A Fixed Point Theorem for a General Contractive Condition of Integral Type in Modular Spaces

S.J. Hosseini Ghoncheh*
Mathematics Department, Takestan Branch, Islamic Azad University, Takestan, Iran
A. Razani, R. Moradi
Mathematics Department, I. Kh. International University, Qazvin, Iran
B.E. Rhoades
Mathematics Department, Indiana University, Bloomington, USA

Abstract

Introduction: Orlicz and Birnbaum considered spaces of functions with growth properties different from the power type growth control, provided by the $L^p$-norm. This generalization found many applications in differential and integral equations with kernels of non-power types. A more abstract approach was given by Nakano in connection with the theory of order spaces. His point of view was generalized by Musielak and Orlicz (1950) who defined what is now commonly known as modular spaces.

Aim: Banach's fixed point theorem is the first and famous fact in the fixed point theory. As an application, one can construct a solution of an ordinary differential equations. The propose of this paper is to establish some analogues of Banach’s fixed point theorem for contractive mappings and mention an application in modular spaces.

Material and Method: the existence of a fixed point for mappings satisfying a contractive condition of integral type in modular spaces is studied. Moreover, we extend the contractive conditions of integral type to solve an integral equation in Musielak-Orlicz space. Our method is based on constructing a contractive operator of integral type in such way, it has a fixed point. This fixed point can be considered as a solution of the integral equation.

Results: Some fixed point theorems in modular spaces are presented. Moreover, as an application of these theorems, the existence of a solution of an integral equation in Musielak-Orlicz space, is proved.

Conclusion: one can consider an integral equation in the modular spaces and can find a solution of it. This shows that one can solve integral equations in a space where members has growth properties different from the power type growth control, provided by the $L^p$-norm.

Keywords: Modular space, Fixed point, Contractive condition of integral type.

Introduction

Alternative to classical $L^p$ spaces are of interest in many applications of functional analysis. Orlicz and Birnbaum considered spaces of functions with growth properties deferent

*Corresponding author
A fixed point theorem for a … Hosseini Ghoncheh and Coworkers

from the power type growth control provided by the $L^p$-norm. This generalization found many applications in deferential and integral equations with kernels of nonpower types. Luxemburg considered measure spaces together with norm and a monotonicity condition. A more abstract approach was given by Nakano in 1950 in connection with the theory of order spaces. His point of view was generalized by Musielak and Orlicz (1950) who defined what is now commonly known as modular spaces.

Material and Method

The propose of this paper is to establish some analogues of Banach’s fixed point theorem for contractive mappings and mention an application. Theorem 1 is a fixed point theorem of integral type in modular spaces when the mapping satisfies certain contractive conditions. Moreover, it is interesting if we replace the restrictive condition ($c>1$ in Theorem 1) with $l=c$. In Theorem 2 the restriction $c>1$ is removed and replaced with a $\rho$-boundedness condition. In Section 3, two other variations of the Banach fixed point theorem are established (Theorem 3 and 4). More precisely, consider the dentition of the quasicontraction of integral type as follows:

**Definition 1.** A mapping $T$ of a metric space $X$ into itself is said to be a quasicontraction of integral type if there exists a number $0 < q < 1$, such that

$$\int_0^1 \varphi(t) dt \leq q \int_0^1 \varphi(t) dt$$

(1)

holds for $x, y \in X$ where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$$

and $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and for all $\varepsilon > 0$

$$\int_0^\varepsilon \varphi(t) dt \to 0.$$

Rhoades [4-6,14-15] proved that if $T$ is a quasi-contraction of integral type on a metric space $X$ satisfying (1), then $T$ has a unique fixed point. We prove the corresponding theorem in the modular space.

Finally, in last section, as an application of Theorem 1, we prove the existence of solutions of integral equations of the type

$$u(t) = \exp(-t) f + \int_0^t \exp(s-t) Tu(s) ds$$

in the modular space $C^\infty_c([0, A], L^p)$, where $L^p$ is the Musielak-Orlicz space, $f$ is a fixed element in $L^p$ and $T$ satisfies a general contractive condition of integral type.

Now, we begin with a brief description of concepts and facts of the theory of modular spaces from [1,3-4,8-9,12,13].

**Definition 2.** Let $X$ be an arbitrary vector space over $K = (\mathbb{R}$ or $\mathbb{C}$).

a) A functional $\rho: X \to [0, \infty]$ is called modular if:

i) $\rho(x) = 0$ iff $x = 0$.

ii) $\rho(ax) = \rho(x)$ for $a \in K$ with $|a| = 1$, for all $x \in X$.

iii) $\rho(ax + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$, for all $x, y \in X$.

If iii) is replaced by:

iii') $\rho(ax + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ for $\alpha, \beta \geq 0, \alpha + \beta = 1$, for all $x, y \in X$. Then the modular $\rho$ is called a convex modular.

b) A modular $\rho$ defines a corresponding modular space, i.e. the space $X_\rho$, given by:

$$X_\rho = \{x \in X | \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}.$$
Let $0 < a < b$. Then, using property (iii) with $y = 0$, one obtains $\rho(ax) = \rho\left(\frac{a}{b}(bx)\right) \leq \rho(bx)$.

