Abstract
The concept of \( \Gamma \)-semihypergroups is a generalization of semigroups, a generalization of semihypergroups and a generalization of \( \Gamma \)-semigroups. In this paper, we study the concept of semiprime ideals in a \( \Gamma \)-semihypergroup and prove some results. Also, we introduce the notion of \( \Gamma \)-hypergroups and closed \( \Gamma \)-subhypergroups. Finally, we study the concept of \( \Gamma \)-semihypergroups associated to binary relations and give necessary and sufficient conditions on a set of binary relations \( \Gamma \) on a non-empty set \( S \) such that \( S \) becomes a \( \Gamma \)-semihypergroup or a \( \Gamma \)-hypergroup.

Keywords: Hypergroup; semihypergroup; \( \Gamma \)-semigroup; \( \Gamma \)-semihypergroup; binary relation

1. Introduction
The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup [1]. Since then, hundreds of papers and several books have been written on this topic, see [2-5]. A recent book on hyperstructures [6] points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a non-empty set. More exactly, let \( H \) be a non-empty set. Then the map \( \circ : H \times H \to P^*(H) \) is called a hyperoperation where \( P^*(H) \) is the family of non-empty subsets of \( H \). The couple \( (H, \circ) \) is called a hypergroupoid.

In the above definition, if \( A \) and \( B \) are two non-empty subsets of \( H \) and \( x \in H \), then we define \( A \circ B = \bigcup_{a \in A, b \in B} a \circ b; \ x \circ A = \{ x \} \circ A \) and \( A \circ x = A \circ \{ x \} \).

A hypergroupoid \( (H, \circ) \) is called a semihypergroup if for every \( x, y, z \in H \), we have \( x \circ (y \circ z) = (x \circ y) \circ z \), and is called a quasihypergroup if for every \( x \in H \), \( x \circ H = H = H \circ x \). This condition is called the reproduction axiom. The couple \( (H, \circ) \) is called a hypergroup if it is a semihypergroup and a quasihypergroup.

The notion of \( \Gamma \)-semigroups was introduced by Sen in [7, 8]. Let \( S \) and \( \Gamma \) be two non-empty sets. Then \( S \) is called a \( \Gamma \)-semigroup if there exists a mapping \( S \times \Gamma \times S \to S \), written \((a, \gamma, b)\) by \( a \gamma b \), such that it satisfies the identities \((a \alpha b) \beta c = a \alpha (b \beta c)\) for all \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \). Let \( S \) be an arbitrary semigroup and \( \Gamma \) a non-empty set. Define a mapping \( S \times \Gamma \times S \to S \) by \( a \alpha b = a^\beta b \) for all \( a, b \in S \) and \( \alpha, \beta \in \Gamma \). It is easy to see that \( S \) is a \( \Gamma \)-semigroup. Thus a semigroup can be considered to be a \( \Gamma \)-semigroup. Many classical notions of semigroups have been extended to \( \Gamma \)-semigroups, see (9, 10).

Let \( S \) be a \( \Gamma \)-semigroup and \( \alpha \) be a fixed element in \( \Gamma \). We define \( a \cdot b = a \alpha b \) for all \( a, b \in S \). Then \((S, \cdot)\) is a semigroup and is denoted by \( S_\alpha \).
2. Preliminaries and basic definitions

The concept of \( \Gamma \)-semihypergroups was introduced by Davvaz et al. [11, 12]. In this section we introduce some preliminaries and basic definitions of \( \Gamma \)-semihypergroups and give some examples.

**Definition 2.1.** Let \( S \) and \( \Gamma \) be two non-empty sets. Then \( S \) is called a \( \Gamma \)-semihypergroup if each \( \gamma \in \Gamma \) is a hyperoperation on \( S \), i.e., \( x\gamma y \subseteq S \) for every \( x, y \in S \), and for every \( \alpha, \beta \in \Gamma \) and \( x, y, z \in S \) we have the associative property that is \( x\alpha(y\beta z) = (x\alpha y)\beta z \).

Let \( A \) and \( B \) be two non-empty subsets of \( S \) and \( \gamma \in \Gamma \). Then we define:

\[
A\gamma B = \cup \{a\gamma b \mid a \in A, b \in B\},
\]

and

\[
A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B = \cup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.
\]

A \( \Gamma \)-semihypergroup \( S \) is called commutative if for every \( x, y \in S \) and for every \( \gamma \in \Gamma \) we have \( x\gamma y = y\gamma x \). A non-empty subset \( A \) of \( S \) is called a \( \Gamma \)-subsemihypergroup of \( S \) if \( A \Gamma A \subseteq A \).

Let \( (S, \cdot) \) be a semihypergroup and let \( \Gamma = \{e\} \). Then \( S \) is a \( \Gamma \)-semihypergroup. So every semihypergroup is a \( \Gamma \)-semihypergroup.

Let \( S \) be a \( \Gamma \)-semihypergroup and \( \alpha \in \Gamma \), if we define \( a \circ b = aba \) for every \( a, b \in S \) then \( (S, \circ) \) becomes a semihypergroup, we denote it by \( S_{\alpha} \).

Now, we give some other examples of \( \Gamma \)-semihypergroups.

**Example 1.** Let \( G \) be a group and \( \Gamma = \{\alpha, \beta\} \). Then for every \( x, y \in G \), we define \( x\alpha y = xy \) and \( x\beta y = G \). Then \( G \) is a \( \Gamma \)-semihypergroup.

**Example 2.** Let \( (S, \leq) \) be a totally ordered set and \( \Gamma \) be a non-empty subset of \( S \). We define

\[
x\gamma y = \{z \in S \mid z \geq \max\{x, \gamma, y\}\},
\]

for every \( x, y \in S \) and \( \gamma \in \Gamma \). Then \( S \) is a \( \Gamma \)-semihypergroup.

**Example 3.** Let \( S \) be a \( \Gamma \)-semigroup and \( P \) be a non-empty subset of \( S \). Let \( \Gamma_{\rho} = \{\alpha_{\rho} : \alpha \in \Gamma\} \).

If we define \( x\alpha_{\rho} y = x\alpha P\beta y \), for every \( x, y \in S \) and \( \alpha \in \Gamma \), then \( S \) is a \( \Gamma_{\rho} \)-semihypergroup.

Let \( S \) be a \( \Gamma \)-semihypergroup. We define a relation \( \rho \) on \( S \times \Gamma \) as follows:

\[
(x, \alpha)\rho(y, \beta) \iff x\alpha = y\beta, \forall s \in S.
\]

Obviously \( \rho \) is an equivalence relation. Let \( [x, \alpha] \) denote the equivalence class containing \( (x, \alpha) \). Let \( M = \{(x, \alpha) : x \in S, \alpha \in \Gamma\} \). We define the hyperoperation \( \circ \) on \( M \) as follows:

\[
[x, \alpha] \circ [y, \beta] = \{[z, \gamma] : z \in x\alpha y\},
\]

for all \( [x, \alpha], [y, \beta] \in M \).

