternary semigroup \( T \). Then \( f \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy bi-ideal (generalized bi-ideal) of \( T \) if and only if \( f_t^\delta(\neq \phi) \) is a bi-ideal (generalized bi-ideal) of \( T \) for all \( t \in (\delta, 1] \).

**Theorem 16.** Let \( 2\delta = 1 + \gamma \) and \( f \) be a fuzzy subset of a ternary semigroup \( T \). Then \( f \) is an \( (\varepsilon, \varepsilon \vee q_\delta) \)-fuzzy ternary subsemigroup of \( T \).
if and only if \([f]^\delta_1(\neq \phi)\) is a ternary subsemigroup of \(T\) for all \(t \in (\gamma,1]\).

**Proof:** Let \(f\) be an \((\varepsilon,\varepsilon \lor q_{\delta})\)-fuzzy ternary subsemigroup of \(T\) and \(x,y,z \in [f]^\delta_1\). Then \(x, y, z \in [f]^\delta_1\). This implies that \(f(x)+t > \gamma\) or \(f(z)+t > \gamma\) and \(f(y)+t > \gamma\), and \(f(x)+t > 2\delta\), \(f(y)+t > 2\delta\), \(f(z)+t > 2\delta\). Thus \(f(x)+t > \gamma\) or \(f(y)+t > \gamma\) or \(f(z)+t > \gamma\) implies that \(f(x)+t > 2\delta\). Similarly, \(f(y)+t > 2\delta\) and \(f(z)+t > 2\delta\).

Conversely, assume that \([f]^\delta_1\) is a ternary subsemigroup of \(T\) for all \(t \in (\gamma,1]\). Let \(x,y,z \in T\) be such that \(\max \{f(x,y),\gamma\} < \min \{f(x),f(y),f(z),\delta\}\). Choose \(t \in (\gamma,1]\) such that \(\max \{f(x,y),\gamma\} < t \leq \min \{f(x),f(y),f(z),\delta\}\). Then \(x, y, z \in f\) such that \((x,y) \in \varepsilon, y, z \in \varepsilon, f\), but \((x,y) \in \varepsilon, \varepsilon \lor q_{\delta} f\), which is a contradiction. Hence \(f\) is an \((\varepsilon,\varepsilon \lor q_{\delta})\)-fuzzy ternary subsemigroup of \(T\).

In a similar manner we can prove the following theorems.

**Theorem 17.** Let \(2\delta = 1+\gamma\) and \(f\) be a fuzzy subset of a ternary semigroup \(T\). Then \(f\) is an \((\varepsilon,\varepsilon \lor q_{\delta})\)-fuzzy ideal (generalized bi-ideal) of \(T\) if and only if \([f]^\delta(\neq \phi)\) is a left (right, lateral) ideal of \(T\) for all \(t \in (\gamma,1]\).

**Theorem 18.** Let \(2\delta = 1+\gamma\) and \(f\) be a fuzzy subset of a ternary semigroup \(T\). Then \(f\) is an \((\varepsilon,\varepsilon \lor q_{\delta})\)-fuzzy ideal (generalized bi-ideal) of \(T\) if and only if \([f]^\delta(\neq \phi)\) is a bi-ideal (generalized bi-ideal) of \(T\) for all \(t \in (\gamma,1]\).

**Remark 2.** Every \((\varepsilon,\varepsilon \lor q_{\delta})\)-fuzzy ideal of a ternary semigroup \(T\) is an \((\varepsilon,\varepsilon \lor q_{\delta})\)-fuzzy ternary subsemigroup of \(T\) but the converse is not true in general.

**Example 3.** Consider the ternary semigroup \(T\) as in Example 1. Define a fuzzy subset \(f\) of \(T\) by:

\[
f(-i) = 0.60, \quad f(0) = 0.40, \quad f(i) = 0.87.
\]

Then

\[
U(f;\tau) = \begin{cases} 
T & \text{if } 0 \leq \tau < 0.48 \\
\{i, i\} & \text{if } 0.48 \leq \tau < 0.59 \\
\{i\} & \text{if } 0.59 \leq \tau < 0.86 \\
\phi & \text{if } 0.86 \leq \tau < 1.
\end{cases}
\]

It is now routine to verify that \(f\) is an \((\varepsilon_{0.48},\varepsilon_{0.48} \lor q_{0.59})\)-fuzzy ternary subsemigroup but not an \((\varepsilon_{0.48},\varepsilon_{0.48} \lor q_{0.59})\)-fuzzy ideal of \(T\).