**Definition 3.** Let $X_\rho$ be a modular space.

a) A sequence $(x_n)_{n \in \mathbb{N}} \in X_\rho$ is said to be:

i) $\rho$-convergent to $x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

ii) $\rho$-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

b) $X_\rho$ is $\rho$-complete if every $\rho$-Cauchy sequence is $\rho$-convergent.

c) A subset $B \subseteq X_\rho$ is said to be $\rho$-closed if for any sequence $(x_n)_{n \in \mathbb{N}} \in B$ and $x_n \to x$ then $x \in B$.

d) A subset $B \subseteq X_\rho$ is called $\rho$-bounded if $\rho(B) = \sup\{\rho(x - y) : x, y \in B\}$.

e) $\rho$ has the Fatou property if: $\rho(x - y) = \inf\{\rho(x_n - y_n) : x_n \to x, y_n \to y\}$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

f) $\rho$ is said to satisfy the $\Delta_2$-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

**A fixed point theorem for mappings satisfying a contractive condition of integral type**

In this section, the existence of a fixed point for mappings satisfying a contractive condition of integral type in modular spaces is studied.

**Theorem 1.** Let $X_\rho$ be a $\rho$-complete modular space, where $\rho$ satisfies the $\Delta_2$-condition, and $B$ a $\rho$-closed subset of $X_\rho$. Suppose, $k, l \in \mathbb{R}^+$, $c > 1$, and $T$ is a selfmap of $B$ satisfying

$$
\int_0^{\rho(\{t \in \mathbb{R}^+ : t \leq x\})} \varphi(t) dt \leq k \int_0^{\rho(\{t \in \mathbb{R}^+ : t \leq y\})} \varphi(t) dt
$$

for some $k \in (0, 1)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and for all $\varepsilon > 0$,

$$
\int_0^\varepsilon \varphi(t) dt = 0
$$

Then $T$ has a unique fixed point.

Proof. Let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{c}{l}$; i.e. $\frac{1}{\alpha} + \frac{1}{c} = 1$. Let $x \in B$ and let us consider the sequence $(T^n x)_{n \in \mathbb{N}} \in B$. For each integer $n \geq 1$, inequality (2) shows that

$$
\int_0^{\rho(\{t \in \mathbb{R}^+ : t \leq x\})} \varphi(t) dt \leq k \int_0^{\rho(\{t \in \mathbb{R}^+ : t \leq y\})} \varphi(t) dt
$$

by induction

$$
\int_0^{\rho(\{t \in \mathbb{R}^+ : t \leq x\})} \varphi(t) dt \leq k^n \int_0^{\rho(\{t \in \mathbb{R}^+ : t \leq x\})} \varphi(t) dt
$$

taking the limit as $n \to \infty$ yields

$$
\lim_n \int_0^{\rho(\{t \in \mathbb{R}^+ : t \leq x\})} \varphi(t) dt \leq 0
$$

which, inequality (3), implies that

$$
\lim_n \rho(c(T^n x - T^{n+1} x)) = 0
$$

we now show that $(T^n x)_{n \in \mathbb{N}}$ is $\rho$-Cauchy. If not, then, there exists an $\varepsilon > 0$ and two sequences of integers $\{n(s)\}$, $\{m(s)\}$, with $n(s) > m(s) \geq s$, and such that

$$
d_s = \rho(T^{n(s)} x - T^{m(s)} x) \geq \varepsilon, \quad s = 1, 2, \ldots
$$
we can assume that
\[ \rho(T^{m(s)}x - T^{n(s)-1}x) < \varepsilon \]  
(7)

In order to show this, suppose \( n(s) \) is the smallest number exceeding \( m(s) \) for which (6) holds and
\[ \sum_{s} = \{ n \in N | \exists m(s) \in N; \rho(T^{n}x - T^{m(s)}x) \geq \varepsilon, n > m(s) \geq s \} \]

obviously, \( \sum_{s} \neq \emptyset \) and since \( \sum_{s} \in N \), then by Well Ordering Principle, the minimum element of \( \sum_{s} \) is denoted by \( n(s) \), and clearly (7) holds.