Since \( (x\alpha y)\beta z = x\alpha(y\beta z) \), for all \( x, y, z \in S \) and \( \alpha, \beta, \gamma \in \Gamma \), then

\[
[x, \alpha] \circ ([y, \beta] \circ [z, \gamma]) = ([x, \alpha] \circ [y, \beta]) \circ [z, \gamma],
\]

for all \( [x, \alpha], [y, \beta], [z, \gamma] \in M \).

Thus the hyperoperation \( \circ \) is associative, so \((M, \circ)\) is a semihypergroup. This semihypergroup is called the left operator semihypergroup of \( S \).

Let \( S \) be a \( \Gamma \)-semihypergroup. If there exist elements \( e \in S \) and \( \delta \in \Gamma \) such that \( e\alpha = x \) for every \( x \in S \), then \( S \) is said to have a left partial unity which is denoted by \( e_{\delta} \). It is easy to check whether \( e_{\delta} \) is a left partial unity of \( S \), then \([e, \delta]\) is a left unity of the left operator semihypergroup \( M \).

**Example 4.** Consider Example 1 and let \( e \) be the identity element of \( G \). Then \( e_{\alpha} = e \) is a left partial unity of the \( \Gamma \)-semihypergroup \( G \).

The concept of \( \Gamma \)-hyperideals of a \( \Gamma \)-semihypergroup was defined and studied in [12].

**Definition 2.2.** A non-empty subset \( I \) of a \( \Gamma \)-semihypergroup \( S \) is called a left (right) \( \Gamma \)-hyperideal, “ideal, for short” of \( S \), if \( S \Gamma I \subseteq I \) (\( I \Gamma S \subseteq I \)). \( S \) is called a left (right) simple \( \Gamma \)-semihypergroup if it has no proper left (right) ideal. \( S \) is simple if \( S \) has no proper left and right ideals.
Let $A$ be a non-empty subset of a $\Gamma$-semi-hypergroup $S$. Then the intersection of all ideals of $S$ containing $A$ is an ideal of $S$ generated by $A$, and denoted by $<A>$.

**Example 5.** Consider Example 4. Put $S=\mathbb{N}$ with natural order. Then the subset $I_n=\{n,n+1,n+2,\cdots\}$ is an ideal of $S$, for every $n \in \mathbb{N}$.

The following lemmas and theorem were proved in [12].

**Lemma 2.3.** Let $S$ be a $\Gamma$-semi-hypergroup. If $A$ is a non-empty subset of $S$, then

$$<A>=A\cup\Delta S\cup S\Delta A\cup S\Delta A\Gamma S.$$

One can see that, if $S$ is a commutative $\Gamma$-semi-hypergroup and $\phi \neq A \subseteq S$, then $<A>=A\cup\Delta S$. If $S$ is a commutative $\Gamma$-semi-hypergroup with left partial unity, then $<A>=\Gamma S$.

**Lemma 2.4.** Let $S$ be a $\Gamma$-semi-hypergroup and $\Lambda$ be a non-empty set such that for every $\lambda \in \Lambda$, $I_{\lambda}$ is an ideal of $S$. Then the following assertions hold:

1. $\bigcup_{\lambda \in \Lambda} I_{\lambda}$ is an ideal of $S$;
2. $\bigcap_{\lambda \in \Lambda} I_{\lambda}$ is an ideal of $S$.

**Definition 2.5.** A proper ideal $P$ of a $\Gamma$-semi-hypergroup $S$ is called a prime ideal, if for every ideal $I$ and $J$ of $S$, if $I\cup J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If a $\Gamma$-semi-hypergroup $S$ is commutative, then a proper ideal $P$ is prime if and only if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a,b \in S$.

**Example 6.** Consider Example 2. Put $S=\Gamma = \{1,2,\cdots,n\}$ for some natural number $n$.

Then, all ideals of $S$ have the form $I_i=\{i,i+1,\cdots,n\}$, for every $i \in S$ and $I_2$ is a prime ideal of $S$.

**Theorem 2.6.** Let $S$ be a $\Gamma$-semi-hypergroup and $P$ be a left ideal of $S$. Then $P$ is a prime ideal of $S$ if and only if for all $x,y \in S$,

$$x\Gamma y \subseteq P \text{ implies that } x \in P \text{ or } y \in P.$$
(2) If $I$ is a right ideal of $S$ then, $I^+$ is a right hyperideal of $M$.

**Theorem 2.10.** [12] Let $S$ be a $\Gamma$-semihypergroup with a left partial unity and $M$ be its left operator semihypergroup. If $I$ is a right ideal of $S$, then $I = (I^+)^*$.

### 3. Semiprime ideals of $\Gamma$-semihypergroups

In this section, we introduce the concept of semiprime ideals of a $\Gamma$-semihypergroup and prove some results.

**Definition 3.1.** Let $S$ be a $\Gamma$-semihypergroup. Then a proper left (right) ideal $P$ of $S$ is said to be a left (right) semiprime ideal, if for any left (right) ideal $A$ of $S$, $A \Gamma A \subseteq P$ implies that $A \subseteq P$. A proper ideal $P$ is called semiprime ideal if $P$ is both left and right semiprime ideal of $S$.

**Example 7.** Let $\Gamma = \{1, 2, 3, \cdots, n\}$ for some $n \in \mathbb{N}$. For every $x, y \in S$ and $\alpha \in \Gamma$ we define the following hyperoperation on $S$

$$x \alpha y = \{s \in S | s \geq \max\{x, \alpha, y\}\}.$$  

Then $S$ is a $\Gamma$-semihypergroup and $I_i = \{i, i+1, \cdots, n\}$ is a semiprime ideal of $S$ for $1 \leq i \leq n$.

**Lemma 3.2** Let $S$ be a $\Gamma$-semihypergroup with a left partial unity and $P$ be a left ideal of $S$. Then $P$ is a left semiprime ideal of $S$ if and only if for every $x, y \in S$ we have

$$x \Gamma y \subseteq P \Rightarrow x \in P.$$ 

**Proof:** Suppose that $P$ is a left semiprime ideal of $S$ and $x \Gamma y \subseteq P$ for $x \in S$. Then $y \Gamma x \subseteq y \Gamma P \subseteq P$. Since $P$ is a left semiprime ideal and $\Gamma x$ is a left ideal of $S$, it follows that $x \in y \Gamma P \subseteq P$.

Conversely, let $A$ be an ideal of $S$ such that $A \Gamma A \subseteq P$. If $a \in A$, then $a \Gamma A \subseteq A \Gamma A \subseteq P$. So, by the above implication $a \in P$ thus $A \subseteq P$.

