ON THE STABILITY OF SOLUTIONS FOR NON-AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS OF THIRD-ORDER

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Abstract – Some sufficient conditions for uniform asymptotic stability of null solution to a certain nonlinear delay differential equation of third order are established. By introducing a Lyapunov functional, a new result is obtained which includes and improves some related results in literature.

Keywords – Stability, non-autonomous differential equations with delay, third order

1. INTRODUCTION

Ideally, one would like to find all the solutions of every differential equation or system of differential equations clearly, which is under consideration. However, as it is well known, there are actually very few equations (beyond linear equations with constant coefficient and even there are difficulties if the order of the equation or system is high) for which we can do this. That is, it is not possible to find the exact solution of every linear or nonlinear differential equation in general, except numerically. In fact, if the solution of any differential equation under investigation is known in the closed form, then one can determine the stability properties of the solution by using the known definitions related to the stability of solutions directly. To date some methods have been developed to obtain information on the qualitative behavior of solutions of differential equations in literature when there is no analytical expression for the solutions. One of these methods is recognized as the Lyapunov’s [1] second method. This technique is also called the direct method because it can be applied to a differential equation directly, without any knowledge of solutions, provided the person using the method is clever in constructing the right auxiliary function or functional. This method is also a general technique and a useful approach in investigating qualitative properties of solutions of ordinary differential equations. The extensions of the Lyapunov’s second (or direct) method to functional differential equations with finite delay have also been performed by using both Lyapunov functions and functionals. This method is widely recognized today as an indispensable tool not only in the theory of stability, but also in the investigation of various other properties of solutions of differential equations with delay or without delay, such as the instability of solutions, boundedness of solutions, existence of periodic solutions, asymptotic behaviors of solutions, etc.. In application of Lyapunov’s theory to concrete problems, the difficulty is always to find a useful auxiliary function or functional, that is, the Lyapunov function or functional which verifies the assumptions of the Lyapunov’s theorems. The construction of Lyapunov functions and functionals remain a general problem in literature. However, with respect to our observations in the literature, till now, many good results have appeared on the stability of solutions to nonlinear third order ordinary differential equations. It is also worth mentioning that there are only a few papers on the same topic to third order
nonlinear differential equations with delay, despite the existence of many papers on the stability of solutions of third order nonlinear differential equations without delay. In particular, one can refer to the papers or books of Burton [2-4], Burton and Zhang [5], Burton and Hattam [6], Burton and Hering [7], Burton and Makay [8], Chukwu [9], Eil’sgol’ts [10], Furumochi [11], Hale [12], Kolmanovskii and Myshkis [13], Krasovskii [14], Li and Wen [15], Makay [16], Qian [17], [18], Reissig et al. [19], Sadek [20], [21], Sinha [22], Tunç [23-31], Yoshizawa [32], Yuanxun et al. [33], Zhu [34] and references cited thereof for some publications performed on the topic, which contain differential equations with delay or without delay. Throughout the aforementioned papers, the Lyapunov’s second (or direct) method is used as a main tool to carryout the proof of the main results established there. With respect to our observations, some results obtained in the literature on the stability of solutions of third order differential equations with delay can be summarized, as follows:

First, in 1973, Sinha [22] established some sufficient conditions for the asymptotic stability of the zero solution of third order differential equation with delay:
\[ \ddot{x}(t) + f(x(t), \dot{x}(t))\dot{x}(t) + g(x(t-r)) + h(x(t-r)) = 0. \]

Second, in 1992, Zhu [34] considered the following third order non-linear delay differential equations:
\[ \ddot{x} + \alpha \dot{x} + \phi(\dot{x}(t-r)) + f(x) = 0 \quad (1) \]
and
\[ \ddot{x} + \alpha \dot{x} + b \dot{x} + f(x(t-r)) = 0. \quad (2) \]

He investigated the asymptotic stability of the null solution of Equation (1) and Equation (2). In two recent papers published in 2003 and 2005, Sadek [20], [21] discussed the asymptotic stability of the solution \( x = 0 \) to the following third order non-linear delay differential equations:
\[ \ddot{x} + a \dot{x} + g(x(t-r(t))) + f(x(t-r(t))) = 0 \]
and
\[ \ddot{x} + a(t) \dot{x} + b(t) \dot{x} + c(t) f(x(t-r(t))) = 0, \quad (3) \]
respectively.

