HYPERRULED SURFACES IN MINKOWSKI 4-SPACE

R. ASLANER

Department of Mathematics, İnönü University, 44280 Malatya, Turkey
E-mail: raslaner@inonu.edu.tr

Abstract – In this paper, the time-like hyperruled surfaces in the Minkowski 4-space and their algebraic invariants are worked. Also some characteristic results are found about these algebraic invariants.

Keywords – Ruled surfaces, rulings, main curvature, scalar curvature, time-like vector

1. INTRODUCTION

The Minkowski space is the space $\mathbb{R}^4$ with the Lorentzian inner product

$$g_0 = -dt^2 + dx^2 + dy^2 + dz^2$$

which is denoted by $\mathbb{R}^4_{1}$. The representation of $g_0$ in the matrix form with respect to the standard basis of $\mathbb{R}^4_{1}$ is $\eta = \text{diag}(-1,1,1,1)$. Suppose that $\mathbb{R}^4_{1}$ is a 4-dimensional vector space over the field of real numbers. A symmetric bilinear form $\beta : \mathbb{R}^4_{1} \times \mathbb{R}^4_{1} \rightarrow \mathbb{R}$ is called

i) positive (resp. negative), definite if and only if $\vec{\omega} \neq \vec{0}$ implies $\beta(\vec{\omega}, \vec{\omega}) > 0$ (resp. $\beta(\vec{\omega}, \vec{\omega}) < 0$) for all $\vec{\omega}$ in $\mathbb{R}^4_{1}$

ii) non-degenerate if and only if $\beta(\vec{\omega}, \vec{z}) = 0$ for all $\vec{z}$ in $\mathbb{R}^4_{1}$, implying that $\vec{\omega} = \vec{0}$, and

iii) indefinite if and only if there exists $\vec{\omega}$ and $\vec{z}$ in $\mathbb{R}^4_{1}$ such that $\beta(\vec{\omega}, \vec{\omega}) > 0$ and $\beta(\vec{z}, \vec{z}) < 0$

[1].

A non-degenerate, symmetric bilinear form $\beta$ is called a scalar product. A scalar product may be positive definite, negative definite or indefinite.

For an indefinite scalar product $\beta$ in $\mathbb{R}^4_{1}$, a nonzero vector $\vec{\omega}$ is said to be

i) space-like if and only if $\beta(\vec{\omega}, \vec{\omega}) > 0$,

ii) time-like if and only if $\beta(\vec{\omega}, \vec{\omega}) < 0$,

iii) null if and only if $\beta(\vec{\omega}, \vec{\omega}) = 0$.

The vector $\vec{0}$ is taken to be space-like. The label space-like, time-like or null is called the causal character of a vector. A curve is called time-like (or space-like) curve if the tangent vector at every point of the curve is a time-like (or space-like) vector. A surface is called time-like surface if each
tangential bundle of the surface is a time-like subspace of $R_4^4$, [1]. A ruled surface is a surface swept out by a straight line $\ell$ moving along a curve $\alpha$. Such a surface has a parametrization in the ruled form

$$\varphi(t,v) = \alpha(t) + ve_1(t),$$

where $\alpha$ is the base curve and $e_1$ is the director vector of $\ell$. The various positions of the generating line $\ell$ are called the rulings of the surface. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable or cylindrical surface. All other ruled surfaces are called skew surfaces [2].

### 2. TIME-LIKE RULED SURFACES

Let

$$\alpha : I \to R_4^4$$

$$t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t))$$

be a differentiable time-like curve in the Minkowski space, where $0 \in I$. A space-like straight line,

$$\ell : R \to R_4^4$$

$$v \to \ell(v) = \alpha(t) + ve_1(t)$$

where $e_1(t)$ is the director vector of $\ell$ at the point $\alpha(t)$ such that $e_1(t)$ and the tangent vector of $\alpha$ are linearly independent at every point of the curve $\alpha$. Since $\ell$ is a space-like straight line $e_1(t)$ and $\dot{e}_1$ denotes the derivative of the vector field $e_1$ along the curve $\alpha$, we have $\langle e_1, e_1 \rangle = 0$.