**Lemma 3.3.** Let $S$ be a $\Gamma$-semihypergroup and $M$ be its left operator semihypergroup. Then the following statements hold:

1. If $P$ is a semiprime ideal of $M$, then $P^+$ is a semiprime ideal of $S$.
2. If $S$ has a left partial unity and $Q$ is a semiprime ideal of $S$, then $Q^+$ is a semiprime ideal of $M$.

**Proof:**

(1) Suppose that $P$ is a semiprime ideal of $M$ and $A$ is an ideal of $S$ such that $A \Gamma A \subseteq P^+$. Then $[A \Gamma A, \Gamma] \subseteq P$ so $[A, \Gamma] \circ [A, \Gamma] \subseteq P$. Since $[A, \Gamma]$ is an ideal of $M$ and $P$ is a semiprime ideal of $M$, it follows that $[A, \Gamma] \subseteq P$ hence $A \subseteq P^+$. Thus $P^+$ is a semiprime ideal of $S$.

(2) Suppose that $Q$ is a semiprime ideal of $S$ and $A$ is an ideal of $M$ such that $A \circ A \subseteq Q^+$. First, we show that $A \Gamma A \subseteq (A \circ A)^+$. Let $t \in A \Gamma A$. Then there exist $x, y \in A^+$ and $\alpha \in \Gamma$ such that $t = xy\alpha$. So $[t, \alpha] \in [x, \alpha] \circ [y, \alpha] \subseteq A \circ A$ for every $\alpha \in \Gamma$. Thus $t \in (A \circ A)^+$, so $A \Gamma A \subseteq (A \circ A)^+$. Now, from $A \circ A \subseteq Q^+$ and Theorem 2.10 we have

$$A \Gamma A \subseteq (A \circ A)^+ \subseteq (Q^+) = Q.$$ 

Since $Q$ is a semiprime ideal and $A^+$ is an ideal of $S$, it follows that $A^+ \subseteq Q$. Thus $A \subseteq (A^+) \subseteq Q^+$.

**Lemma 3.4.** Let $P_i$ be a prime ideal of a $\Gamma$-semihypergroup $S$ for every $i \in I$ and let $P = \bigcap_{i \in I} P_i$. Then if $P \neq \emptyset$, then $P$ is a semiprime ideal of $S$.

**Proof:** It is immediate.

**Lemma 3.5.** Let $T$ be a $\Gamma$-subsemihypergroup and $I$ be an ideal of the $\Gamma$-semihypergroup $S$ such that $I \Gamma T = \emptyset$. Then $T$ is contained in a $\Gamma$-subsemihypergroup that is maximal with respect to the property of not meeting $I$. 

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Proof: Since the set \( A = \{ K | T \leq K \leq S \text{ and } K \cap I = \emptyset \} \) is non-empty, it follows that by Zorn's lemma, \( A \) has a maximal element that satisfies the theorem.

Lemma 3.6. Let \( T \) be a commutative \( \Gamma \)-subsemihypergroup and \( I \) be an ideal of the \( \Gamma \)-semihypergroup \( S \) such that \( I \cap T = \emptyset \). Then there exists a prime ideal of \( S \), say \( P \), such that \( I \subseteq P \) and \( P \cap T = \emptyset \).

Proof: By Zorn's lemma, there exists an ideal \( P \) such that \( P \) is maximal with respect to properties of \( I \subseteq P \) and \( P \cap T = \emptyset \). We claim that \( x, y \in S \setminus P \). Then, we show that \( x \Gamma S \Gamma y \subset P \). Since \( x, y \notin P \) and \( P \) is maximal, it follows that \( \langle P, x \rangle \cap T \neq \emptyset \) and \( \langle P, y \rangle \cap T \neq \emptyset \).

Thus, there exist \( s, t \in S \) such that \( s \leq \langle P, x \rangle \cap T \) and \( t \leq \langle P, y \rangle \cap T \). From the property \( P \cap T = \emptyset \), we have only four cases:

(i) \( s \in s, \alpha x \) and \( t \in t, \beta y \) for some \( s, t \in S \) and \( \alpha, \beta \in \Gamma \),
(ii) \( s \in s, \alpha x \) and \( t = y \) for some \( s \in S \) and \( \alpha \in \Gamma \),
(iii) \( s = x \) and \( t \in t, \beta y \) for some \( t \in S \) and \( \beta \in \Gamma \),
(iv) \( s = x \) and \( t = y \).

If (i) holds, then \( s \Gamma t \subseteq (s, \alpha x) \Gamma (t, \beta y) \subseteq x \Gamma S \Gamma y \).

Now, since \( T \) is a \( \Gamma \)-subsemihypergroup, it follows that \( s \Gamma t \subseteq T \). Thus \( x \Gamma S \Gamma y \subset P \).

Similarly, in the other cases we conclude that \( x \Gamma S \Gamma y \subset P \). Therefore, \( P \) is a prime ideal of \( S \).

Let \( S \) be a \( \Gamma \)-semihypergroup and \( I \) be an ideal of \( S \). A prime ideal \( P \) of \( S \) is called a minimal prime ideal belonging to \( I \), if \( I \subseteq P \) and there is no prime ideal containing \( I \) and properly contained in \( P \).

Corollary 3.7. If \( Q \) is a prime ideal containing an ideal \( I \), then there exists a minimal prime ideal belonging to \( I \) which is contained in \( Q \).

Definition 3.8. Let \( S \) be a \( \Gamma \)-semihypergroup and \( I \) be an ideal of \( S \). Then the prime radical of \( I \) is defined as the intersection of all prime ideals of \( S \) containing \( I \) and is denoted by \( \sqrt{I} \).

Proposition 3.9. Let \( S \) be a \( \Gamma \)-semihypergroup and \( I \) be an ideal of \( S \). Then the following statements hold:

1. \( \sqrt{I} \) is a semiprime ideal of \( S \);
2. \( \sqrt{I} = \cap \{ P | P \text{ is a minimal prime ideal belonging to } I \} \).

Proof: (1) It is straightforward.
(2) It is taken from Corollary 3.7.

4. \( \Gamma \)-hypergroups

In this section we study the concept of \( \Gamma \)-hypergroups and give some examples. Also, we introduce the concept of closed \( \Gamma \)-subhypergroups of a \( \Gamma \)-hypergroup.

Definition 4.1. A \( \Gamma \)-semihypergroup \( S \) is called a \( \Gamma \)-hypergroup if \( (S, \alpha) \) is a hypergroup for every \( \alpha \in \Gamma \).

