Adaptive and Sliding Mode Control for Non-Linear Systems

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Abstract: This paper deals with a globally convergent adaptive and sliding mode control of a cart-pole inverted pendulum for trajectory tracking in the presence of a bounded measurement noise and parameter uncertainty. Two kinds of controllers have been used for evaluation of tracking error in presence of a bounded noise; as a result, we want to compare that at what time we can see the convergence of tracking error and which controller can perform better? Simulation results on a cart-pole inverted pendulum are shown for trajectory tracking in presence of impulse disturbance.

Keywords: Adaptive Control, Sliding Mode Control, Cart Pole Inverted Pendulum, Trajectory Tracking, Uncertainty.


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1 Introduction

Adaptive and sliding mode control [1-4] is widely accepted as a powerful method of tackling uncertain non-linear systems.

Model-based adaptive controller has received more attention in the last years because it makes it possible to cope with the above variations using linear like techniques do to the linear parameterization property of the mode. In general, the adaptive control laws designed use a non-linear term having the same structure as the regressor of the model in which appear the actual state variables, their desired values or both. (Slotine & Li, 1986; Ortega & Spong, 1989; Whitcomb, Rizzi & Koditschek, 1993; Alonge, D'Ippolito & Raimondi, 1999).

Of course, a drawback of model-based adaptive control is due to its high sensitivity to high frequency unmodeled dynamics such as, for example, unstructured uncertainty or structural vibrations and low frequency unmodeled dynamic such as coulomb friction. (Rohrs, Valavani, Athans & Stein, 1985; Astrom, 1984).In the sliding mode control (SMC), when the upper bounds of the uncertainties are known, this control strategy can be designed to drive the system into pre-defined sliding plane. Once the sliding plane is hit, the system responses will only be governed by the Plane and are insensitive to the parameter uncertainty. As a result, SMC offers good robustness against parameter uncertainties. However, since the controller will switch between two structures during operation, the system will undergo the oscillation near the sliding plane. This phenomenon of chattering [5, 6, 8] is a major drawback of SMC. A common method to alleviate chattering is to insert a boundary layer [5-8] near the sliding plane so that a continuous control replaces the discontinuous one when the system is inside the boundary layer.

In this paper, section 2 and 3 give a brief review on conventional adaptive control and SMC for a general n-th order non-linear system with uncertain parameter and bounded measurement noise, respectively. Then in section 4, these ways are applied to a non-linear system and ultimately, it's simulated on a cart-pole inverted pendulum as a plant to show its ability and merits. A conclusion will be drawn in section 4.

2 Mathematical Model of Adaptive Control with Parameter Uncertainty and Measurement Noise

Consider an n-th order non-linear system in companion form:

\[ x^{*} = af(x) + cb(x)u + T_d(t) \]  \hspace{1cm} (1)

Where \( x = [x, \dot{x}, ..., x^{(n-1)}]^T \) is the state vector, and \( f(x), b(x) \) are known non-linear functions of the state and time, and the parameters \( a, c \) are unknown constants and \( T_d(t) \) is the external disturbance and the disturbance is assumed to be bounded as \( |T_d(t)| \leq D(t) \).

It's assumed that the state is measured, and that the sign of \( c \) is known. The objective of the adaptive control design to make the output asymptotically tracks a desired output \( x_d(t) \) despite the parameter uncertainty.

To facilitate the adaptive controller derivation, let us rewrite Eq. (1), as

\[ \left( \frac{m}{b(x)} \right)x^{*} + \left( \frac{q}{b(x)} \right)f(x) - \left( \frac{T_d}{cb(x)} \right) = u \] \hspace{1cm} (2)

Where

\[ m = \left( \frac{1}{c} \right) \quad ; \quad q = \left( \frac{-a}{c} \right) \] \hspace{1cm} (3)

To earn a suitable control law, a combined error is defined:

\[ S = e^{(n-1)} + \lambda_{(n-2)}e^{(n-2)} + ... + \lambda_0e = \Delta(p)e \] \hspace{1cm} (4)