Recently, the author in [28] proved a similar result on the asymptotic stability of the solution \( x = 0 \) of the delay differential equation:
\[ \ddot{x} + \varphi(x, \dot{x})\dot{x} + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = 0. \]

In the present paper, we consider the nonlinear delay differential equation of third order:
\[ \ddot{x}(t) + a(t)\varphi(x(t), \dot{x}(t))\dot{x}(t) + b(t)\psi(x(t), \dot{x}(t)) + c(t)h(x(t-r(t))) = 0, \quad (4) \]
where \( r \) is a positive constant; \( a(t), b(t) \) and \( c(t) \) are positive and continuously differentiable functions on \([0, \infty)\); \( \varphi(x, \dot{x}), \psi(x, \dot{x}) \) and \( h(x) \) are continuous functions on their respective domains; \( \psi(x,0) = h(0) = 0. \) The derivatives \( \frac{\partial}{\partial x} \varphi(x, \dot{x}) \equiv \varphi_x(x, \dot{x}) \) and \( \frac{\partial}{\partial x} \psi(x, \dot{x}) \equiv \psi_x(x, \dot{x}) \) exist and are also continuous, and differentiability of the function \( h \) is also assumed. Throughout the paper \( x(t), y(t) \) and \( z(t) \) are abbreviated as \( x, y \) and \( z \), respectively. We prove here the uniform asymptotic stability of the null solution of Equation (4). It is worth mentioning that Equation (2) and
Equation (3) are special cases of our equation (4). The motivation for the present work has been inspired basically by the paper of Sadek [21], Zhu [34] and the papers mentioned above.

2. PRELIMINARIES

We will give some basic definitions and important stability criteria for the general non-autonomous delay differential system. Consider the general non-autonomous delay differential system

\[ \dot{x} = f(t, x_t), \quad x(t + \theta) = x(t), \quad -r \leq \theta \leq 0, \quad t \geq 0, \]

where \( f : [0, \infty) \times C_H \to \mathbb{R}^n \) is a continuous mapping, \( f(t, 0) = 0 \), and we suppose that \( f \) takes closed bounded sets into bounded sets of \( \mathbb{R}^n \). Here \( \left( C, \| \cdot \| \right) \) is the Banach space of continuous functions \( \phi : [-r, 0] \to \mathbb{R}^n \) with supremum norm, \( r > 0 \); \( C_H \) is the open \( H \)-ball in \( C \); \( C_H := \{ \phi \in \left( C [-r, 0], \mathbb{R}^n \right) : \| \phi \| < H \} \). Standard existence theory, [3], shows that if \( \phi \in C_H \) and \( t \geq 0 \), then there is at least one continuous solution \( x(t, t_0, \phi) \) such that on \( [t_0, t_0 + \alpha] \) satisfying Equation (5) for \( t > t_0 \), \( x(t, \phi) = \phi \) and \( \alpha \) is a positive constant. If there is a closed subset \( B \subset C_H \) such that the solution remains in \( B \), then \( \alpha = \infty \). Further, the symbol \( \| \cdot \| \) will denote the norm in \( \mathbb{R}^n \) with \( |x| = \max_{i \leq i \leq n} |x_i| \).

**Definition 1.** (See [3].) A continuous function \( W : [0, \infty) \to [0, \infty) \) with \( W(0) = 0 \), \( W(s) > 0 \) if \( s > 0 \), and \( W \) strictly increasing is a wedge. (We denote wedges by \( W \) or \( iW \), where \( i \) an integer.)

**Definition 2.** (See [3].) Let \( D \) be an open set in \( \mathbb{R}^n \) with \( 0 \in D \). A function \( V : [0, \infty) \times D \to [0, \infty) \) is called positive definite if \( V(t, 0) = 0 \) and if there is a wedge \( W_1 \) with \( V(t, x) \geq W_1(|x|) \), and is called decrescent if there is a wedge \( W_2 \) with \( V(t, x) \leq W_2(|x|) \).

**Definition 3.** (See [8].) A continuous functional \( V : [0, \infty) \times C_H \to [0, \infty) \), which is locally Lipschitzian in \( \phi \), is called a Lyapunov functional for Equation (5) if there is a wedge \( W \) with

(i) \( W(|\phi(0)|) \leq V(t, \phi), \quad V(t, 0) = 0 \), and

(ii) \( \dot{V}(t, x_t) = \lim \sup_{h \to 0} \frac{1}{h} \left[ V(t + h, x_{t+h}(t_0, \phi)) - V(t, x(t_0, \phi)) \right] \leq 0 \).