Example 8. Let \( S = \{ a, b, c, d \} \) and \( \Gamma = \{ \alpha, \beta \} \).
We define the hyperoperations \( \alpha \) and \( \beta \) as follows:

\[
\begin{array}{cccc}
\alpha & a & b & c & d \\
\hline
a & \{a, b\} & \{b, c\} & \{c, d\} & \{a, d\} \\
b & \{b, c\} & \{c, d\} & \{a, d\} & \{a, b\} \\
c & \{c, d\} & \{a, d\} & \{a, b\} & \{b, c\} \\
d & \{a, d\} & \{a, b\} & \{b, c\} & \{c, d\}
\end{array}
\]

\[
\begin{array}{cccc}
\beta & a & b & c & d \\
\hline
a & \{b, c\} & \{c, d\} & \{a, d\} & \{a, b\} \\
b & \{c, d\} & \{a, d\} & \{a, b\} & \{b, c\} \\
c & \{a, d\} & \{a, b\} & \{b, c\} & \{c, d\} \\
d & \{a, b\} & \{b, c\} & \{c, d\} & \{a, d\}
\end{array}
\]

Then \( S \) is a \( \Gamma \)-hypergroup.

Example 9. Let \( S \) be a non-empty set and \( \Gamma = \{ \alpha, \beta \} \). Then for every \( x, y \in S \) and \( \alpha, \beta \in \Gamma \) we define \( x \alpha y = S \) and \( x \beta y = \{x, y\} \).
Then \( S \) is a \( \Gamma \)-hypergroup.

Example 10. Let \( (S, \cdot) \) be a group. Let \( \Gamma \subseteq P(S) \). We define \( x \alpha y = x \cdot \alpha \cdot y \) for every
Then \( S \) is a \( \Gamma \)-hypergroup.

**Example 11.** Let \((S, \diamond)\) be a hypergroup and \( \emptyset \neq \Gamma \subseteq S \). We define \( x\alpha y = x \diamond \alpha \diamond y \) for every \( x, y \in S \) and \( \alpha \in \Gamma \). Then \( S \) is a \( \Gamma \)-hypergroup.

**Example 12.** Let \((G, \cdot)\) be a group and \( \{A_{g}\}_{g \in G} \) be a collection of disjoint sets. Consider \( S = \bigcup_{g \in G} A_{g} \) and \( \Gamma = G \). For \( x, y \in S \) there exist \( g_{x}, g_{y} \in G \) such that \( x \in A_{g_{x}} \) and \( y \in A_{g_{y}} \). We define \( x \alpha y = A_{g_{x}\alpha g_{y}} \). Then \( S \) is a \( \Gamma \)-hypergroup.

**Theorem 4.2.** [12] Let \( S \) be a \( \Gamma \)-group and \( P \) be a \( \Gamma \)-subgroup of \( S \). Let \( \Gamma' = \{ \gamma | \gamma \in \Gamma \} \). Now, for every \( x, y \in S \) and \( \alpha' \in \Gamma \) we define \( x \alpha' y = x \alpha y \cup P \). Then, \( S \) is a \( \Gamma' \)-hypergroup.

**Theorem 4.3.** Let \( S \) be a \( \Gamma \)-semihypergroup. Then \( S \) is a simple \( \Gamma \)-semihypergroup if and only if \( S_{\alpha} \) is a hypergroup for every \( \alpha \in \Gamma \).

**Proof:** Suppose that \( S_{\alpha} \) is a hypergroup and \( I \) is a left (right) ideal of \( S_{\alpha} \). If \( x \in I \), then the reproduction axiom implies that \( x \alpha S = S = S \alpha x \). On the other hand, we have \( S \alpha x \subseteq I \) \((x \alpha S \subseteq I)\). Therefore, \( I = S \).

Conversely, suppose that \( S \) is left and right simple. Then for every \( x \in S \) and \( \alpha \in \Gamma \), put \( I = x \alpha S \). Thus, \( I \) is a right ideal of \( S \), for

\[
\Gamma S = (x \alpha S) \Gamma S = x \alpha (S \Gamma S) \subseteq x \alpha S = I
\]

so \( x \alpha S = S \). Similarly, we have \( S = S \alpha x \). Therefore, \( S \) is a \( \Gamma \)-hypergroup.

**Corollary 4.4.** If \( S_{\alpha} \) is a hypergroup for some \( \alpha \in \Gamma \), then for every \( \alpha \in \Gamma \), \( S_{\alpha} \) is a hypergroup.

**Definition 4.5.** A subset \( H \) of a \( \Gamma \)-hypergroup is called a \( \Gamma \)-subhypergroup if for every \( h, k \in H \) and \( \alpha \in \Gamma \) we have \( h \alpha k \subseteq H \) and \( h \alpha H = H = H \alpha h \).

**Definition 4.6.** Let \( S \) be a \( \Gamma \)-hypergroup. Then a subset \( H \) of \( S \) is called closed if for every \( h, k \in H \), \( x \in S \) and \( \alpha \in \Gamma \) we have the following implication

\[
h \in x \alpha H \Rightarrow x \in H.
\]

**Example 14.** Consider \((\mathbb{Z}, +)\) and let \( \Gamma = \{ \alpha, \beta \} \) where \( \alpha = \{-1, 1\} \) and \( \beta = \{-2, +2\} \). If for every \( x, y \in \mathbb{Z} \) we define:

\[
x \alpha y = \{ x + y - 1, x + y + 1 \}, x \beta y = \{ x + y - 2, x + y + 2 \}.
\]

Then \( \mathbb{Z} \) is a \( \Gamma \)-hypergroup and \( H = 2\mathbb{Z} \) is a closed subset of \( \mathbb{Z} \).

**Example 15.** Consider \((\mathbb{Z}, +)\) and let \( \Gamma = \{ \alpha, \beta \} \) where \( \alpha = \{-2, 2\} \) and \( \beta = \{-4, 4\} \). If for every \( x, y \in \mathbb{Z} \) we define:

\[
x \alpha y = \{ x + y - 2, x + y + 2 \}, x \beta y = \{ x + y - 4, x + y + 4 \}.
\]

Then \( \mathbb{Z} \) is a \( \Gamma \)-hypergroup and \( H = 2\mathbb{Z} \) is a closed \( \Gamma \)-subhypergroup of \( \mathbb{Z} \).

Let \( S \) be a \( \Gamma \)-hypergroup. Then two new hyperoperations may be defined on \( S \) as follows:

\[
a / b = \{ x \in S | a \in x \alpha b, \alpha \in \Gamma \} \quad \text{and} \quad a \setminus b = \{ x \in S | a \in b \alpha x, \alpha \in \Gamma \}.
\]

If \( A \) and \( B \) are non-empty subsets of \( S \), then

\[
A / B = \bigcup_{a \in A, b \in B} a / b \quad \text{and} \quad A \setminus B = \bigcup_{a \in A, b \in B} a \setminus b.
\]

**Lemma 4.7.** Let \( S \) be a \( \Gamma \)-hypergroup, \( A, B, C \) and \( D \) be non-empty subsets of \( S \) and \( x, y \in S \). Then the following assertions hold:

1. If \( A \subseteq B \) and \( C \subseteq D \), then \( A / C \subseteq B / D \);
2. \( (A / B) / C = A / (C \Gamma B) \);
3. \( (A \setminus B) \setminus C = A \setminus (B \setminus C) \);
4. \( y \in x \setminus (x / y) \);
(5) \( y \in x/(x \setminus y) \);

(6) If \( A \) is a closed subset of \( S \), then \( A/A \subseteq A \);  
(7) \( A \subseteq (A\Gamma B)/B \);  
(8) If \( H \) is a \( \Gamma \)-subhypergroup, then \( H \subseteq H/H \).