When $\ell$ moves along $\alpha$, it generates a ruled surface given by the chart $(I \times R, \varphi)$, where

$$\varphi : I \times R \to R_4^4$$

$$(t,v) \to \varphi(t,v) = \alpha(t) + ve_1(t).$$

This ruled surface will be denoted by $M$. Taking the derivatives of $\varphi$ with respect to $t$ and $v$, we have

$$\varphi_t = \dot{\alpha}(t) + ve_1(t)$$

$$\varphi_v = e_1(t).$$

Note that $\text{rank}[\varphi_t, \varphi_v] = \text{rank}[\dot{\alpha} + v\dot{e}_1, e_1] = 2$.

So $M$ is 2-manifold in the Minkowski space $R_4^4$.

### 3. TIME-LIKE HYPERRULED SURFACES IN THE MINKOWSKI SPACE $R_4^4$

Throughout this section we assume that

$$1 \leq i, j \leq 2 \text{ and } 0 \leq m, n \leq 2.$$
then $M$ is a 3-manifold in $R^4_1$. In this case $M$ is called a \textit{hyperruled surface} and can be (locally) represented by the chart $(U, \varphi)$, where $U = I \times R^2$ and

$$\varphi : I \times R^2 \rightarrow R^4_1$$

$$(t, v) \rightarrow \varphi(t, v) = \alpha(t) + v^i e_i(t), \quad v = (v^1, v^2).$$

(4)

Suppose that the base curve $\alpha$ is an orthogonal trajectory of the generating plane $E_z(t)$. If

$$\text{rank}[e_0, e_1, e_2, \dot{e_1}, \dot{e_2}] = 4 - k$$

(5)

Then

i) if $k = 0$ in (5), then $M$ is called non-developable,

ii) if $k = 1$ in (5), then $M$ is called developable,

where $e_0$ is the unit tangent vector field of the base curve $\alpha$, which is a time-like curve, and $\dot{e}_i$ is the derivative of the vector fields $e_i$ along $\alpha$.

We begin with some properties of a general pseudo-Riemann manifold $M$. Suppose that $\overline{D}$ is the Levi-Civita connection on $R^4_1$, while $D$ is the Levi-Civita connection of $M$. Then, for any vector fields $X, Y$ on $M$, we have the Gauss equation:

$$\overline{D}_X Y = D_X Y + V(X, Y)$$

(6)

where $V$ is the second fundamental form of $M$.

If the $\xi$ is the unit normal vector field on $M$, we have the Weingarten equation giving the tangential and normal components of $\overline{D}_X \xi$;

$$\overline{D}_X \xi = -A_\xi(X) + D^\perp_{X} \xi,$$

(7)

where $A_\xi$ is determined at each point of a self-adjoint linear map on $\chi(M)$, and $D^\perp$ is a metric connection in the normal bundle of $M$ [3].

Let $X, Y \in \chi(M)$ and $\xi \in \chi(M^\perp)$. Then, by combining (6), (7) and the Minkowski inner product on $R^4_1$, denoted by $<\cdot, \cdot>$, yield that

$$<V(X, Y), \xi> = <Y, A_\xi(X)>.$$

(8)

Assume that $\{e_0, e_1, e_2\}$ is an orthonormal base field of the tangential bundle of $M$ and $\xi$ is the unit normal vector field of $M$. Then we have the following Weingarten equation

$$\overline{D}_{e_m} \xi = a^m_n e_n + b_m \xi,$$

(9)

where the Einstein summation is used. $a^m_n$'s are coefficients of the matrix $A_\xi$, and

$$a^m_n = <\overline{D}_{e_m} \xi, e_n> = <\xi, \overline{D}_{e_m} e_n>.$$

Since the generating space $E_z(t)$ of $M$ is a space-like subspace in $R^4_1$, we have that

$$<e_i, e_j> = \delta_{ij}$$

and

$$\overline{D}_{e_i} e_j = 0,$$

which imply that $a^i_0 = 0$ and

$$a^m_0 = <\overline{D}_{e_m} \xi, e_n> = <\xi, \overline{D}_{e_m} e_n> = -<\xi, \dot{e}_n> = -a_n,$$

so we may write the matrix $A_\xi$ as
Lemma 3.1. Consider the orthonormal base fields $e_0, e_1, e_2$ of $M$. Then the Riemannian curvature $\kappa_{\sigma}(e_i, e_0)$ in the two-dimensional direction $\sigma$ of $\chi(M)$, spanned by the vector fields $e_i$ and $e_0$, is given by

$$\kappa_{\sigma}(e_i, e_0) = -\langle \overline{D}_{e_i}e_0, \overline{D}_{e_i}e_0 \rangle.$$  \hspace{1cm} (10)