**Definition 4.** (See [8].) Let \( f(t, 0) = 0 \). The zero solution of Equation (5) is:

(i) stable if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( [t \geq 0, |\phi| < \delta, t \geq t_0] \) implies that \( |x(t, t_0, \phi)| < \varepsilon \).

(ii) asymptotic stable if it is stable and if for each \( t \geq 0 \) there is an \( \eta > 0 \) such that \( |\phi| < \eta \) implies that \( x(t, t_0, \phi) \to 0 \) as \( t \to \infty \).

**Theorem 1.** (See ([7].)) If there is a Lyapunov functional \( V(t, \phi) \) for Equation (5) and wedges \( W_1 \), \( W_2 \) and \( W_3 \) such that

(i) \( W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi|) \), (where \( W_1(r) \) and \( W_2(r) \) are wedges,)

(ii) \( \dot{V}(t, x_t) \leq -W_3(|x(t)|) \), (where \( W_3(r) \) is a wedge,)

then the zero solution of Equation (5) is uniformly asymptotically stable.

**Example 1.** Consider the following nonlinear second order delay differential equation:
\[ \dot{x}(t) + \frac{t^2 + 2}{t^2 + 1} \phi(x(t - r)) \dot{x}(t) + \frac{e^t + 1}{e^t} x(t) = 0, \quad (6) \]

where \( r \) is a positive constant, \( t \in [0, \infty) \), \( \phi \) is a continuous function such that \( \phi(x) \geq a > 0 \) for all \( x \), and the derivative \( \frac{d}{dx} \phi(x) \equiv \phi'(x) \) exists and is also continuous. Equation (6) can be transformed to the following system:

\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\frac{t^2 + 2}{t^2 + 1} \phi(x) y + \frac{t^2 + 2}{t^2 + 1} \int_{t-r}^{t} \phi'(x(s)) y(s) ds - \frac{e^t + 1}{e^t} x.
\end{align*} \quad (7) \]

We define the Lyapunov functional

\[ V_0(t, x_r, y_r) = (1 + e^{-t}) \left( \frac{x^2}{2} + \frac{y^2}{2} + \lambda \int_{t-r}^{t} \int_{t-s}^{t} y^2(\eta) d\eta ds \right) \quad (8) \]

to verify the stability of the solution \( x = 0 \) of Equation (6), where \( \lambda \) is a positive constant which will be determined later. It is clear that the Lyapunov functional \( V_0(t, x_r, y_r) \) is positive definite: \( V_0(t, 0, 0) = 0 \), and we have from (8) that

\[ 0 < \frac{x^2}{2} + \frac{y^2}{2} \leq V_0(t, x_r, y_r), \]

and

\[ V_0(t, x_r, y_r) \leq x^2 + \frac{y^2}{2} + \lambda \int_{t-r}^{t} \int_{t-s}^{t} y^2(\eta) d\eta ds . \]

The time derivative of the functional \( \dot{V} = V_0(t, x_r, y_r) \) in (8) with respect to the system (7) can be calculated as follows:

\[ \dot{V}_0 = \frac{d}{dt} V_0(t, x_r, y_r) = -\frac{e^{-t}}{2} x^2 - \frac{t^2 + 2}{t^2 + 1} \phi(x) y^2 \]

\[ + \frac{t^2 + 2}{t^2 + 1} y^2 \int_{t-r}^{t} \phi'(x(s)) y(s) ds \]

\[ + \lambda y^2 r - \lambda \int_{t-r}^{t} y^2(s) ds . \]

The assumption \( \phi(x) \geq a > 0 \) implies that

\[ \dot{V}_0 \leq -\frac{e^{-t}}{2} x^2 - a y^2 + \lambda y^2 r - \lambda \int_{t-r}^{t} y^2(s) ds \]

\[ = -\frac{e^{-t}}{2} x^2 - (a - \lambda r) y^2 - \lambda \int_{t-r}^{t} y^2(s) ds . \]

Let \( \lambda = L \). Then we have for a positive constant \( \alpha \) that

\[ \dot{V}_0 \leq -\frac{e^{-t}}{2} x^2 - \alpha y^2 \leq 0 \text{ provided } r < \frac{a}{L}. \]
Thus, subject to the above discussion and Theorem 1, we conclude that the null solution of Equation (6) is uniform asymptotic stable.

3. MAIN RESULT

A system equivalent to Equation (4) is

\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= z,
\end{align*} \]

\[ \begin{align*}
\dot{z} &= -a(t)\varphi(x,y)z - b(t)\psi(x,y) - c(t)h(x) + c(t)\int_{t-r}^{t} h'(x(s))y(s)ds.
\end{align*} \]  \tag{9}

Our main result is the following theorem.