**Proof:** (1) It is immediate.

(2) Suppose that \( x \in (A/B)/C \). Then, there exist \( a \in A, b \in B \) and \( c \in C \) such that \( x \in (a/b)/c \).

So, we have

\[
x \in (a/b)/c \quad \Rightarrow \exists y \in a/b, x \in y/c
\]

\[
\Rightarrow a \in y\Gamma b, y \in x\Gamma c
\]

\[
\Rightarrow a \in (x\Gamma c)\Gamma b = x\Gamma(c\Gamma b)
\]

\[
\Rightarrow \exists z \in c\Gamma b, a \in x\Gamma z
\]

\[
\Rightarrow x \in a/z \subseteq a/(c\Gamma b) \subseteq A/(C\Gamma B).
\]

Thus, \( (A/B)/C \subseteq A/(C\Gamma B) \).

Conversely, suppose that \( x \in A/(C\Gamma B) \). Then there exist \( a \in A, b \in B \) and \( c \in C \) such that \( x \in a/(c\Gamma b) \). So there exists \( y \in c\Gamma b \) such that \( x \in a/y \). So \( a \in x\Gamma y \subseteq x\Gamma(c\Gamma b) = (x\Gamma c)\Gamma b \).

Thus there exists \( z \in x\Gamma c \) such that \( a \in z\Gamma b \) and so \( x \in z/c, z \in a/b \). Therefore, \( x \in (A/B)/C \).

(3) It is similar to (2).

(4) Let \( a \in x/y \neq \emptyset \). Then \( x \in a\Gamma y \), so \( y \in x \setminus a \subseteq x \setminus (x/y) \).

(5) it is similar to (4).

(6) If \( x \in A/A \), then \( x \in a_1/a_2 \). So \( a_1 \in x\Gamma a_2 \subseteq x\Gamma A \cap A \). Since \( A \) is a closed subset of \( S \), it follows that \( x \in A \). Therefore, \( A/A \subseteq A \).

(7) Suppose that \( x \in A \) and \( y \in x\Gamma B \). Then \( x \in y/B \subseteq (A\Gamma B)/B \).

(8) Suppose that \( H \) is a \( \Gamma \)-subhypergroup and \( h \in H \). Then there exists \( h_1 \in H \) such that \( h_1 \in h_1\Gamma h_2 \) thus \( h_1 \in h_1/h_2 \subseteq H/H \), so \( H \subseteq H/H \).

**Theorem 4.8.** Let \( S \) be a \( \Gamma \)-hypergroup and \( H \) be a \( \Gamma \)-subhypergroup of \( S \). Then \( H \) is a closed \( \Gamma \)-subhypergroup if and only if \( H = H/H \).

**Proof:** Suppose that \( H \) is a closed \( \Gamma \)-subhypergroup. Then, by the previous lemma, \( H \subseteq H/H \subseteq H \). Thus \( H = H/H \).

Conversely, suppose that \( H/H = H \). If \( y \in x \alpha \cap H \), for \( h \in H \), then \( x \in y/h \subseteq H/H = H \). Therefore, \( H \) is a closed \( \Gamma \)-subhypergroup of \( S \).

**Example 16.** Let \( G \) be a group with a non trivial center. Let \( P, Q \subseteq Z(G) \) and put \( \Gamma = \{\alpha, \beta\} \).

For every \( x, y \in G \) we define \( x\alpha y = xyP \) and \( x\beta y = xyQ \). Then \( G \) is a \( \Gamma \)-hypergroup.

Let \( a, b \in G \). Then

\[
a/b = \{x \in G | a \in x\alpha b\}
\]

\[
= \{x \in G | a \in x\alpha b \cap x\beta b\}
\]

\[
= \{x \in G | a \in x\beta P \cup x\beta Q\}
\]

\[
= ab^{-1} P^{-1} \cup ab^{-1} Q^{-1}.
\]

If \( H \) is a \( \Gamma \)-subhypergroup of \( G \) containing \( P \) and \( Q \), then for every \( a, b \in H \) we have \( a/b = ab^{-1} P^{-1} \cup ab^{-1} Q^{-1} \subseteq H \), so by the above theorem, \( H \) is a closed \( \Gamma \)-subhypergroup of \( G \).

**Lemma 4.9.** Let \( S \) be a \( \Gamma \)-semihypergroup and \( H \) and \( K \) be two closed \( \Gamma \)-subhypergroups of \( S \). Then \( < H \cup K > \subseteq \langle H \Gamma K \rangle \).

**Proof:** Since \( H\Gamma K \subseteq < H \cup K > \), it follows that \( < H\Gamma K > \subseteq < H \cup K > \). Now, we prove the converse of inclusion. Since \( H \) and \( K \) are closed \( \Gamma \)-subhypergroups of \( S \), it follows that \( H\Gamma K \) is a closed subset of \( S \). Now, by the previous theorem and Lemma 4.7, we have

\[
H = H/H \subseteq (H\Gamma K)/(H\Gamma K)/H
\]

\[
= (H\Gamma K)/(H\Gamma K) \subseteq < H\Gamma K >.
\]

Similarly, \( K \subseteq < H\Gamma K > \). Therefore, \( < H \cup K > \subseteq < H\Gamma K > \).

5. \( \Gamma \)-semihypergroups associated to binary relations

The connections between hyperstructures and binary relations have been analyzed by many
researchers, such as Rosenberg [13], Corsini [14], Cristea and Stefănescu [15] and others [16, 17, 18].

In this section we associate to a set of binary relations on a non-empty set \( S \), say \( \Gamma \), a partial \( \Gamma \)-hypergroupoid and get necessary and sufficient conditions such that it is a \( \Gamma \)-semihypergroup or a \( \Gamma \)-hypergroup.

Rosenberg [13] has associated a partial hypergroupoid \( H_\Gamma \), with a binary relation \( R \) defined on a non-empty set \( H \), where, for any \( \langle x, y \rangle \in H \),
\[
\{ z \in H \mid (x, z) \in R \}, \quad x \circ y = x \circ x \cup y \circ y.
\]

An element \( x \in H \) is called an outer element for \( R \) if there exists \( h \in H \) such that \( (h, x) \not\in R^2 \).

Rosenberg proved the next theorem.

**Theorem 5.1.** [13] \( H_\Gamma \) is a hypergroup if and only if

1. \( R \) has full domain;
2. \( R \) has full range;
3. \( R \subseteq R^2 \);
4. If \( (a, x) \in R^2 \), then \( (a, x) \in R \), whenever \( x \) is an outer element.

