"Research Note"

THE MODEL UPDATING OF MASS AND STIFFNESS MATRICES*

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Abstract – Modeling is one of the most fundamental tools used to simulate the complex world. The goal of modeling is to modify a numerical method for updating linear eigenvalues problems to reflect measured spectral information. In this method, we will produce the mass and stiffness matrices.

Keywords – Model updating, optimization method, linear pencil

1. INTRODUCTION

The problem of finding scalars \( \lambda \in \mathbb{C} \) and nontrivial vectors \( x \in \mathbb{C}^n \) such that

\[
Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0,
\]

where \( M, C \) and \( K \) are given \( n \times n \) real matrices, is known as the quadratic eigenvalue problem (QEP). The nonzero vectors \( x \) and the corresponding scalars \( \lambda \) are called eigenvectors and eigenvalues of the QEP, respectively. Recently the QEP has received much attention because its information has repeatedly arisen in many different disciplines, vibroacoustics, fluid mechanics and signal processing. Tisseur and Meerbergen have published a survey on QEP [1]. However, due to lack of reliable computational methods to handle distributed parameter systems, a finite element method is generally used for systems of an analytical model (finite element model), namely,

\[
Q_a(\lambda)x = (\lambda^2 M_a + \lambda C_a + K_a)x = 0,
\]

where \( M_a, C_a \) and \( K_a \) represent the measured mass, damping and stiffness, respectively, and all real \( n \times n \) symmetric matrices, and so \( M_a \) is positive definite [2]. In the past decades, Baruch, Bar-Itzack [3], Berman, Nagy [4, 5] and Wei [6, 7, 8] considered variant aspects of finite element model updating by using measured data for undamped structured systems (i.e \( C = C_a = 0 \)).

The purpose of this paper is to develop an algorithm for the computation of solutions \( M_a \) and \( K_a \), where the penalty function

\[
J = \| M - M_a \|_F^2 + \| K - K_a \|_F^2
\]

is minimized, subject to

\[
M\Phi A - K\Phi = 0, \quad M^T = M, \quad K^T = K,
\]

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where, \((\Lambda, \Phi) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}\) are newly measured eigenpairs and \(M\) is positive definite. The fundamentals of the approach are to utilize the parametric representation of \((M, K)\) that Chu, Datta, Lin and Xu developed earlier [9], and to rewrite the objective function as an unconstrained optimization in terms of the free parameters.

2. SOLVING AN IQEP WITHOUT DAMPING

We first consider the self–adjoint pencil

\[ \lambda^2 M_a + K_a, \]

where \(M_a\) is assumed to be positive definite. It is clear that \(\lambda^2\) is real. By defining \(\mu := -\lambda^2\), we can rewrite the quadratic pencil as a linear pencil

\[ L_a(\lambda) = \mu M_a - K_a, \]

effectively reducing the number of eigenvalues for the system (4) to \(n\). We shall make a practical assumption that all eigenvalues are distinct. Such an assumption can be deemed reasonable, because multiple roots are sensitive to perturbation and, hence, are hardly observable in real applications.

Let \((\Lambda, \Phi) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}\) be a given pair of matrices, where

\[ \Lambda = \text{diag} \{ \mu_1, \mu_2, \ldots, \mu_n \}, \quad \Phi = \text{diag} \{ \Phi_1, \Phi_2, \ldots, \Phi_n \}, \]

and since \(\Lambda\) has only simple eigenvalues, \(\Phi\) is a matrix of full rank. We try to find a general form of real positive definite symmetric matrix \(M\) and a symmetric matrix \(K\) that satisfy in (3a).

A general solution of the above problem is given in the theorem that Chu, Datta, Lin and Xu developed [9].

3. OPTIMIZATION METHOD

In this section we shall develop an algorithm for solving the optimization problem described in (3). We will first solve our optimization problem.

**Optimization problem:** Given \((\Lambda, \Phi) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}\) in (5) and let

\[ x = [x_1, x_2, \ldots, x_n]^T, \]

be the vector corresponding to the matrix \(\Gamma\) where:

\[ \Gamma = \text{diag} \{ x_1, x_2, \ldots, x_n \}. \]

We first substitute parametric representation of \((M, K)\) in [4] into (3) and obtain an unconstrained optimization problem. Our optimization problem is of the following form: Minimize

\[ f(x) = \| M - M_a \|_F^2 + \| K - K_a \|_F^2 = \| \Phi^{-T} \Gamma \Phi^{-1} - M_a \|_F^2 + \| \Phi^{-T} \Lambda^T \Gamma \Phi^{-1} - K_a \|_F^2 = \sum_{j=1}^n f_j(x) \]

for \(x\), with

\[ f_j(x) = \| \Phi^{-T} \Gamma \Phi^{-1} - (M_a)_j \|_F^2 + \| \Phi^{-T} \Lambda^T \Gamma \Phi^{-1} - (K_a)_j \|_F^2, \]

where \((M_a)_j, (K_a)_j\) and \(\Phi_j^{-1} = [\Phi_{1j}, \Phi_{2j}, \ldots, \Phi_{nj}]^T\) are the column \(j\) of matrices \(M_a, K_a\) and \(\Phi^{-1}\). The vector \(\Gamma \Phi_j^{-1}\) in (8a) can be rewritten by...
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\[ \Gamma \Phi_j^{-1} = D_j \mathbf{x} \quad , \quad j=1, 2, \ldots, n, \]  
(9)

where

\[ D_j = \text{diag}\{\Phi_{1j}, \Phi_{2j}, \ldots, \Phi_{nj}\}. \]  
(10)