**Theorem 2.** In addition to the basic assumptions imposed on the functions \( \varphi, \psi, h, a(t), b(t) \) and \( c(t) \), we suppose that there exist positive constants \( A, B, C, a, b, a_0, b_0, c_0, h_0, h_1, \alpha, \mu \) such that the following conditions hold:

(i) \( A \geq a(t) \geq a_0 > 0, \ B \geq b(t) \geq b_0 > 0, \ C \geq c(t) \geq c_0 > 0 \) for all \( t \in [0, \infty) \),

(ii) \( h(0) = 0, \ \frac{h(x)}{x} \geq h_0 > 0, \ (x \neq 0) \), and \( h'(x) \leq h_1 \leq 1 \),

(iii) \( a_0b_0 - C > 0, \ \psi(x,0) = 0, \ 0 \leq \frac{\psi'(x,y)}{y} - b \leq \alpha \ (b \geq 1, \ y \neq 0) \) and \( \psi'(x,y) \leq 0 \),

(iv) \( \frac{1}{2} \left[ \frac{b'(t)}{b + \alpha} + \frac{c'(t)}{c + \alpha} - \frac{c'(t)}{2} + \frac{\alpha}{2} \right] < \frac{a_0b_0 - C}{2b} \),

for all \( t \in [0, \infty) \), where \( \mu = \frac{a_0b_0 + C}{2b_0} \),

(v) \( \int_{0}^{\infty} c'(t)dt < \infty \), \( c'(t) \to 0 \) as \( t \to \infty \),

(vi) \( 0 \leq \varphi(x,y) - a \leq \varepsilon \ (a \geq 1) \) and \( \varphi_{xx}(x,y) \leq 0 \).

Then the null solution of Equation (4) is uniform asymptotic stable, provided

\[ r < \min \left\{ \frac{2c_0h_0}{h_1C}, \frac{a_0b_0 + C}{h_1C}, \frac{(a_0b_0 - C)b + 4a_0C(1-h_1)}{2h_1C[1 + 2\mu + 2a_0^2 + a_0(a_0b_0 - C)C]} \right\} \]

with \( \mu = \frac{a_0b_0 + C}{2b_0} \).

**Remark:** Theorem 2 includes the results of Sadek [21] and Zhu [34, Theorem 2.1] and improves them to Equation (4).

**Proof of Theorem 2.** To prove this theorem, we define a Lyapunov functional \( V_3 = V_3(t,x,y,z) : \)

\[ V_3(t,x,y,z) = e^{-\int_{t_0}^{t} \gamma(s)ds} \left[ V_1(t,x,y,z) + V_2(t,x,y,z) \right] \]

where \( \gamma(t) = \int_{t_0}^{t} c'(s)ds \), and it is assumed that \( \int_{0}^{\infty} c'(t)dt \leq N < \infty \),

\[ V_1(t,x,y,z) = \mu c(t) \int_{0}^{x} h(\xi)d\xi + c(t)h(x)y + b(t) \int_{0}^{y} \psi(x,\eta)d\eta + \mu a(t) \int_{0}^{y} \varphi(x,\eta)d\eta \]

\[ V_2(t,x,y,z) = \int_{t-r}^{t} \left[ h'(x(s))y(s)ds \right]. \]
+ µyz + \frac{1}{2} z^2 + \lambda \int_{-r}^{0} \int_{r+s}^{y} y^2 (\theta) d\theta ds,
(11)
and
\begin{align*}
V_2(t,x,y,z) &= a_0^2 c(t) \int_{0}^{x} h(\xi) d\xi + b_0^2 (a_0 b_0 - C)x^2 + a_0 c(t) h(x) y + b(t) \int_{0}^{y} \psi(x,\eta) d\eta \\
&\quad + (a_0 b_0 - C) x (z + a_0 y) + \frac{a_0}{2} z^2 + a_0^2 y z + a_0^3 \int_{0}^{y} \phi(x,\eta) \eta d\eta.
(12)
\end{align*}