Let \( R \) be a binary relation on a non-empty set \( S \). Then an element \( Sx \) is called a semiouter element for the relation \( R \) if there exists \( h \in S \) such that \( (h, x) \not\in R \).

Let \( R \) be a binary relation on a non-empty set \( S \), \( A \subseteq S \) and \( x, y \in S \). Then we use the following notations:
\[
L^R_x = R(x) = \{ z \in S \mid (x, z) \in R \}; \quad R(x, y) = \{ z \in S \mid (x, z) \in R \lor (y, z) \in R \}; \quad R(A) = \{ z \in S \mid (a, z) \in R, \exists a \in A \}; \quad R^{-1}(A) = \{ z \in S \mid (z, a) \in R, \exists a \in A \}.
\]

**Definition 5.2.** Let \( S \) be a non-empty set and \( \mathcal{R} \) be a set of binary relations on \( S \). Then for every \( \alpha \in \mathcal{R} \) we can associate a hyperoperation \( \circ_{\alpha} \) on \( S \) as follows:
\[
x \circ_{\alpha} y = \alpha(x, y) = L^\alpha_x \cup L^\alpha_y, \quad \forall x, y \in S.
\]

So \( (S, \circ_{\alpha}) \) is a partial hypergroupoid. Now, let \( \Gamma = \{ \alpha \mid \alpha \in \mathcal{R} \} \). Then \( S \) is a partial \( \Gamma \)-hypergroupoid and is denoted by \( S_\Gamma \).

To simplify, we write \( \circ_{\alpha} \) by \( \alpha \) and consider \( \Gamma = \mathcal{R} \), in this way for every \( \alpha \in \Gamma \) and \( x, y \in S \) we have
\[
x \circ \gamma = x \circ_{\alpha} \gamma = \alpha(x, y) = L^\alpha_x \cup L^\alpha_y.
\]

It is easy to see that if for every \( \alpha \in \Gamma \) we have \( \alpha^{-1}(S) = S \), then \( S_\Gamma \) is a \( \Gamma \)-hypergroupoid.

**Example 17.** Let \( S = \{1, 2, 3, 4, 5\} \) and \( \Gamma = \{\alpha, \beta, \gamma\} \) such that
\[
\alpha = \{(1,1), (1,2), (2,4), (3,4), (4,5), (4,4), (5,2), (4,4), (4,5), (3,3), (4,1), (5,4), (5,3)\},
\]
\[
\beta = \{(1,1), (1,3), (1,4), (2,5), (3,3), (4,1), (5,4), (5,3)\},
\]
\[
\gamma = \{(1,1), (2,3), (3,4), (4,5), (5,1), (5,5)\}.
\]

Then \( S_\Gamma \) is a \( \Gamma \)-hypergroupoid.

**Lemma 5.3.** Let \( S \) be a non-empty set and \( \Gamma \) be a set of binary relations on \( S \) such that \( S_\Gamma \) is a \( \Gamma \)-hypergroupoid. Then the following assertions hold:
1. \( S_\Gamma \) is a commutative \( \Gamma \)-hypergroupoid;
2. For every \( x \in S \) and \( \alpha \in \Gamma \), \( x \circ \alpha = \alpha(x) \);
3. For every \( x, y, z \in S \) and \( \alpha, \beta \in \Gamma \), \( x \circ (y \circ z) = \alpha(x) \circ \beta(y, z) \);
4. For every \( x, y, z \in S \) and \( \alpha, \beta \in \Gamma \), \( (x \circ y) \circ z = \alpha \circ \beta(x, y) \cup \beta(z) \).

**Proof:** The proof is straightforward.

In the following we provide some conditions on \( \Gamma \) such that \( S_\Gamma \) be a \( \Gamma \)-semihypergroup.

**Theorem 5.4.** Let \( S \) be a non-empty set and \( \Gamma \) be a set of binary relations on \( S \) such that \( S_\Gamma \) be a \( \Gamma \)-hypergroupoid. Then \( S_\Gamma \) is a \( \Gamma \)-semihypergroup if and only if the following conditions hold:

1. \( \text{(}\Gamma \text{SH}1) \) For every \( \alpha, \beta \in \Gamma \), \( \alpha \subseteq \alpha \beta \);
2. \( \text{(}\Gamma \text{SH}2) \) If \( x \) is a semiouter element for the relation \( \alpha \beta \) and \( (a, x) \in \beta \alpha \), then \( (a, x) \in \beta \) for every \( a \in S \) and \( \alpha, \beta \in \Gamma \);
If \( x \) is a semiouter element for the relations \( \alpha \beta \) and \( \beta \) and \((a, x) \in \beta \alpha\), then \((a, x) \in \alpha \beta\), for every \( a \in S \) and \( \alpha, \beta \in \Gamma \).

**Proof:** Suppose that \( S_\Gamma \) is a \( \Gamma \)-semihypergroup. We prove the conditions \((\Gamma SH1)\), \((\Gamma SH2)\) and \((\Gamma SH3)\) of the theorem.

\((\Gamma SH1)\) Let \( x, y \in S \) and \( \alpha, \beta \in \Gamma \) such that \( y \in \alpha(x) \). Then we consider two cases:

Case (i) \( y \in \beta(y) \). Then \( y \in \alpha \beta(x) \).

Case (ii) \( y \notin \beta(y) \). Then we have \((x \alpha(x)y) = x \alpha(x) \beta(y)\) so the associativity axiom and the previous lemma conclude that \( \alpha \beta(x) \cup \beta(y) = \alpha(x) \cup \beta \alpha(x) \cup \beta \alpha(y) \).

Now, since \( y \in \alpha(x) \) and \( y \notin \beta(y) \), it follows that \( y \in \alpha \beta(x) \). Therefore, \( \alpha \subseteq \alpha \beta \).

\((\Gamma SH2)\) Suppose that \( x \) is a semiouter element for the relation \( \alpha \beta \) and \( x \in \beta \alpha(a) \). So there exists \( h \in S \) such that \( x \notin \alpha \beta(h) \). Thus the associativity axiom and the previous lemma conclude that \( (h \alpha h \beta h = h \alpha (h \beta h \alpha) \), thus \( \alpha \beta(h) \cup \beta(a) = \alpha(h) \cup \beta \alpha(h) \cup \beta \alpha(a) \).

Since \( x \in \beta \alpha(a) \) and \( x \notin \alpha \beta(h) \), it follows that \( x \in \beta \alpha(a) \).

\((\Gamma SH3)\) Suppose that \( x \) is a semiouter element for the relations \( \alpha \beta \) and \( \beta \) and let \( x \in \beta \alpha(a) \).

So there exist \( h, t \in S \) such that \( (h, x) \notin \alpha \beta \) and \( (t, x) \notin \beta \). Now, we have \( h \alpha(x \beta h) = (h \alpha a) \beta t \) thus \( \alpha(h) \cup \beta \alpha(a, t) = \alpha \beta(a, h) \cup \beta \alpha(t) \).