Now,
\[ \rho(T^{m(s)}x - T^{n(s)-1}x) = \rho(T^{m(s)}x - T^{m(s)}x + T^{m(s)}x - T^{n(s)-1}x) \]
\[ \leq \rho(ad(T^{m(s)}x - T^{m(s)}x) + (c(T^{m(s)}x - T^{n(s)-1}x)) \]

\( \Delta_{2} \)-condition and inequality (5) shows that
\[ \lim_{n} \rho(ad(T^{m(s)}x - T^{m(s)}x)) = 0 \]

therefore
\[ \lim_{n} \int_{0}^{\varepsilon} \rho(T^{m(s)}x - T^{n(s)-1}x) \phi(t) dt \leq \int_{0}^{\varepsilon} \rho(T^{m(s)}x - T^{n(s)-1}x) \phi(t) dt \]

(8)
also by using inequality (6), one may have
\[ \int_{0}^{\varepsilon} \phi(t) dt \leq \int_{0}^{\varepsilon} \rho(c(T^{m(s)}x - T^{n(s)-1}x)) \phi(t) dt \]

(9)
thus inequalities Thus inequalities (3), (5), (8) and (9), prove that
\[ \int_{0}^{\varepsilon} \phi(t) dt \leq \int_{0}^{\varepsilon} \rho(T^{m(s)}x - T^{n(s)-1}x) \phi(t) dt \]
\[ \leq k \int_{0}^{\varepsilon} \rho(T^{m(s)}x - T^{n(s)-1}x) \phi(t) dt \]
\[ \leq k \int_{0}^{\varepsilon} \phi(t) dt \]

(10)
which is a contradiction. Therefore by \( \Delta_{2} \)-condition \( (T^{n}x)_{n \in N} \) is \( \rho \)-Cauchy. Since \( X_{\rho} \) is \( \rho \)-complete and \( B \) is \( \rho \)-closed, there exists a \( z \in B \) such that
\[ \rho(c(T^{n}x - z)) \to 0 \quad \text{as} \quad n \to \infty. \]

we now prove that \( z \) is a unique fixed point of \( T \). In order to prove this, one may write
\[ \rho \left( \frac{c}{2} (Tz - z) \right) = \rho \left( \frac{c}{2} (Tz - T^{n}z + T^{n}z - z) \right) \]
\[ \leq \rho \left( c(Tz - T^{n}z) + \rho \left( c(T^{n}z - z) \right) \right) \]
and so
\[ \int_{0}^{\varepsilon} \rho \left( \frac{c}{2} (Tz - z) \right) \phi(t) dt \leq \int_{0}^{\varepsilon} \rho \left( c(Tz - T^{n}z) + \rho \left( c(T^{n}z - x - z) \right) \right) \phi(t) dt \]
\[ \int_{0}^{\varepsilon} \phi(t) dt \leq 0 \]

(11)

taking the limit as \( n \to \infty \) yields
\[ \int_{0}^{\varepsilon} \frac{c}{2} (Tz - z) \phi(t) dt = 0 \]

which implies that
\[ \int_{0}^{\varepsilon} \frac{c}{2} (Tz - z) \phi(t) dt = 0 \]

now, inequality (3) shows that \( \rho \left( \frac{c}{2} (Tz - z) \right) = 0 \) or \( Tz = z \). This means that \( z \) is a fixed point of \( T \).
in order to prove the uniqueness, suppose that \( z \) and \( w \) are two arbitrary fixed points of \( T \). then,
\[
\int_0^\infty \rho(c(z-w)) \varphi(t)dt = \int_0^\infty \rho((Tz-Tw)) \varphi(t)dt
\leq k \int_0^\infty \rho((z-w)) \varphi(t)dt
\leq \int_0^\infty \rho(c(z-w)) \varphi(t)dt
\]
which implies (by (3)) \( \rho(c(z-w)) = 0 \), and hence \( z = w \).

**Remark 1.** If one set \( \varphi(t) = 1 \) in Theorem 2, then Theorem 2 of \([8]\) will be obtained.

**Remark 2.** If \( \rho = l \) or \( \rho = l = 1 \), then Theorem 1 is not valid. But if we suppose, \( B \) is a \( \rho \)-bounded subset of \( X_\rho \), then we can prove the next theorem.

**Theorem 2.** Let \( X_\rho \) be a \( \rho \)-complete modular space, where \( \rho \) satisfies the \( \Delta_2 \)-condition. Suppose \( B \) is a \( \rho \)-closed and \( \rho \)-bounded subset of \( X_\rho \) and \( T \) is a selfmap of \( B \) satisfying
\[
\int_0^\infty \rho((Tz-Ty)) \varphi(t)dt \leq k \int_0^\infty \rho((z-y)) \varphi(t)dt
\]
for all \( x, y \in B \), where \( c, k \in \mathbb{R}^+ \) with \( k \in (0, 1) \), and \( \varphi : \mathbb{R} \to \mathbb{R}^+ \) is a Lebesgue integrable mapping which is summable, nonnegative and for all \( \varepsilon > 0 \)
\[
\int_0^\infty \varphi(t)dt \neq 0
\]
Then \( T \) has a unique fixed point.