Differentiating the functional \( V_1(t,x,y,z) \) with respect to \( t \) along trajectories of the system (9), we get
\begin{align*}
\dot{V}_1 &= \frac{d}{dt} V_1(t,x,y,z) = c(t) h'(x) y^2 + \mu y^2 \left[ \frac{\mu b(t)}{y} \psi(x,y) \right] y^2 \\
&\quad - a(t) \varphi(x,y) z^2 + \mu a(t) y \int_{0}^{y} \psi(x,\eta) \eta d\eta \frac{1}{2} \left[ h(x) + \frac{y}{\mu} \right]^2 \\
&\quad + \mu c'(t) \int_{0}^{y} \left[ 1 - h'(\xi) \right] h(\xi) d\xi + c(t) (\mu y + z) \int_{0}^{y} h''(x(s)) y(s) ds + b(t) y \int_{0}^{y} \psi(x,\eta) d\eta \\
&\quad - \lambda r y^2 \lambda \int_{-r}^{y} y^2(s) ds + b'(t) \int_{0}^{y} \psi(x,\eta) d\eta + \mu a'(t) \int_{0}^{y} \phi(x,\eta) \eta d\eta - \frac{1}{2} \frac{c'(t) y^2}{\mu}.
(13)
\end{align*}

On the other hand, by noting (13), from the assumptions \( a(t) \geq a_0, \quad c(t) \leq C, \quad \varphi(x,y) \leq 0, \quad 0 < \varphi(x,y) \leq a + \varepsilon \quad (a \geq 1), \quad 0 < b \leq \frac{y(x,y)}{\mu} \quad (b \geq 1), \quad \psi(x,y) \leq 0, \quad h'(x) \leq h_1, \quad \mu = \frac{a_0 b_0 + C}{2b_0} \)
of Theorem 2 and inequality \( 2|\mu| \leq u^2 + v^2 \), it follows the existence of the following inequalities:
\begin{align*}
\dot{V}_1 &\leq - \left[ a(t) \varphi(x,y) - \mu - \frac{Ch(r)}{2} \right] z^2 - \left[ \mu b(t) \frac{\psi(x,y)}{y} - c(t) h'(x) - \frac{\mu Ch(r)}{2} \right] y^2 \\
&\quad + \left[ \frac{Ch}{2} + \frac{\mu Ch}{2} - \lambda \right] \int_{-r}^{y} y^2(s) ds + \mu c'(t) \int_{0}^{y} \left[ 1 - h'(\xi) \right] h(\xi) d\xi \\
&\quad + \frac{1}{2} \mu c'(t) \left[ h(x) + \frac{y}{\mu} \right]^2 \\
&\quad + \left[ b'(t) \int_{0}^{y} \psi(x,\eta) d\eta + \mu a'(t) \int_{0}^{y} \phi(x,\eta) \eta d\eta - \frac{1}{2} \frac{c'(t) y^2}{\mu} \right] \\
&\leq - \left[ a_0 a - \mu - \frac{Ch(r)}{2} \right] z^2 - \left[ \mu b_0 b - Ch_1 - \frac{\mu Ch(r)}{2} - \lambda r \right] y^2 \\
&\quad + \left[ \frac{Ch_1}{2} - \frac{Ch}{2} - \lambda \right] \int_{-r}^{y} y^2(s) ds + \mu c'(t) \int_{0}^{y} \left[ 1 - h'(\xi) \right] h(\xi) d\xi.
\end{align*}
\[
+ \frac{1}{2} \mu c'(t) \left[ h(x) + \frac{y}{\mu} \right]^2 + \frac{1}{2} \left[ h'(t) \left( b + \alpha \right) + \mu \left( a + \epsilon \right) \left( a - \frac{1}{\mu} c'(t) \right) \right] y^2.
\] (14)

The time derivative of \( V_2(t, x, y, z, h) \) along trajectories of the system (9) gives the following relation:

\[
\dot{V}_2 = \frac{d}{dt} V_2(t, x, y, z, t) = - (a_0 b_0 - C) c(t) h(x) x - a_0 C y^2 + a_0 c(t) h'(x) y^2 \\
+ a_0 c'(t) \int_0^x \left[ 1 - h'(\xi) \right] h(\xi) d\xi + \frac{1}{2} a_0^2 c'(t) h^2(x) \\
+ a_0 c'(t) h(x) y + b'(t) \int_0^y \psi(x, \eta) d\eta + b(t) y \int_0^y \psi(x, \eta) d\eta \\
+ a_0 y \int \phi_\xi(x, \eta) d\eta + (a_0 b_0 - C) c(t) x \int_{t-r}^t h'(x(s)) y(s) ds \\
+ a c(t) z \int_{t-r}^t h'(x(s)) y(s) ds + a_0^2 c(t) y \int_{t-r}^t h'(x(s)) y(s) ds.
\] (15)