Proof: Suppose that $R$ is the curvature tensor of $M$, then

$$\kappa_{\sigma}(e_i, e_0) = \langle e_i, R(e_i, e_0)e_0 \rangle.$$ 

But we see from the Gauss equation that

$$\langle e_i, R(e_i, e_0)e_0 \rangle = \langle V(e_i, e_0), V(e_0, e_0) \rangle = -\langle V(e_i, e_0), V(e_0, e_0) \rangle$$

and we know that $V(e_i, e_i) = 0$. Moreover, we have

$$\langle \overline{D}_{e_j}e_0, e_j \rangle = \langle e_j, \overline{D}_{e_j}e_0 \rangle = 0 \Rightarrow \overline{D}_{e_j}e_0 \perp e_j$$

and

$$\langle \overline{D}_{e_i}e_0, e_0 \rangle = \langle e_0, \overline{D}_{e_i}e_0 \rangle = 0 \Rightarrow \overline{D}_{e_i}e_0 \perp e_0.$$  

This means that $\overline{D}_{e_i}e_0$ is a normal vector field or

$$\overline{D}_{e_i}e_0 = V(e_i, e_0)$$  \hspace{1cm} (11)

which completes the proof.

4. The Algebraic Invariants of the Hyper RULED Surfaces in the Space $R^4_1$

Let $M$ be a time-like hyperruled surface in the Minkowski 4-space $R^4_1$. Then the space of tangent vector fields of $M$ denoted by $\chi(M)$, is a time-like vector subspace of $R^4_1$ over the field of real numbers. Let $A$ be linear operator on $\chi(M)$. A characteristic value of $A$ is a scalar $\lambda$ in $R$ such that there exists a non-zero vector field $X$ in $\chi(M)$, with $A(X) = \lambda X$, where $X$ is called the characteristic vector of $A$ corresponding to $\lambda$. The set of all $X$’s is called the characteristic space of $A$.

The function $f(\lambda) = \det(A - \lambda \, \epsilon)$ is called the characteristic polynomial of $A$, where $\epsilon = diag(-1,1,1)$ is the matrix of the induced metric on $\chi(M)$. In order to find the roots of the characteristic polynomial we must solve the characteristic equation $\det(A - \lambda \, \epsilon) = 0$

$$\begin{vmatrix}
  a_0 + \lambda & -a_1 & -a_2 \\
  a_1 & -\lambda & 0 \\
  a_2 & 0 & -\lambda 
\end{vmatrix} = (a_0 + \lambda)\lambda^2 - a_1^2 \lambda - a_2^2 \lambda = 0.$$
or
\[ \lambda \left( \lambda^2 + a_0 \lambda - a_1^2 - a_2^2 \right) = 0. \]

This implies that 
\[ \lambda = 0 \quad \text{and} \quad \lambda^2 + a_0 \lambda - (a_1^2 + a_2^2) = 0 \]
Since \( \Delta = a_0^2 + 4(a_1^2 + a_2^2) > 0 \), the solution of the characteristic equation are
\[ \lambda_1 = 0, \quad \lambda_2 = -\frac{1}{2}(a_0 + \sqrt{\Delta}) \quad \text{and} \quad \lambda_3 = \frac{1}{2}(-a_0 + \sqrt{\Delta}). \]

Thus we may give the following result:

**Result 4.1.** Let \( M \) be a time-like hyperruled surface in \( \mathbb{R}^4 \). If \( \lambda_2 = \lambda_3 \), then \( M \) is minimal and developable.

**Proof:** Let \( \lambda_2 = \lambda_3 \), then \( \Delta = 0 \), which implies that \( a_0 = a_1 = a_2 = 0 \).
Thus \( a_0 = 0 \) implies that \( \text{tr}A = 0 \), and so \( M \) is minimal.
By lemma 1, \( a_i = 0 \) implies that \( \kappa(e_i, e_0) = 0 \) and so \( M \) is developable.