**Proof.** Let \( x \in B, m, n \in \mathbb{N} \). Then
\[
\int_0^\infty \rho(c(T^{n+m-1}x-T^{m}x)) \varphi(t)dt \leq k \int_0^\infty \rho(c(T^{n+m-1}x-T^{m}x)) \varphi(t)dt
\leq k \int_0^\infty \rho(c(T^{n+m-1}x-T^{m}x)) \varphi(t)dt
\leq k \int_0^\infty \rho(c(T^{n}x-T^{m}x)) \varphi(t)dt
\]
Since \( B \) is \( \rho \)-bounded,
\[
\lim_{m,n} \int_0^\infty \rho(c(T^{n+m-1}x-T^{m}x)) \varphi(t)dt = 0
\]
which implies that \( \lim_{m,n} \rho(c(T^{m+n}x-T^{m}x)) = 0 \). Therefore by \( \Delta_2 \)-condition \((T^nx)_{n\in\mathbb{N}} \) is \( \rho \)-Cauchy. Since \( X_\rho \) is \( \rho \)-complete and \( B \) is \( \rho \)-closed, there exists a \( z \in B \) such that \( \lim \rho(c(T^nx-z)) = 0 \).

We now prove that \( z \) is a unique fixed point of \( T \). In order to prove this, one may write
\[
\rho \left( \frac{c}{2} (Tz-z) \right) \leq \rho \left( c (Tz-T^nx + T^nx - z) \right)
\leq \rho \left( c (Tz-T^nx) + \rho(T^nx - z) \right)
\]
And so
\[
\int_0^\infty \rho \left( \frac{c}{2} (Tz-z) \right) \kappa(t)dt \leq \int_0^\infty \rho \left( \frac{c}{2} (Tz-z) \right) \rho \left( c (T^nx - z) \right) \varphi(t)dt
\]
Therefore taking the limit as \( n \to \infty \) yield
\[
\int_0^\infty \rho \left( \frac{c}{2} (Tz-z) \right) \varphi(t)dt \leq 0
\]
which implies that \( \rho \left( \frac{c}{2} (Tz-z) \right) = 0 \) or \( Tz = z \). This means that \( z \) is a fixed point of \( T \). In order to prove uniqueness, suppose \( z \) and \( w \) are two arbitrary fixed points.
A fixed point theorem for quasi-contraction mapping of integral type

In this section, we extend the contractive conditions of integral type which are introduced in pervious Section.

**Definition 4.** A mapping $T \colon X \to X$ of a modular space $X$ into itself is said to be a $(c, l, q)$-generalized contraction of integral type if there exists $0 < q < 1$ and $c, l \in \mathbb{R}^+$ with $c > l$, such that for all $x, y \in X$

$$\int_0^\rho\left(\frac{c(x - Ty)}{\rho(c(x - Tx))}\right) \phi(t) dt \leq q \int_0^\rho\left(\frac{l(x - TTy)}{\rho(l(x - TTx))}\right) \phi(t) dt$$

(12)

Where

$$m(x, y) = \max \left\{ \rho(l(x - y)), \rho(l(x - Tx)), \rho(l(Ty - y)) \right\}$$

and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping

which is summable, nonnegative for all $\varepsilon > 0 \int_0^\varepsilon \varphi(t) dt \neq 0$.

**Theorem 3.** Let $X \varphi$ be a modular space where $\rho$ satisfies the Fatou Property and $\Delta_2$-condition. Suppose $(c, l, q)$-generalized contraction of integral type selfmap of $X$.

Then $T$ has a unique fixed point $z \in X$.

Proof. At first suppose $c > 2l$ and set $y = Tx$ in (12) then

$$m(x, Tx) = \max \left\{ \rho(l(x - Tx)), \rho(l(Tx - T^2x)) \right\}$$

Let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{c}{l}$; i.e. $\frac{c}{l} + \frac{1}{\alpha} = 1$. Property (iii) of

Definition 1 shows that

$$\frac{\rho(l(x - T^2x))}{2} \leq \rho(c(x - Tx)) + \rho(\alpha(Tx - T^2x))$$

$$\leq \max \left\{ \rho(c(x - Tx)), \rho(\alpha(Tx - T^2x)) \right\}$$

which is hold, because $c > 2l$ shows that $\alpha \leq c$. So

$$m(x, Tx) = \max \left\{ \rho(c(x - Tx)), \rho(c(Tx - T^2x)) \right\}$$

Suppose, there exists a value of $x$ such that the maximum of the $\rho(c(Tx - T^2x))$ hand side of the above inequality is then, from (12) we have

$$\int_0^{\rho(c(Tx - T^2x))} \phi(t) dt \leq q \int_0^{\rho(c(Tx - T^2x))} \phi(t) dt$$

which implies that $Tx$ is a fixed point of $T$. Therefore, for all $x$ such that $Tx \neq T^2x$ we have

$$\int_0^{\rho(c(Tx - T^2x))} \phi(t) dt \leq q \int_0^{\rho(c(Tx - T^2x))} \phi(t) dt$$

Replacing $x$ by $Tx$ gives

$$\int_0^{\rho(c(Tx - T^2x))} \phi(t) dt \leq q \int_0^{\rho(c(Tx - T^2x))} \phi(t) dt \leq q^2 \int_0^{\rho(c(Tx - T^2x))} \phi(t) dt$$
Continuing this process gives

\[ \int_0^\rho \varphi(t) dt \leq q \int_0^\rho \varphi(t) dt \]

Taking the limit as \( n \to \infty \) implies that \( \lim_{n} \rho(c(T^{n+1}x - T^n x)) = 0 \) for \( c \geq 2l \).