Taking into account (15), the assumptions \( h(x) \geq h_0 > 0 \) \((x \neq 0)\), \( h'(x) \leq h_1 \), \( 0 \leq \psi(x, y) - b \leq \alpha \) \((y \neq 0)\), \( \psi_\xi(x, y) \leq 0 \), \( y \phi_\xi(x, y) \leq 0 \) of Theorem 2 and inequality \( 2 |\mu| \leq u^2 + v^2 \) imply that

\[
\dot{V}_2 \leq - (a_0 b_0 - C) \left[ c_0 h_0 - \frac{Ch_1 r}{2} \right] x^2 - a_0 C \left[ (1 - h_1) - \frac{a_0 h_1 r}{2} \right] y^2 \\
+ a_0 Ch_1 r \left[ \frac{(a_0 b_0 - C) Ch_1}{2} + \frac{a_0 Ch_1}{2} + \frac{a_0 Ch_1}{2} \right] \int_{t-r}^t y^2(s) ds \\
+ a_0^2 c'(t) \int_0 \left[ 1 - h'(\xi) \right] h(\xi) d\xi + \frac{1}{2} a_0^2 c'(t) \left[ h(x) + \frac{y}{a_0} \right]^2 \\
+ \frac{|c'(t)| (b + \alpha)}{2} \frac{\epsilon}{2} c(t) y^2.
\] (16)

From the estimates (14) and (16) and the fact \( \mu = \frac{a_0 b_0 + C}{2b_0} \), \( a \geq 1 \) and \( h_1 \leq 1 \), we obtain

\[
\dot{V}_1 + \dot{V}_2 \leq - (a_0 b_0 - C) \left[ c_0 h_0 \right] x^2 - \left[ a_0 a - \mu - \frac{Ch_1 r}{2} - \frac{a_0 Ch_1 r}{2} \right] y^2 \\
- \left[ \mu b_0 b - Ch_1 + a_0 C (1 - h_1) - \frac{a_0^2 Ch_1 r}{2} - \frac{\mu Ch_1 r}{2} - \lambda r \right] y^2 \\
+ \left[ \frac{Ch_1}{2} + \frac{\mu Ch_1}{2} + \frac{(a_0 b_0 - C) Ch_1}{2} + \frac{a_0 Ch_1}{2} + \frac{a_0 Ch_1}{2} - \lambda \right] \int_{t-r}^t y^2(s) ds.
\]
\[
\begin{align*}
+ (\mu + a_0^2) c'(t) \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} \mu c'(t) \left[ h(x) + \frac{y}{\mu} \right]^2 \\
+ \frac{1}{2} \left[ b'(t) \left( b + \alpha \right) + \frac{c'(t)(b + \alpha)}{2} - \frac{c'(t)}{2} + \frac{\mu a'(t)(a + \varepsilon)}{2} - \frac{1}{\mu} c'(t) \right] y^2 \\
+ \frac{1}{2} a_0^2 c'(t) \left[ h(x) + \frac{y}{a_0} \right]^2 \\
\leq - (a_0 b_0 - C) \left[ c_0 h_0 - \frac{C h_1 r}{2} \right] x^2 - \left[ a_0 - \mu - \frac{C h_1 r}{2} - \frac{a_0 C h_1 r}{2} \right] z^2 \\
- \left[ \frac{(a_0 b_0 - C)b}{2} + a_0 C (1 - h_1) - \frac{a_0^2 C h_1 r}{2} - \frac{\mu C h_1 r}{2} - \lambda r \right] y^2 \\
+ \left[ \frac{C h_1}{2} + \frac{\mu C h_1}{2} + \frac{(a_0 b_0 - C) C h_1}{2} + \frac{a_0 C h_1}{2} + \frac{a_0^2 C h_1}{2} - \lambda \right] \int_0^x y^2(s) ds \\
+ (\mu + a_0^2) c'(t) \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} \mu c'(t) \left[ h(x) + \frac{y}{\mu} \right]^2 \\
+ \frac{1}{2} \left[ b'(t) \left( b + \alpha \right) + \frac{c'(t)(b + \alpha)}{2} - \frac{c'(t)}{2} + \frac{\mu a'(t)(a + \varepsilon)}{2} - \frac{1}{\mu} c'(t) \right] y^2 \\
+ \frac{1}{2} a_0^2 c'(t) \left[ h(x) + \frac{y}{a_0} \right]^2. 
\end{align*}
\]