**Definition 11.** A fuzzy subset \(f\) of a ternary semigroup \(T\) is called an \((\varepsilon,\varepsilon \lor q_{\delta})\)-fuzzy quasi-ideal of \(T\) if

\[
\max \{f(x,y),\gamma\} \geq \min \{f(x),f(y),f(z),\delta\}
\]

and

\[
\max \{f(x),\gamma\} \geq \min \{f(x \lor f, y \lor f), f(z \lor f), f(t \lor f), f(x), f(y), f(z), \delta\}
\]
for all $x \in T$, where $T$ is the fuzzy subset of $T$ mapping every element of $T$ on 1.

**Lemma 3.** A non-empty subset $Q$ of a ternary semigroup $T$ is a quasi-ideal of $T$ if and only if the characteristic function $C_Q$ of $Q$ is an $(\varepsilon_x, \varepsilon_x \vee q_\delta)$-fuzzy quasi-ideal of $T$.

**Proof:** Suppose $Q$ is a quasi-ideal of $T$ and $C_Q$, the characteristic function of $Q$. Let $x \in T$. If $x \notin Q$, then $x \notin TTQ$ or $x \notin TQT$ or $x \notin QTQ$. If $x \notin TTQ$ or $x \notin TQT$ or $x \notin QTQ$, then $(T \circ T \circ C_Q)(x) = 0$ or $(T \circ C_Q \circ T)(x) = 0$ or $(C_Q \circ T \circ T)(x) = 0$ and therefore,

$$
\min \{ (T \circ T \circ C_Q)(x), (T \circ C_Q \circ T)(x), (C_Q \circ T \circ T)(x), \delta \} = 0 \leq \max \{ C_Q(x), \gamma \}.
$$

If $x \in Q$, then

$$
\max \{ C_Q(x), \gamma \} = 1 \geq \min \{ (T \circ T \circ C_Q)(x), (T \circ C_Q \circ T)(x), (C_Q \circ T \circ T)(x), \delta \}.
$$

Similarly,

$$
\max \{ C_Q(x), \gamma \} = 1 \geq \min \{ (T \circ T \circ C_Q)(x), (T \circ C_Q \circ T)(x), (C_Q \circ T \circ T)(x), \delta \}.
$$

Hence, $C_Q$ is an $(\varepsilon_x, \varepsilon_x \vee q_\delta)$-fuzzy quasi-ideal of $T$.

Conversely, assume that $C_Q$ is an $(\varepsilon_x, \varepsilon_x \vee q_\delta)$-fuzzy quasi-ideal of $T$. We show that $Q$ is a quasi-ideal of $T$. Let $a \in TTQ \cap TQT \cap QTQ$. Then $a \in TTQ$ and $a \in TQT$, and $a \in QTQ$. This implies that there exist $x, y, z \in Q$ and $s_1, s_2, t_1, t_2, t_3 \in T$ such that $a = s_1 t_1 x$ and $a = s_2 y t_2$ and $a = z s_3 t_3$. Thus,

$$
(C_Q \circ T \circ T)(a) = \bigvee_{a \in T \cap a \in T} \{ C_Q(p) \land T(q) \land T(r) \} \geq C_Q(z) \land T(s_1) \land T(t_1) = C_Q(z) \land 1 \land 1 = C_Q(z) = 1.
$$

So $(C_Q \circ T \circ T)(a) = 1$. Similarly, $(T \circ T \circ C_Q)(a) = 1$ and $(T \circ C_Q \circ T)(a) = 1$. Now,

$$
\max \{ C_Q(a), \gamma \} \geq \min \{ (T \circ T \circ C_Q)(a), (T \circ C_Q \circ T)(a), (C_Q \circ T \circ T)(a), \delta \} = \gamma.
$$

Thus $C_Q(a) = 1$. This implies that $a \in Q$.

Therefore, $TTQ \cap TQT \cap QTQ \subseteq Q$. Similarly, we can show that $TTQ \cap TQT \cap QTQ \subseteq Q$. Hence, $Q$ is a quasi-ideal of $T$.