By substituting (9) into (8a) we compute

\[ \nabla f_j(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_j}{\partial x_1} & \frac{\partial f_j}{\partial x_2} & \cdots & \frac{\partial f_j}{\partial x_n} \end{bmatrix}^T \]
\[ = 2(\Phi^{-T} D_j)^T (\Phi^{-T} D_j \mathbf{x} - (M_a)_j) + 2(\Phi^{-T} \Lambda^T D_j)^T (\Phi^{-T} \Lambda^T D_j \mathbf{x} - (K_a)_j). \]  
(11)

Consequently,

\[ \nabla f(\mathbf{x}) = \sum_{j=1}^n \nabla f_j(\mathbf{x}) = \]
\[ 2 \sum_{j=1}^n [D_j^T (\Phi^T \Phi)^{-1} D_j \mathbf{x} - (\Phi^{-T} D_j)^T (M_a)_j] + D_j^T \Lambda (\Phi^T \Phi)^{-1} \Lambda^T D_j \mathbf{x} - (\Phi^{-T} \Lambda^T D_j)^T (K_a)_j]. \]
(12)

Setting \( \nabla f(\mathbf{x}) = 0 \), we end up with the following linear system of equation

\[ P \mathbf{x} = b, \]  
(13)

where

\[ P = \sum_{j=1}^n [D_j^T (\Phi^T \Phi)^{-1} D_j + D_j^T \Lambda (\Phi^T \Phi)^{-1} \Lambda^T D_j] \]  
(14)
\[ b = \sum_{j=1}^n [(\Phi^{-T} D_j)^T (M_a)_j + (\Phi^{-T} \Lambda^T D_j)^T (K_a)_j]. \]  
(15)

Since the function \( f(\mathbf{x}) \) in (8) must have an optimum value, the linear system of equation (13) is a consistent system.

**Algorithm:** Given \( L(\lambda) = \lambda M_a - K_a \) and \( (\Lambda, \Phi) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \) as in (5). The optimal solutions \( M \) and \( K \) in (3) are computed by

**Step 1.** Solve \( P \mathbf{x} = b \) for \( \mathbf{x} = [x_1, x_2, \ldots, x_n]^T \) where \( P \) and \( b \) are given by (14) and (15).

**Step 2.** Compute \( M \) and \( K \) with

\[ M = \Phi^{-T} \Gamma \Phi^{-1}, \quad K = \Phi^{-T} \Gamma \Lambda \Phi^{-1}. \]

### 4. NUMERICAL RESULTS

In this section, we will present a numerical example to show that our algorithm is reliable. We will report all numbers in 16 significant digits.

**Example 1.** To generate test data, we first randomly generate a \( 3 \times 3 \) real symmetric

\[ L(\lambda) = \lambda M_a - K_a, \]  
where

\[ M_a = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 7 \end{bmatrix}, \]  
\[ K_a = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \]

and compute its ‘exact’ eigenpairs \((\Lambda, \Phi)\). We conclude that

\[ \Lambda = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_n\}, \quad \Phi = [\Phi_1, \Phi_2, \ldots, \Phi_n], \]
with
\[ \mu_1 = 1.603022689155528e + 000, \quad \mu_2 = 1.000000000000000e + 000, \]
\[ \mu_3 = 3.969773108444730e - 001 \]
and the corresponding eigenvectors
\begin{align*}
\Phi_1 &= \begin{pmatrix} 1.000000000000000e + 000 \\ -5.190356543872574e - 001 \\ -3.807130877451422e - 002 \end{pmatrix}, \\
\Phi_2 &= \begin{pmatrix} 9.999999999999999e - 001 \\ 1.000000000000000e + 000 \\ 6.549108341133741e - 017 \end{pmatrix}, \\
\Phi_3 &= \begin{pmatrix} 9.999999999999999e - 001 \\ 1.000000000000000e + 000 \end{pmatrix}.
\end{align*}

The algorithm should theoretically give the optimal solution \( M = M_a \) and \( K = K_a \). The numerical result of the relative errors computed by the algorithm are estimated
\[ \frac{\| M - M_a \|_F}{\| M \|_F} = 1.095122344567561e-015, \quad \frac{\| K - K_a \|_F}{\| K \|_F} = 1.5261926477674955e - 015. \]

5. CONCLUSION

One common procedure to improve the discrepancy between a mathematical model and the corresponding real-world system is to modify the model parameters in such a way to achieve a good correspondence between the analytical solution and the real data. In this paper we have considered a model updating of self-adjoint linear pencils using all measured natural frequencies and mode shapes. The model updating problem is cast as a generalized inverse eigenvalues problem with prescribed eigenpairs. We have used a parametric representation of the solution to the IQEP in which symmetry is required of the matrices involved. The example which is given is used to demonstrate that the algorithm is reliable. Also, the efficiency of the algorithm will be preserved for large \( n \).

REFERENCES