Let
\[
\lambda = \left[ \frac{C h_1}{2} + \frac{\mu C h_1}{2} + \frac{(a_0 b_0 - C) C h_1}{2} + \frac{a_0 C h_1}{2} + \frac{a_0^2 C h_1}{2} - \lambda \right].
\]

In view of \( \mu = \frac{(a_0 b_0 - C)}{2b_0} \), the assumptions of Theorem 2 and (18), it follows from (17) that
\[
\dot{V}_1 + \dot{V}_2 \leq - (a_0 b_0 - C) \left[ c_0 h_0 - \frac{C h_1 r}{2} \right] x^2 - \left[ a_0 b_0 - C - \frac{(1 + a_0) C h_1 r}{2} \right] z^2 \\
- \left[ \frac{(a_0 b_0 - C)b}{2} + 4a_0 C (1 - h_1) - \frac{C h_1}{2} \right] \left[ 1 + 2\mu + 2a_0^2 + a_0 + (a_0 b_0 - C) C \right] y^2 \\
+ (\mu + a_0^2) c'(t) \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} \mu c'(t) \left[ h(x) + \frac{y}{\mu} \right]^2 \\
+ \frac{1}{2} a_0^2 c'(t) \left[ h(x) + \frac{y}{a_0} \right]^2.
\]
Now, we consider the expression
\[
S_1 \equiv \mu c(t) \int_0^x h(\xi) d\xi + c(t) h(x) y + b(t) \int_0^x \psi(x, \eta) d\eta \\
+ \mu a(t) \int_0^x \varphi(x, \eta) d\eta + \mu y z + \frac{1}{2} z^2,
\]
the terms of \( S_1 \) are contained in (11). The assumptions of Theorem 2 and the fact \( \mu = \frac{a_0 b_0 + C}{2b_0} \) yields that
\[
S_1 \geq \mu c(t) \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} \mu c(t) \left[ h(x) + \frac{y}{\mu} \right]^2 + \frac{1}{2} b(t) y^2 \\
+ \frac{\mu}{2} a(t) y^2 + \mu y z + \frac{1}{2} z^2 - \frac{1}{2\mu} c(t) y^2 \\
= \mu c(t) \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} \mu c(t) \left[ h(x) + \frac{y}{\mu} \right]^2 + \frac{1}{2} (\mu y + z)^2 \\
+ \frac{1}{2\mu} \left[ \mu b(t) - c(t) + \mu^2 \{ a(t) - \mu \} \right] y^2 \\
\geq \mu c_0 \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} \mu c_0 \left[ h(x) + \frac{y}{\mu} \right]^2 + \frac{1}{2} (\mu y + z)^2 \\
+ \frac{1}{2\mu} \left[ \mu b_0 - C + \mu^2 (a_0 - \mu) \right] y^2 \\
\geq \mu c_0 \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} \mu c_0 \left[ h(x) + \frac{y}{\mu} \right]^2 + \frac{1}{2} (\mu y + z)^2 \\
+ \frac{1}{2\mu} \left[ \frac{a_0 b_0 - C}{2} - C + \mu^2 \left( \frac{a_0 b_0 - C}{2 b_0} \right) \right] y^2.
\]
Next, subject to the assumptions of Theorem 2, an easy calculation gives that
\[
V_2(t, x, y, z) \geq a_0^2 c(t) \int_0^x [1 - h'(\xi)] h(\xi) d\xi + \frac{1}{2} a_0^2 c(t) \left[ h(x) + \frac{y}{a_0} \right]^2
\]
$$+ \frac{1}{2} a_0 \left( (z + a_0 y) + \frac{(a_0 b_0 - C)x}{a_0} \right)^2 + \frac{(a_0 b_0 - C)C}{2a_0} x^2.$$ 

Combining the preceding estimates, we have

$$V_1 + V_2 \geq (\mu + a_0^2) c_0 \int_0^x \left[ 1 - h'(\xi) \right] h(\xi) d\xi + \frac{1}{2} \mu c_0 \left[ h(x) + \frac{y}{\mu} \right]^2$$

$$+ \frac{1}{2} a_0^2 c_0 \left[ h(x) + \frac{y}{a_0} \right]^2 + \frac{1}{2\mu} \left[ a_0 b_0 - C \right] + \mu \left[ a_0 b_0 - C \right] \left[ \frac{a_0 b_0 - C}{2b_0} \right] y^2$$

$$+ \frac{1}{2} \left( \mu v + z \right)^2.$$ 

(20)

Hence, we can find a continuous function $W_1(\|\phi(0)\|)$ such that

$$W_1(\|\phi(0)\|) \geq 0 \text{ and } W_1(\|\phi(0)\|) \leq V_1(t, \phi).$$

Similarly, subject to the assumptions of Theorem 2, the existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $V_3(t, \phi) \leq W_2(\|\phi\|)$ can be verified.