Since \( x \in \beta \alpha(a) \), \( x \notin \alpha \beta(h) \) and \( x \notin \beta \alpha(t) \), it follows that \( x \in \alpha \beta(a) \).

Conversely, suppose that \( S \) is a non-empty set and \( \Gamma \) be a set of binary relations on \( S \) such that \( S_\Gamma \) is a \( \Gamma \)-hypergroupoid and the conditions \((\Gamma SH1)\), \((\Gamma SH2)\) and \((\Gamma SH3)\) of the theorem are satisfied. We prove the associativity axiom for \( S_\Gamma \).

Let \( x, y, z, t \in S \) and \( \alpha, \beta \in \Gamma \) such that \( t \in x \alpha(y \beta z) = \alpha(x) \cup \beta \alpha(y, z) \). Then we have three cases:

Case (i) \( t \in \alpha(x) \). Then by the condition \((\Gamma SH1)\) \( t \in \alpha \beta(x) \).

Case (ii) \( t \in \beta \alpha(x) \). Then if \( t \notin \alpha \beta(x) \cup \beta(z) \), then \( t \) is a semiouter element for the relations \( \alpha \beta \) and \( \beta \). So by the condition \((\Gamma SH3)\) \( t \in \alpha \beta(x) \).

Case (iii) \( t \in \beta \alpha(z) \). Then if \( t \notin \alpha \beta(x) \), then \( t \) is a semiouter element for the relation \( \alpha \beta \) so by the condition \((\Gamma SH2)\), \( t \in \beta(z) \). Thus \( x \alpha(y \beta z) \subseteq (x \alpha) \beta z \). In the same way, we can prove the converse inclusion. Therefore, \( S_\Gamma \) is a \( \Gamma \)-semihypergroup.

**Example 18.** Let \( S = \{1,2,3\} \) and \( \Gamma = \{\alpha, \beta\} \) such that \( \alpha = \{(1,2), (2,2), (2,3), (3,3)\} \) and \( \beta = \{(1,3), (2,2), (3,2), (3,3)\} \). Then we have the table of hyperoperations \( \alpha \) and \( \beta \) as follows:

\[
\begin{array}{c|ccc}
\alpha & 1 & 2 & 3 \\
\hline
1 & \{2\} & \{2,3\} & \{2,3\} \\
2 & \{2,3\} & \{2,3\} & \{2,3\} \\
3 & \{2,3\} & \{2,3\} & \{3\} \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\beta & 1 & 2 & 3 \\
\hline
1 & \{3\} & \{2,3\} & \{2,3\} \\
2 & \{2,3\} & \{2\} & \{2,3\} \\
3 & \{2,3\} & \{2,3\} & \{2,3\} \\
\end{array}
\]

Then \( S_\Gamma \) is a \( \Gamma \)-semihypergroup.

**Theorem 5.5.** Let \( S \) be a non-empty set and \( \Gamma \) be a set of binary relations on \( S \) such that \( S_\Gamma \) is a \( \Gamma \)-semihypergroup. Then \( S_\Gamma \) is a \( \Gamma \)-hypergroup if and only if \( \alpha(S) = S \) for every \( \alpha \in \Gamma \).

**Proof:** Suppose that \( S_\Gamma \) is a \( \Gamma \)-hypergroup. Then \( S_\alpha \) is a hypergroup for every \( \alpha \in \Gamma \). So by Theorem 5.1, \( \alpha \) has full range, thus \( \alpha(S) = S \).

Conversely, suppose that \( \alpha(S) = S \) for every \( \alpha \in \Gamma \) so \( S_\alpha \) is a hypergroup. Therefore, \( S_\Gamma \) is a \( \Gamma \)-hypergroup.

**Example 19.** Let \( S = \{1,2,3\} \) and \( \Gamma = \{\alpha, \beta\} \) such that \( \alpha = \Delta_S \cup \{(2,1), (3,2)\} \) and \( \beta = \Delta_S \cup \{(3,1)\} \), where \( \Delta_S \) is the diagonal
relation on $S$. Then we have the table of hyperoperations $\alpha$ and $\beta$ as follows:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${1,2}$</td>
<td>$S$</td>
</tr>
<tr>
<td>2</td>
<td>${1,2}$</td>
<td>${1,2}$</td>
<td>$S$</td>
</tr>
<tr>
<td>3</td>
<td>$S$</td>
<td>$S$</td>
<td>${2,3}$</td>
</tr>
</tbody>
</table>

Then $S$ is a $\Gamma$-hypergroup.

**Lemma 5.6.** Let $S$ be a non-empty set and $\Gamma$ be a set of binary relations on $S$ such that $S_\Gamma$ is a $\Gamma$-semihypergroup. Then $I = \Gamma(S) = \bigcup_{\alpha \in I} \alpha(S)$ is a minimal ideal of $S_\Gamma$.

**Proof:** Suppose that $a \in I$, $s \in S$ and $\alpha \in \Gamma$. Then we have $saa = \alpha(a) \cup \alpha(s) \subseteq \alpha(S) \subseteq I$. So $I$ is an ideal of $S_\Gamma$. Furthermore, if $J$ is an ideal of $S_\Gamma$ and $b \in J$, then for every $s \in S$ and $\alpha \in \Gamma$, $sab = \alpha(s) \cup \alpha(b) \subseteq J$. So $\alpha(S) \subseteq J$ hence $I \subseteq J$.

**Proposition 5.7.** Let $S$ be a non-empty set and $\Gamma$ be a set of binary relations on $S$ such that $S_\Gamma$ is a $\Gamma$-semihypergroup. Let $\Gamma_\cup = \{\alpha \cup \beta | \alpha, \beta \in \Gamma\}$. Then $S_{\Gamma_\cup}$ is a $\Gamma_\cup$-semihypergroup.

**Proof:** We prove that $S_{\Gamma_\cup}$ satisfies the conditions $(\Gamma \text{ SH1}), (\Gamma \text{ SH2})$ and $(\Gamma \text{ SH3})$ of Theorem 5.4. Suppose that $\theta', \varphi' \in \Gamma_\cup$. Then there exist $\alpha, \beta, \delta, \gamma \in \Gamma$, such that $\theta' = \alpha \cup \beta$ and $\varphi' = \delta \cup \gamma$. Since $S_\Gamma$ is a $\Gamma$-semihypergroup, it follows that $\alpha \subseteq \alpha \delta \cup \alpha \gamma$ and $\beta \subseteq \beta \delta \cup \beta \gamma$. Thus

$$\theta' = \alpha \cup \beta \subseteq \alpha \delta \cup \alpha \gamma \cup \beta \delta \cup \beta \gamma = (\alpha \cup \beta)(\delta \cup \gamma) = \theta' \varphi'.$$

So the condition $(\Gamma \text{ SH1})$ holds.