**Theorem 19.** Every $(\varepsilon_x, \varepsilon_x \vee q_\delta)$-fuzzy right (left, lateral) ideal of a ternary semigroup $T$ is an $(\varepsilon_x, \varepsilon_x \vee q_\delta)$-fuzzy quasi-ideal of $T$.

**Proof:** Let $f$ be an $(\varepsilon_x, \varepsilon_x \vee q_\delta)$-fuzzy right ideal of $T$ and $x \in T$. Then

$$
(f \circ T \circ T)(x) = \bigvee_{x \in T \cap x \in T} \{ f(u) \land T(v) \land T(w) \} = \bigvee_{x \in T \cap x \in T} f(u).
$$

This implies that

$$
(f \circ T \circ T)(x) \land \delta = \bigvee_{x \in T \cap x \in T} f(u) \land \delta = \bigvee_{x \in T \cap x \in T} f(u) \land \delta \leq \bigvee_{x \in T \cap x \in T} f(u) \land \gamma = f(x) \land \gamma.
$$

This implies that $f(x) \land \gamma \geq (f \circ T \circ T)(x) \land \delta$. This implies that

$$
\max \{ f(x), \gamma \} \geq \min \{ (f \circ T \circ T)(x), (T \circ T \circ f)(x), (T \circ C_Q \circ T)(x), \delta \}.
$$

Similarly,

$$
\max \{ f(x), \gamma \} \geq \min \{ (f \circ T \circ T)(x), (T \circ T \circ f)(x), (T \circ C_Q \circ T)(x), \delta \}.
$$

Hence $f$ is an $(\varepsilon_x, \varepsilon_x \vee q_\delta)$-fuzzy quasi-ideal of $T$.

Similarly, we can prove the case of left and lateral ideal of $T$.

The converse of Theorem 19 does not hold in general, as shown in the following example.

**Example 4.** Let

$$
T = \left\{ \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array} \right\},
$$

$$
\text{Example 4.}$$
where $T$ is a ternary semigroup under ternary matrix multiplication. Let $f$ be a fuzzy subset of $T$ defined by:

$$
f(x) = \begin{cases}
0.7 & \text{if } x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
0.5 & \text{if } x = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \\
0.4 & \text{if } x = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \\
0.3 & \text{if } x = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \\
0.2 & \text{if } x = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.
\end{cases}
$$

Then we have the following:

$$
U(f,z) = \begin{cases}
T & \text{if } t \leq 0.2, \\
\bigwedge_{s \leq t \leq 0.3} \bigwedge_{s \leq t \leq 0.5} \bigwedge_{s \leq t \leq 0.7} \bigwedge_{s < t} \end{cases}
$$

This shows that $f$ is an $(e_{0.4}, e_{0.4} \vee q_{0.7})$-fuzzy quasi-ideal, but neither $(e_{0.4}, e_{0.4} \vee q_{0.7})$-fuzzy left nor $(e_{0.4}, e_{0.4} \vee q_{0.7})$-fuzzy right, nor $(e_{0.4}, e_{0.4} \vee q_{0.7})$-fuzzy lateral ideal of $T$.

**Theorem 20.** Every $(e_{y}, e_{y} \vee q_{y})$-fuzzy quasi-ideal of a ternary semigroup $T$ is an $(e_{y}, e_{y} \vee q_{y})$-fuzzy bi-ideal of $T$.

**Proof:** Let $f$ be an $(e_{y}, e_{y} \vee q_{y})$-fuzzy quasi-ideal of $T$ and $u, v, x, y, z \in T$. Then

$$
\max \left\{ f(xz), f(yz) \right\} \geq \min \left\{ f(x) \wedge T(y) \wedge T(z), f(x) \wedge T(y) \wedge f(z), f(x) \wedge f(y) \wedge T(z) \right\}
$$

This implies that

$$
\max \left\{ f(xyz), y \right\} \geq \min \left\{ f(x), f(y), f(z), \delta \right\}
$$

Also,

$$
\max \left\{ f(xyz), y \right\} \geq \min \left\{ f(x) \wedge T(y) \wedge f(z), f(x) \wedge T(y) \wedge f(z) \wedge \delta \right\}
$$

This implies that

$$
\max \left\{ f(xyz), y \right\} \geq \min \left\{ f(x), f(y), f(z), \delta \right\}
$$

Hence, $f$ is an $(e_{y}, e_{y} \vee q_{y})$-fuzzy bi-ideal of $T$.