Now, suppose \( l < c' < 2l \). Since \( \rho \) is an increasing function, then one may write

\[ \rho(c(T^{n+1}x - T^n x)) \leq \rho(c(T^{n+1}x - T^n x)) \text{whenever } c' < 2l \leq c. \]

Taking the limit from both sides of this inequality shows that \( \lim_{n} \rho(c(T^{n+1}x - T^n x)) = 0 \) for \( l < c' < 2l \). Thus we have \( \lim_{n} \rho(c'(T^{n+1}x - T^n x)) = 0 \) for any \( c > 1 \).

We now show that \( (T^n x)_{n=0}^\infty \) is \( \rho \)-Cauchy. If not, (by using the same method in the proof of Theorem 1) there exists an \( \varepsilon > 0 \) and two subsequences \( \{m(s)\} \) and \( \{n(s)\} \) such that

\[ \rho( T^{m(s)} x - T^{n(s)} x ) < \varepsilon \]

and we can assume

\[ \rho( T^{m(p)} x - T^{n(p)} x ) \geq \varepsilon \]

But

\[ m(T^{m(p)} x, T^{n(p)} x) = \max \left\{ \rho(T^{m(p)} x - T^{n(p)} x) \right\} \]

Moreover,

\[ \rho(T^{m(p)} x - T^{n(p)} x) = \rho(T^{m(p)} x - T^{m(p)} x) + T^{m(p)} x - T^{n(p)} x) \]

\[ \Delta_2 \text{-condition shows that} \]

\[ \lim_{\varepsilon} \int_0^\varepsilon \rho( T^{m(p)} x - T^{n(p)} x ) \varphi(t) dt \leq \int_0^\varepsilon \rho( T^{m(p)} x - T^{n(p)} x ) \varphi(t) dt \]

On the other hand,

\[ \int_0^\varepsilon \rho( T^{m(p)} x - T^{n(p)} x ) \varphi(t) dt \leq \int_0^\varepsilon \rho( T^{m(p)} x - T^{n(p)} x ) \varphi(t) dt \]

Therefore

\[ \int_0^\varepsilon \rho( T^{m(p)} x - T^{n(p)} x ) \varphi(t) dt \leq \int_0^\varepsilon \rho( T^{m(p)} x - T^{n(p)} x ) \varphi(t) dt \]

\[ \leq q \int_0^\varepsilon \rho( T^{m(p)} x - T^{n(p)} x ) \varphi(t) dt \]

which is a contradiction. Therefore, by \( \Delta_2 \)-condition, \( (T^n x)_{n=0}^\infty \) is \( \rho \)-Cauchy. Since \( X_\rho \) is a \( \rho \)-complete, there exists a \( z \in X_\rho \) such that \( \rho(c(T^n x - z)) \to 0 \) as \( n \to \infty \).

We now prove that \( z \) is a fixed point of \( T \). Note that

\[ \int_0^\rho \varphi(t) dt \leq q \int_0^{m(z,T^n x)} \varphi(t) dt \]

Where

\[ m(z,T^n x) = \max \{ \rho(T^{m(z,T^n x)} x - T^{n+1} x), \rho(T^{m(z,T^n x)} x - T^n x), \rho(T^{m(z,T^n x)} x - T^n x) \} \]

Since

\[ \rho(T^{m(z,T^n x)} x - T^{n+1} x) \]

\[ \rho(T^{m(z,T^n x)} x - T^n x) \]

\[ \rho(T^{m(z,T^n x)} x - T^n x) \]

www.SID.ir
This means that
\[ m(z,T^n x) \leq \max \{ 1/2 (\rho(l(Tz-z)), \rho(l(Tz-z)), \rho(l(T^n x - T^n x))) \} \]
Now, by applying Fatou property, one may write
\[ \rho(c(Tz-z)) \leq \liminf \rho(c(Tz-T^n x)) \]
and since \( \rho(c(T^n x-z)) \to 0 \) and \( \rho(c(T^n x-T^{n+1} x)) \to 0 \) when \( n \to \infty \). Therefore, by taking \( \liminf \) from the both sides of (13), one can prove
\[ \int_0^{\rho(c(Tz-z))} \varphi(t) dt \leq q \int_0^{\rho(c(Tz-T^n x))} \varphi(t) dt \]