Where \( e \) is the output tracking error and \( \Delta(p) = p^{(n-1)} + \lambda_{(n-2)}p^{(n-2)} + ... + \lambda_0 \) is a stable (Hurwitz) polynomial in the Laplace variable \( p \), note that \( s \) can be rewritten as

\[ S = x^{(n-1)} - x_r^{(n-1)} \] \hspace{1cm} (5)

Where \( x_r^{(n-1)} \) is defined as

\[ x_r^{(n-1)} = x_d^{(n-1)} - \lambda_{(n-2)}e^{(n-2)} - ... - \lambda_0e \] \hspace{1cm} (6)

Then the control law

\[ u = \left( \frac{1}{b(x)} \right)(m \ddot{x}^*_r - ks + \ddot{q}f(x)) \] \hspace{1cm} (7)

Where \( k \) is a constant of the same sign as \( m \). Note that \( x^*_r \), the so-called reference value of \( x^*_r \), is obtained by modifying \( x^*_d \) according to the tracking error.

The tracking error from this control law can be easily shown to be

\[ m\ddot{s} + ks = \ddot{m}x^*_r + \dddot{q}f(x) \] \hspace{1cm} (8)

Then adaption law can be shown that

\[ \dot{m} = -\gamma \text{sgn}(m)xx^*_r \] \hspace{1cm} (9)
\[
\dot{q} = -\gamma \text{sgn}(m)sf(x)
\]

Specifically, using the Lyapunov function candidate
\[
V = |m|^2 + \gamma^{-1}[\tilde{m}^2 + \tilde{q}^2]
\]

It is straightforward to verify that
\[
\dot{V} = -2k|s|^2
\]

And therefore the globally tracking convergence of the adaptive control system can be easily shown [1, 4].

**Assumption.** All the variables appearing in the system are bounded and position and velocity errors converge globally asymptotically to zero if:

1. \(x_1, \dot{x}_1\) are bounded;
2. \(\gamma\) is positive definite matrix;

### 3 MATHEMATICAL MODEL OF SMC WITH PARAMETER UNCERTAINTY AND MEASUREMENT NOISE

Consider the single-input dynamic system again; eq. (1), that now with sliding mode control will be designed. Let us define a time varying \(S(t)\) in the state space \(\mathbb{R}^n\) by the scalar equation
\[
S(x,t) = \left(\frac{d}{dt} + \lambda\right)^{(n-1)}
\]

And \(\lambda\) is a strictly positive constant, whose choice shall be interpret later. For instance, if \(n \geq 2\)
\[
S = \dot{x} + \lambda x
\]
i.e., \(s\) is simply a weighted sum of the position error and the velocity error. In this paper, \(f(x)\) is a vector of which the elements are non-linear functions of \(x\) which are not exactly known but estimated as \(f_{eq}(x) = [f_{eq1}(x), f_{eq2}(x), \ldots, f_{eqn}(x)]^T\). The stimation error in \(f(x)\) is assumed to be bounded by a known function \(F(x) = [F_1(x), F_2(x), \ldots, F_n(x)]\) such that
\[
|f(x) - f_{eq}(x)| \leq F(x)
\]

The sliding mode controller is designed as follows. Define a sliding plane
\[
\sigma = sx = 0
\]
Where \(s = [s_1, s_2, \ldots, s_n]\) is a constant vector also define an operator \([\|\|^2\]\) such that
\[
|s|^2 = [|s_1|^2 + |s_2|^2 + \ldots + |s_n|^2]
\]

The vector \(s\) is to be designed such that when \(xx = 0\), the resultant system is stable. The control is to drive the system of eq. (1) to the stable sliding plane. Therefore, [9]
\[
\dot{\sigma} = sx = sf_f(x) + scb(x)u
\]

When the system is operating on the sliding plane, eq. (18) equals zero. Then the best estimation of the control for the estimated plant \(f_{eq}\) is given by
\[
u = -[scb(x)]^{-1}af_{eq}(x)
\]

Where \(scb(x)\) is assumed to be non-linear singular. Hence the best estimation of the equivalent control is
\[
u_{eq} = -af_{eq}(x)
\]

To tackle the uncertainty of \(f\), a discontinuous control \(u_d\) is added to the control input such that
\[
u = [scb(x)]^{-1}(u_{eq} + u_d)
\]
\[
u_d = -k \text{sgn}(\sigma)
\]