Taking into consideration the above discussion and (20), we get

$$\frac{d}{dt} V_3(t, x, y, z, t) = e^{-\frac{\gamma(t)}{c_0}} \left[ \frac{d}{dt} \left( V_1 + V_2 - \frac{|e(t)|}{c_0} (V_1 + V_2) \right) \right].$$ 

(21)

Making use of the inequalities (19), (20) and $c'(t) - |c'(t)| \leq 0$, it follows that

$$\frac{d}{dt} \left( V_1 + V_2 - \frac{|e(t)|}{c_0} (V_1 + V_2) \right) \leq - \frac{(a_0 b_0 - C)}{c_0} \left[ c_0 h_0 - \frac{Ch}{2} \right] x^2 - \left[ \frac{a_0 b_0 - C}{2b_0} - \frac{(1 + a_0)Ch}{2} \right] z^2$$

$$- \left[ \frac{(a_0 b_0 - C)b + 4a_0 C(1 - h_1)}{4} - \frac{Ch}{2} \right] \left[ 1 + 2\mu + 2a_0^2 + a_0 + (a_0 b_0 - C) \right] r y^2.$$ 

(22)

Hence, we have from (21) and (22) for some constant $\alpha > 0$ that

$$\frac{d}{dt} V_3(t, x, y, z, t) \leq - \alpha e^{-\frac{\gamma(t)}{c_0}} (x^2 + y^2 + z^2) \leq - W_3(\|x(t)\|)$$

provided

$$r < \min \left\{ \frac{2c_0 h_0}{h C}, \frac{a_0 b_0 - C}{(1 + a_0) b_0 h C}, \frac{(a_0 b_0 - C)b + 4a_0 C(1 - h_1)}{2h C} \left[ 1 + 2\mu + 2a_0^2 + a_0 + (a_0 b_0 - C) \right]\right\}.$$
The proof of Theorem 2 is now complete.

**Example:** As a special case of Equation (4), we consider the following third order nonlinear delay differential equation

\[
x'' + \left(4 + \frac{1}{1 + (x')^2}\right)x' + 9x' + \frac{x'}{1 + (x')^2} + \frac{1}{4}x(t-r) = 0.
\]  

(23)

We consider a system equivalent to (23):

\[
x' = y,
\]

\[
y' = z,
\]

\[
z' = -\left(4 + \frac{1}{1 + y^2}\right)z - 9y - \frac{y'}{1 + y^2}
\]

\[
= \frac{x}{4} + \frac{1}{4} \int_{t-r}^{t} y(s) ds.
\]

One can follow the following:

\[
\phi(y) = 4 + \frac{1}{1 + y^2},
\]

\[
\phi(y) - 4 = \frac{1}{1 + y^2},
\]

\[
0 \leq \frac{1}{1 + y^2} \leq 1,
\]

that is,

\[
0 \leq \phi(y) - 4 \leq 1,
\]

\[
a = 4 > 1,
\]

\[
\varepsilon = 1;
\]

\[
\psi(y) = 9y + \frac{y}{1 + y^2}, \quad \psi(0) = 0,
\]

\[
\frac{\psi(y)}{y} = 9 + \frac{1}{1 + y^2}, \quad (y \neq 0),
\]

\[
\frac{\psi(y)}{y} - 9 = \frac{1}{1 + y^2},
\]

\[
0 \leq \frac{1}{1 + y^2} \leq 1,
\]

that is,
\[ 0 \leq \frac{\psi(y)}{y} - 9 \leq 1, \ (y \neq 0), \ b = 9 > 1, \ \alpha = 1; \]

\[ h(x) = \frac{x}{4}, \ h(0) = 0, \ \frac{h(x)}{x} = \frac{1}{4} \geq \frac{1}{5} = h_0 > 0, \]

\[ h'(x) = \frac{1}{4} \leq \frac{1}{2} = h_1 < 1 \text{ for all } x. \]

Thus, all the assumptions Theorem 2 holds for Equation (23). That is, the null solution of Equation (23) is uniform asymptotic stable.

The special case of this paper was only presented at the 37th International Annual Iranian Mathematics Conference, Tabriz University, September 2-5, 2006, Tabriz-IRAN

REFERENCES