Therefore
\[ \int_0^{\rho(c(Tz-z))} \varphi(t) dt \leq q \int_0^{\rho(c(Tz-T^n x))} \varphi(t) dt \]
which implies that \( \rho(c(Tz-z)) = 0 \), and hence that \( Tz = z \). This means that \( z \) is a fixed point of \( T \). To establish the uniqueness, let \( z \) and \( w \) be two arbitrary fixed points of \( T \).
Then
\[ m(z, w) = \max \{ \rho(l(z-w)), 0, 0, 1/2 (\rho(l(z-w)) + \rho(l(w-z))) \} = \rho(l(z-w)) \]
Therefore
\[ \int_0^{\rho(c(z-w))} \varphi(t) dt \leq q \int_0^{\rho(c(z-w))} \varphi(t) dt \]
which implies that \( z = w \).
We now state a fixed point theorem for a quasi-contraction of integral type map in modular spaces. But before this we need some definitions and lemmas as follows:
Let \( T \) be a mapping of a modular space \( X \) into itself. For \( A \subset X \), set \( \delta^A \) = \( \sup \{ \rho(x-y) : x, y \in A \} \), as before. For each \( x \in X \), define
\[ O(x, n) = \{ x, T x, ..., T^n x \}, n = 1, 2, ... \text{ and } O(x, \infty) = \{ x, T x, ... \}. \]
\( O(x, n) \) is called the \( n \)th orbit of \( x \).

**Definition 5.** A mapping \( T \) of a modular space \( X \) into itself is said to be quasi-contraction of integral type if there exists a number \( 0 < q < 1 \), and a number \( c > 1 \), such that
\[ \int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \leq q \int_0^{m(x,y)} \varphi(t) dt \]
Where \( m(x, y) = \max \{ \rho(x-y), \rho(x-Tx), \rho(y-Ty), \rho(x-Ty), \rho(y-Tx) \} \), holds for every \( x, y \in X \) and \( \varphi : R \to R^+ \) is a Lebesgue integrable mapping which is summable, nonnegative and for all \( \varepsilon > 0 \), \( \int_0^\varepsilon \varphi(t) dt > 0 \).

Now, the following lemmas for quasi-contraction of integral type maps are presented.

**Lemma 1.** Let \( T \) be a quasi-contraction of integral type on \( X \) and \( n \) be any positive integer. Then
\[ \int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \leq q \int_0^{\rho(c(Tx-Ty))} \varphi(t) dt \]
hold for every \( x, y \in X \) and all positive integers \( i, j \in \{ 1, 2, ..., n \} \).

Proof. Let \( x \in X \) be arbitrary, and all positive integers \( i, j \in \{ 1, 2, ..., n \} \).
Then \( T^{-i} x, T^{-j} x, T^{-i} x, T^{-j} x \in O(x, n) \) and, since \( T \) is a quasi-contraction of integral type,
\[
\int_0^\infty \rho\left(e^{(T^i x - T^i x)}\right) \varphi(t) \, dt = \int_0^\infty \rho\left(e^{(T^j x - T^j x)}\right) \varphi(t) \, dt
\]
\[
\leq q \max \left\{ \int_0^\infty \rho\left(e^{(T^{i+1} x - T^{i+1} x)}\right) \varphi(t) \, dt, \int_0^\infty \rho\left(e^{(T^{j+1} x - T^{j+1} x)}\right) \varphi(t) \, dt \right\} \leq q \int_0^\infty \delta[O(x,n)] \varphi(t) \, dt
\]

This proves the lemma.

**Lemma 2.** Let \( T \) be a quasi-contraction of integral type and \( x \in X_p \). Then for every positive integer \( n \) there exists a positive integer \( k \leq n \), such that \( \rho\left(e^{(x - T^i x)}\right) = \delta[O(x,n)] \).

Proof. If \( x \) is a fixed point of \( T \), then the proof is complete. Suppose, \( x \) is not a fixed point of \( T \). Then \( \delta[O(x,n)] > 0 \), and there exists integers \( p, p' \) with \( 0 \leq p \leq p' \leq n \) such that
\[
\rho\left(e^{(T^p x - T^{p'} x)}\right) = \delta[O(x,n)].
\]

Assume that \( p > 0 \). For any \( c > 1 \), using property (iii) of Definition 2 implies that
\[
\delta[O(x,n)] = \rho\left(e^{(T^p x - T^{p'} x)}\right) \leq \rho\left(e^{(T^p x - T^{p'} x)}\right).
\]

Now Lemma 1 prove
\[
\int_0^{\delta[O(x,n)]} \varphi(t) \, dt = \int_0^{\delta[O(x,n)]} \varphi(t) \, dt \leq q \int_0^{\delta[O(x,n)]} \varphi(t) \, dt
\]
which implies that \( \delta[O(x,n)] = 0 \), and this is a contradiction. Therefore \( p = 0 \).

**Theorem 4.** Let \( X_p \) be a \( p \)-complete modular space. Suppose \( T \) is a quasi-contraction of integral type selfmap of \( X_p \). If there exists a point \( x \in X_p \) with bounded orbit, then \( T \) has a unique fixed point \( z \in X_p \).

Proof. Let \( x \) be an arbitrary point of \( X_p \). We show that the sequence of iterates \( \{T^n x\}_{n \in \mathbb{N}} \) is a \( p \)-Cauchy sequence.