Suppose that $x \in S$ is a semiouter element for the relation $\theta'$ and let $(a, x) \in \varphi' \theta'$. Then there exists $h \in S$ such that $(h, x) \notin \theta' \varphi'$. Thus $x$ is a semiouter element for the relations $\alpha \delta, \alpha \gamma, \beta \delta$ and $\beta \gamma$. Since $(a, x) \in \varphi' \theta'$, it follows that $(a, x) \in \delta \alpha$, $(a, x) \in \alpha \gamma$, $(a, x) \in \beta \delta$ or $(a, x) \in \gamma \beta$. From the condition $(\Gamma \text{ SH2})$ for $S_\Gamma$ we conclude that $(a, x) \in \delta$, $(a, x) \in \gamma$, $(a, x) \in \delta$ or $(a, x) \in \gamma$. Thus $(a, x) \in \delta \cup \gamma = \varphi'$ and the condition $(\Gamma \text{ SH2})$ holds.

Suppose that $x \in S$ is a semiouter element for the relations $\theta' \varphi'$ and $\varphi'$ and let $(a, x) \in \varphi' \theta'$. Then there exist $h, t \in S$ such that $(h, x) \notin \theta' \varphi'$ and $(t, x) \notin \varphi'$. So $x$ is a semiouter element for the relations $\alpha \delta, \alpha \gamma, \beta \delta, \beta \gamma, \delta$ and $\gamma$. Thus if $(a, x) \in \alpha \delta, (a, x) \in \alpha \gamma, (a, x) \in \beta \delta$ or $(a, x) \in \gamma \beta$, then from the condition $(\Gamma \text{ SH3})$ for $S_\Gamma$ we conclude that $(a, x) \in \alpha \delta$, $(a, x) \in \alpha \gamma$, $(a, x) \in \beta \delta$ or $(a, x) \in \gamma \beta$, respectively, and the condition $(\Gamma \text{ SH3})$ holds. Therefore, $S_{\Gamma_\cup}$ is a $\Gamma_\cup$-semihypergroup.

Let $S_\Gamma$ be a hypergroupoid associated to a binary relation $R$. Let $\Gamma_\varphi = \{\alpha_i | i \in \mathbb{N}\}$. Now, for every $x, y \in S$ and $\alpha_i \in \Gamma_\varphi$ we define

$x\alpha_i y = \{z | (x, z) \in R^i \lor (y, z) \in R^i\} = L^i_x \cup L^i_y$.

Then $S$ is a $\Gamma_\varphi$-hypergroupoid and denoted by $S_{\Gamma_\varphi}$. In the following we verify conditions such that $S$ is a $\Gamma_\varphi$-semihypergroup.

**Lemma 5.8.** Let $S_\varphi$ be a semihypergroup associated to a binary relation $R$. Then if $(z, t) \in R^{i+j}$ and $(x, t) \notin R^{i+j}$, then $(z, t) \in R^i$, for every $x, z, t \in S$ and $i, j \in \mathbb{N}$.

**Proof:** We prove by mathematical induction on $i + j$. If $i + j = 2$, $(z, t) \in R^2$ and $(x, t) \notin R^2$, then $t$ is an outer element for $R$ so $(z, t) \in R$. 

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Suppose that the result holds for \( i + j - 1 \). Now, let \((z, t) \in R^{i+j}\) and \((x, t) \notin R^{i+j}\). Then there exists \( s \in S \) such that \((z, s) \in R^{2}\) and \((s, t) \in R^{i+j-1}\). Thus \((x, s) \notin R^{2}\), that is, \( s \) is an outer element for \( R \) and so \((z, s) \in R\). Therefore, \((z, t) \in R^{i+j}\). Now, we have \((z, t) \in R^{i+j-1}\) and \((x, t) \notin R^{i+j-1}\) thus \((z, t) \in R^{i+j}\).

**Lemma 5.9.** Let \( S_{R} \) be a semihypergroup associated to a binary relation \( R \). Then \( S_{R}^{i} \) is a \( \Gamma_{R} \)-semihypergroup.

**Proof:** We prove the associativity law. Suppose that \( x, y, z \in S_{R} \) and \( \alpha_{i} \alpha_{j} \in \Gamma \). Then

\[
x_{\alpha_{i}}(y_{\alpha_{j}}z) = L_{x}^{i+j} \cup L_{x}^{i} \cup L_{x}^{j}
\]

and

\[
(x_{\alpha_{i}}y)_{\alpha_{j}}z = L_{x}^{i} \cup L_{x}^{i} \cup L_{x}^{j}
\]

If \( t \in L_{x}^{j+i} \) and \( t \notin L_{x}^{i+j} \), then by the previous lemma \( t \in L_{x}^{j+i} \subseteq (x_{\alpha_{i}}y)_{\alpha_{j}}z \). Therefore, \( x_{\alpha_{i}}(y_{\alpha_{j}}z) \subseteq (x_{\alpha_{i}}y)_{\alpha_{j}}z \). In a similar way we have the inverse inclusion.

**Example 20.** Let \( S = \{1,2,3\} \) and \( R = \{(1,2),(1,3),(2,2),(3,2)\} \). Then \( S_{R}^{i} \) is a semihypergroup. Let \( \Gamma_{R} = \{\alpha_{1}, \alpha_{2}\} \). Then we have the following hyperoperations:

<table>
<thead>
<tr>
<th>( \alpha_{1} )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{1,3}</td>
<td>( S )</td>
<td>( S )</td>
</tr>
<tr>
<td>2</td>
<td>( S )</td>
<td>{2}</td>
<td>{2,3}</td>
</tr>
<tr>
<td>3</td>
<td>( S )</td>
<td>{2,3}</td>
<td>{2}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \alpha_{2} )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( S )</td>
<td>( S )</td>
<td>( S )</td>
</tr>
<tr>
<td>2</td>
<td>( S )</td>
<td>{2}</td>
<td>{2,3}</td>
</tr>
<tr>
<td>3</td>
<td>( S )</td>
<td>{2,3}</td>
<td>{2}</td>
</tr>
</tbody>
</table>

Then \( S_{R}^{i} \) is a \( \Gamma_{R} \)-semihypergroup.

**6. Conclusion**

In this work, we presented the concept of semiprime ideals in a \( \Gamma \)-semihypergroup and proved some results. Also, we introduced the notion of \( \Gamma \)-hyperrings and closed \( \Gamma \)-subhypergroups. Finally, we defined the concept of \( \Gamma \)-semihypergroups and \( \Gamma \)-hypergroups associated to a set of binary relations. Then we find the necessary and sufficient conditions on a set of binary relations \( \Gamma \) on a non-empty set \( S \) such that \( S \) becomes a \( \Gamma \)-semihypergroup or a \( \Gamma \)-hypergroup.

Our future research will consider \( \Gamma \)-semihypergroups associated to binary relations.

**References**