Let us find the characteristic vector corresponding to characteristic values \( \lambda_1, \lambda_2, \lambda_3 \) of the matrix \( A \). The vector field \( X_1 \) corresponding to \( \lambda_1 \) is obtained by the solution of the equation
\[ AX_1 = 0 \iff X_1(t) = \left( 0, t, \frac{a_1}{a_2} t \right). \]

Similarly, the vector fields \( X_2 \) and \( X_3 \), corresponding to \( \lambda_2 \) and \( \lambda_3 \), are obtained by the solutions of the equations
\[ AX_2 = \lambda_2 X_2 \iff X_2(t) = \left( t, -\frac{a_1}{\lambda_2} t, -\frac{a_2}{\lambda_2} t \right) \]
and
\[ AX_3 = \lambda_3 X_3 \iff X_3(t) = \left( t, -\frac{a_1}{\lambda_3} t, -\frac{a_2}{\lambda_3} t \right). \]

Since the vector fields \( X_k(t) \), \( k = 1, 2, 3 \) have one arbitrary parameter, the dimension of the characteristic space is equal to 1. Therefore, we can choose an orthonormal base field \( \phi = \{ X_1, X_2, X_3 \} \) of \( \chi(M) \) corresponding to characteristic values \( \lambda_1, \lambda_2, \lambda_3 \).
If we denote the matrix of the linear map \( A \) with respect to the orthogonal base \( \phi \) by \( S \), then we observe that
\[ S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \]

\( S \) is called as the *Weingarten (or Shape) operator* of \( M \) with respect to the base \( \phi \).
Thus we can state the following results:
**Result 4.2.** Let $M$ be a time-like hyperruled surface in $R^4_1$, and $S$ be the shape operator of $M$. Then

i) The main curvature of $M$ is $\|H\| = -\frac{a_0}{3}$.

ii) The Gauss curvature of $M$ is $\kappa = 0$.

**Definition 4.1.** Let $M$ be a time-like hyperruled surface with curvature tensor $R$ in $R^4_1$. If $\{e_0, e_1, e_2\}$ is an orthonormal base field of $\mathcal{X}(M)$, then the Ricci curvature tensor $S$ is defined by

$$S : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M),$$

$$(X, Y) \rightarrow S(X, Y) = \sum_{m} e_m < R(e_m, X)e_m >$$

where

$$\varepsilon_m = \begin{cases} -1, & m = 0 \\ 1, & m = 1, 2. \end{cases}$$

The scalar curvature of $M$ is defined by

$$r = \sum_m S(e_m, e_m),$$

and the scalar normal curvature of $M$ is defined by

$$r_n = \sum_{\sigma, \nu} \{ A_{\xi, \nu} A_{\xi, \nu} - A_{\xi, \nu} A_{\xi, \nu} \}_{\sigma, \nu} \in \{1, 2\}, [4].$$

Thus we can find the following results for the time-like hyperruled surfaces in the Minkowski 4-space $R^4_1$:

**Result 4.3.** Let $M$ be a time-like hyperruled surface with a base curve $\alpha$ and the generating space $E_2(t)$ spanning by the vectors $e_2(t)$ in the Minkowski 4-space $R^4_1$. Then the scalar curvature of $M$ is

$$r = -2 \sum_i a_i^2,$$

where $a_i = <\xi, \dot{e}_i>$ and $\xi \in \mathcal{X}(M^+)$. 

**Proof:** Let $\{e_0, e_1, e_2\}$ be an orthonormal base field of $M$. Then

$$r = \sum_m S(e_m, e_m) = S(e_0, e_0) + \sum_i S(e_i, e_i)$$

$$S(e_0, e_0) = \sum_m < R(e_m, e_0)e_0, e_m >$$

$$= \sum_i < R(e_i, e_0)e_0, e_i >$$

$$= \sum_i \kappa(e_i, e_0) = -\sum_i a_i^2$$

$$S(e_i, e_i) = \sum_m < R(e_m, e_i)e_i, e_m > = \kappa(e_i, e_0) = -a_i^2$$
which implies that \( S(e_0, e_0) = -\sum S(e_i, e_i) \).

So we have \( r = -2\sum S(e_i, e_i) = -2\sum a_i^2 \).

Thus we derive the following results for a time-like hyperruled surface in \( R_4^+ \):

i) i) \( r = 0 \) if \( M \) is developable,

ii) ii) \( r = 0 \) and \( M \) is minimal if \( M \) is hyperplane,

iii) The scalar normal curvature of \( M \) is always zero.

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