Let \( n, m \in \mathbb{N} \) and \( n < m \). The definition of \( T \) and Lemma 3.4 show that
\[
\int_0^{\rho\left(e^{(T^p x - T^p x)}\right)} \varphi(t) \, dt = \int_0^{\rho\left(e^{(T^{n+1} x - T^{n+1} x)}\right)} \varphi(t) \, dt
\]
\[
\leq q \int_0^{\delta[p^{n+1} x, m+1]} \varphi(t) \, dt
\]

Then, by Lemma 2, there exists an integer \( k_i \), with \( 1 \leq k_i \leq m - n + 1 \), such that
\[
\delta[O(T^{n-1} x, m-n+1)] = \rho\left(e^{(T^{n-1} x - T^{n} x)}\right)
\]

Lemma 1 prove that
\[
\int_0^{\rho\left(e^{(T^{n+1} x - T^{n+1} x)}\right)} \varphi(t) \, dt = \int_0^{\rho\left(e^{(T^{n+2} x - T^{n+2} x)}\right)} \varphi(t) \, dt
\]
\[
\leq q \int_0^{\delta[p^{n+2} x, k_{i+1}]} \varphi(t) \, dt
\]
\[
\leq q^2 \int_0^{\delta[p^{n+2} x, m+2]} \varphi(t) \, dt
\]

Therefore the following system of inequalities is obtained.

By applying induction, one may write
\[
\int_0^{\rho\left(e^{(T^n x - T^n x)}\right)} \varphi(t) \, dt \leq q^n \int_0^{\delta[O(m,n)]} \varphi(t) \, dt
\]

Taking the limit as \( m, n \to \infty \) (since the orbit of \( x \) is bounded) one obtains implies that,
\[
\lim_{m,n \to \infty} \int_0^{\rho\left(e^{(T^n x - T^n x)}\right)} \varphi(t) \, dt \leq 0,
\]

Which implies that
\[
\rho\left(e^{(T^n x - T^n x)}\right) \to 0
\]
and this means that \( (r^s)_{n=1}^\infty \) is \( \rho \)-Cauchy. Since \( X \) is a \( \rho \)-complete, there exists a \( z \in X \) such that \( \rho(\phi(T^n x - z)) \to 0 \) as \( n \to \infty \). We now show that \( z \) is a fixed point of \( T \).

In order to prove this, one may write

\[
\int_0^{r^s z} \phi(t) dt \leq q \int_0^{m(r^s z)} \phi(t) dt
\]

\[
\leq q \max \{ \int_0^{r^s z} \phi(t) dt, \int_0^{r^s T y} \phi(t) dt \}
\]

Taking the limit as \( n \to \infty \), one obtains

\[
\int_0^{(r^s z)} \phi(t) dt \leq q \int_0^{(r^s z)} \phi(t) dt
\]

which implies that \( \rho(c(T^n z - z)) = 0 \) or \( T z = z \). This means that \( z \) is a fixed point of \( T \).

In order to prove the uniqueness, suppose that \( z \) and \( w \) are two arbitrary fixed points of \( T \). Then

\[
\int_0^{c(T^n z - w)} \phi(t) dt = q \int_0^{c(T^n z - w)} \phi(t) dt
\]

\[
\leq q \max \{ \int_0^{c(T^n z - w)} \phi(t) dt, 0 \}
\]

which implies that \( \rho(c(z - w)) = 0 \) and \( z = w \).

**An application**

In this section, the existence of a solution to the integral equations in the Musielak-Orlicz space is studied \([10,11]\) for the definition of the Musielak-Orlicz space). Consider the following integral equation:

\[
u(t) = \exp(-t)f + \int_0^t \exp(s-t)Tu(s) ds\tag{14}
\]

Where

(i) \( B \) is a convex and \( \rho \)-closed subset of a Musielak-Orlicz space \( L^\rho \), where \( \rho \) satisfies the \( \Delta 2 \)-condition.

(ii) \( T: B \to B \) is such that, for each \( u, v \in L^\rho \), there exist \( c_0, k, l \in \mathbb{R}^+ \) with \( c_0 > \max(1, l) \), \( k \in (0, 1) \) and

\[
\int_0^{\rho(c_0(T u - v))} \varphi(z) dz \leq k \int_0^{\rho(l(u-v))} \varphi(z) dz
\]

where \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue integrable mapping which is summable, nonnegative and for all \( \varepsilon > 0 \), \( \int_0^\varepsilon \varphi(z) dz > 0 \).

(iii) \( f \in B \) is a fixed element in \( L^\rho \).

**Theorem 5** Under conditions (i)-(iii), for each \( A > 0 \), integral equation (14) has a solution \( u \in C^\rho = C([0, A], L^\rho) \), where \( C^\rho \) is the modular space of continuous mappings from \([0, A]\] into \( L^\rho \).