**Definition 12.** Let $f, g$ and $h$ be fuzzy subsets of a ternary semigroup $T$. We define the fuzzy subsets $(f)^{\delta}, f \wedge_{y} g, f \vee_{y} g$ and $f \delta_{y} g, g_{y} h \delta_{y} h$ as follows:

1. $(f)^{\delta}(x) = (f(x) \vee y) \wedge \delta$
(2) \((f \wedge \delta g)(x) = ((f(x) \wedge g(x)) \vee \gamma) \land \delta\)
(3) \((f \vee \delta g)(x) = ((f(x) \vee g(x)) \vee \gamma) \land \delta\)
(4) \((f \circ \delta g \circ \delta h)(x) = ((f \circ g \circ h)(x) \vee \gamma) \land \delta\)
for all \(x \in T\).

**Lemma 4.** Let \(f, g\) and \(h\) be fuzzy subsets of a ternary semigroup \(T\). Then the following hold.

1. \((f \wedge \delta g) = (f)'' \wedge (g)''\)
2. \((f \vee \delta g) = (f)'' \vee (g)''\)
3. \(f \circ \delta g \circ \delta h = (f)'' \circ (g)'' \circ (h)''\).

**Proof:** The proofs of (1) and (2) are obvious.

Let \(x \in T\). Consider,
\[
(f \circ \delta g \circ \delta h)(x) = ((f \circ g \circ h)(x) \vee \gamma) \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ (f(u) \circ g(v) \circ h(w)) \vee \gamma \right] \land \delta \right\}
\]
\[
= \left\{ \bigvee_{x \in T} \left[ (f(u) \vee \gamma) \wedge (g(v) \wedge \delta) \vee \gamma \right] \right\} \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ (f(u) \vee \gamma) \wedge (g(v) \vee \gamma) \right] \right\} \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ f(u) \wedge g(v) \wedge h(w) \vee \gamma \right] \right\} \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ f(u) \wedge g(v) \wedge h(w) \right] \right\} \land \delta
\]

Hence \(f \circ \delta g \circ \delta h = (f)'' \circ (g)'' \circ (h)''\).

**Theorem 21.** Let \(f\) be an \((\varepsilon, \varepsilon, \gamma)\)-fuzzy right ideal, \(g\) a \((\varepsilon, \varepsilon, \gamma)\)-fuzzy lateral ideal and \(h\) an \((\varepsilon, \varepsilon, \gamma)\)-fuzzy left ideal of a ternary semigroup \(T\). Then \(f \circ \delta g \circ \delta h \leq f \wedge \delta g \wedge \delta h\).

**Proof:** Suppose \(f\) is an \((\varepsilon, \varepsilon, \gamma)\)-fuzzy right ideal and \(g\) an \((\varepsilon, \varepsilon, \gamma)\)-fuzzy left ideal of \(T\). Let \(x \in T\). Consider,
\[
(f \circ \delta g \circ \delta h)(x) = ((f \circ g \circ h)(x) \vee \gamma) \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ (f(u) \circ g(v) \circ h(w)) \vee \gamma \right] \land \delta \right\}
\]
\[
= \left\{ \bigvee_{x \in T} \left[ (f(u) \vee \gamma) \wedge (g(v) \wedge \delta) \vee \gamma \right] \right\} \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ (f(u) \vee \gamma) \wedge (g(v) \vee \gamma) \right] \right\} \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ f(u) \wedge g(v) \wedge h(w) \vee \gamma \right] \right\} \land \delta
\]
\[
= \left\{ \bigvee_{x \in T} \left[ f(u) \wedge g(v) \wedge h(w) \right] \right\} \land \delta
\]

Thus, \(f \circ \delta g \circ \delta h \leq f \wedge \delta g \wedge \delta h\).

The proof of the following theorem is straightforward and we omit the detail.

**Theorem 23.** A non-empty subset \(A\) of a ternary semigroup \(T\) is a (left, right, two sided) ideal of \(T\) if and only if \((C_{A})''\) is an \((\varepsilon, \varepsilon, \gamma)\)-fuzzy right (left, right, two sided) ideal of \(T\), where \(C_{A}\) is the characteristic function of \(A\).

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