Where \(k\) is positive constant that satisfy the Lyapunov function.
\[
V = \frac{1}{2}\sigma^2(x)
\]

But eq. (22) will have high-frequency switching near the sliding surface, due to the 'sgn' function involved. Thus, in order to reduce the chattering phenomena, we replace \(\text{sgn}(\sigma)\) with \(\text{sat}(\sigma)\) as follows:
\[
u = [scb(x)]^{-1}(u_{eq} - k\text{sat}(\sigma))
\]

### 4 SIMULATION

In order to verify the peculiarities of the previously discussed control laws, an application example is proposed. A cart-pole inverted pendulum is shown in Fig. 1. The state-space equations are as follows: [10]
\[
\dot{x}_1 = x_2
\]
\[
\dot{x}_2 = \left(\frac{g \sin(x_1)_1 - (a - \alpha x_1 \sin(2x_1)) - a \cos(x_1)_1 u}{2}\right)
\]

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where \( x_1 \) denotes the angle (in rad) of the pendulum from the vertical axes, \( x_2 \) is the angular velocity (in rad \( s^{-1} \)), \( m=0.5\text{kg} \) is the mass of the pendulum, \( M=1\text{kg} \) is the mass of the cart, \( g=9.81 \text{ is the acceleration due to gravity, } a = \left( \frac{1}{M + m} \right) \), \( 2l=1\text{m} \) is the length of the pendulum, and \( u \) is the force applied to the cart. It's assumed to be \( x_0 = \left[ \frac{\pi}{3}, 0 \right]^T \) is the initial condition and \( x_d = 0 \) is desired trajectory.

\[
\begin{align*}
\dot{x}_1 &= \frac{\cos x_2}{\cos x_1} \sin x_1 - \frac{g}{2l} \sin x_1 - u \\
\dot{x}_2 &= \frac{\sin x_1}{\cos x_1} \left( \frac{x_2}{\cos x_1} - \frac{g}{2l} \right) - \frac{\sin x_1}{\cos x_1}u
\end{align*}
\]

\[a = \left( \frac{1}{M + m} \right), 2l=1\text{m} \]

Fig. 1 Cart-pole inverted pendulum

Figures 2 & 3 Show that the angle and angular velocity of pendulum will be converged approximately after 3 seconds with adaptive controller and 6 seconds with SMC. As a result, adaptive controller can perform better than SMC in presence of bounded noise. This bounded disturbance is imposed in \( t=2 \). As you see, after this time, all of the figures are converging asymptotically.

Fig. 2 Angle of pendulum (“a” and “b” are denoted for adaptive and SMC, respectively)

Fig. 3 Angular velocity of pendulum (“a” and “b” are denoted for adaptive and SMC, respectively)
Fig. 4  Tracking error ("a" and "b" are denoted for adaptive and SMC, respectively)

Fig. 5  Control law ("a" and "b" are denoted for adaptive and SMC, respectively)

Fig. 6  Convergent of parameters for adaptive control
However, there is a little chatter that depends on \textit{sgn}' function in adaptive controller; of course, in SMC with using \textit{sat}' function this problem is alleviated. Also tracking error in Fig. 1 has a good convergence after 3 seconds.

In equations (7) and (24) each of the control laws for two mentioned controllers has been extracted. In Figure 5 adaptive and sliding mode control laws, are compared. These figures show that choosing of control law is fortunately good. Likewise, in Fig. 6 It is turn out that all of the parameters (m, q) have a nice convergence to desired values, that is requested.

5 CONCLUSION

In this paper, two approaches are considered for designing a control law which includes parameter uncertainty and bounded disturbance. The first approach is based on a detailed, though approximate, modeling of the dynamic of cart-pole inverted pendulum starting from which the design of an adaptive control law is affected in presence of bounded disturbance and parameter uncertainty. The second approach is based on the assumption that dynamics of the system starting from which the design of a sliding mode control law is affected in presence of bounded disturbance and parameter uncertainty. Simulation results showed that using both of controllers is suitable and tracking error converging globally asymptotically to zero. Likewise parameters converge to there's real value.

Of course, figures show that adaptive controller has a better convergent than SMC and there is a good settlement in tracking error with adaptive controller.

REFERENCES