Proof. Define the operator \( S \) over \( C(1, B) \), by

\[Su(t) = \exp(-t)f + \int_0^t \exp(s-t)Tu(s) ds\]
where $I = [0, A]$, $u \in C(I, B)$ and $t \in I$. We now establish two properties of $S$.

(a) Note that $S : C(I, B) \rightarrow C(I, B)$.

(i) $Su$ is continuous from $I = [0, A]$ into $\left(L^p, \| \cdot \|_p \right)$. Let $t_0$ and $t_n \in I$ with $t_n \rightarrow t_0$,

since $u$ is $\rho$-continuous, $Tu$ is $\rho$-continuous at $t_0$ and by $\Delta_2$-condition $Tu$ is $\| \cdot \|_p$–continuous at $t_0$. Hence $Su$ is $\| \cdot \|_p$–continuous at $t_0$.

(ii) $Su(t) \in B$ for all $t \in I$. It is well known that, in the Banach space $\left(L^p, \| \cdot \|_p \right)$,

\[
\int_0^t \exp(s-t) Tu(s) \, ds \in \left( \int_0^t \exp(s-t) \, ds \right) \overline{COB}(Tu(s); 0 \leq s \leq t)
\]

\[
\leq (1 - \exp(-t)) \overline{COB}
\]

where $[1,7] \overline{COB}$ is the closed convex hull of $B$ in $\left(L^p, \| \cdot \|_p \right)$

Since $B$ is convex and $\rho$-closed, $\overline{COB} = \overline{B} \subset \overline{B}^\rho = B$

where $\overline{B}$ denotes the closure of $B$ in the sense of $\| \cdot \|_p$. Hence

\[
Su(t) = \exp(-t)f + \int_0^t \exp(s-t) Tu(s) \, ds \in (\exp(-t)B + (1 - \exp(-t))B) \subseteq B
\]

for all $t \in I$.

(b) From the definition of $S$,

\[
\lambda \left( Su(t) - Sv(t) \right) = \lambda \left( \int_0^t \exp \left( s-t \right) (Tu(s) - Tv(s)) \, ds \right)
\]

where $u, v \in C(I, B)$ and $\lambda > 0$.

To complete the proof of Theorem 4 we shall need the following lemma from [1].

But, before it we recall $\rho_a(u) = \sup_{x \in [a,1]} \rho(u(t))$ here $a \geq 0$ and $u \in C^\rho$.

**Lemma 3.** Let $x \in C^\rho$ and $0 < \lambda \leq \frac{\exp A}{\exp A - 1}$. Then

\[
\rho \left( \int_0^t \exp(s-t) x(s) \, ds \right) \leq \lambda \frac{\exp(at) - \exp(-t)}{1 + a} \rho_a(x).
\]

Proof. Now, Lemma 3 implies that

\[
\rho(c(Su(t) - Sv(t)) \leq \lambda \frac{\exp(at) - \exp(-t)}{1 + a} \rho_a(Tu - Tv).
\]

But, since

\[
\exp(-at) \rho(c(Su(t) - Sv(t)) \leq \lambda \frac{1 - \exp(-(1+a)t)}{1 + a} \rho_a(Tu - Tv).
\]

thus for all $t \in I$

\[
\rho_a(c(Su - Sv)) \leq \lambda \frac{1 - \exp(-(1+a)t)}{1 + a} \rho_a(Tu - Tv).
\]

\[
\leq \lambda \frac{1 - \exp(-(1+a)A)}{1 + a} \rho_a(Tu - Tv).
\]

Then

\[
\rho_a(c(Su - Sv)) \leq \lambda \rho_a(Tu - Tv).
\]

Since $c_0 \neq \max(1,t)$ and the convexity of $\rho$ implies
A fixed point theorem for a … Hosseini Ghoncheh and Coworkers

\[ \rho_a(c(Su - Sv)) \leq \frac{\lambda}{\lambda_0} \rho_a(\lambda_0(Tu - Tv)). \]

Consider \( \lambda \) such that \( 1 < \lambda \leq \frac{\exp A}{\exp A - 1} \) and \( l \pi \lambda \pi \lambda_0 \) therefore

\[ \int_0^{\rho_a(c(Su - Sv))} \phi(z)dz \leq k \int_0^{\rho_a(l(u - v))} \phi(z)dz \leq k \int_0^{\rho_a(l(u - v))} \phi(z)dz \]

Theorem 1 shows that \( S \) has a fixed point which is a solution of the integral equation (14). Thus the proof is completed.

\[ d_s = \rho(\|T^{m(s)}x - T^{m(s)}x\|) \geq \varepsilon \quad s = 1, 2, \ldots \]

We can assume that

\[ \rho(\|T^{m(s)}x - T^{n(s)}x\|) < \varepsilon \]

Conclusion

One can consider an integral equation in the modular spaces and can find a solution of it. This shows that one can solve integral equations in a space where members has growth properties different from the power type growth control, provided by the \( L^p \)-norm.

References: