SET-NORM EXHAUSTIVE SET MULTIFUNCTIONS

A. CROITORU AND A. GAVRILUȚ

Abstract. In this paper we present some properties of set-norm exhaustive set multifunctions and also of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity.

1. Introduction

The subject of the present paper concerns fuzzy set multifunctions. Non-additive set functions and fuzzy sets have been intensively studied by many authors (e.g., Asahina [1], Choquet [4], Daneshgar and Hashemi [7], Denneberg [9], Drewnowski [10], Dubois and Prade [11], Funiokova [12], Li [16], Merghadi and Aliouche [17], Pap [18], Precupanu [19], Shafer [20], Sugeno [21], Suzuki [22], Zadeh [23], Vaezpour and Karimi [24], Wen, Shi and Li [25], Wu and Bo [26]), due to its applications in statistics, economy, theory of games, human decision making.

In our previous papers [5,6,13-15] we extended and studied different concepts (such as pseudo-atom, Darboux property, continuity, exhaustivity, regularity) to the set-valued case.

In [5] we introduced the notion of set-norm on the family of non-empty subsets of a real linear space and studied different notions of continuous set multifunctions with respect to a set-norm.

This paper contains three sections. In the second section, properties of set-norm exhaustive set multifunctions are presented. These set multifunctions (such as probability multimeasures) are used in control, robotics, decision theory (in Bayesian estimation) or in statistical inference (Dempster [8]). In the third section, we present different properties of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. Our results generalize to set-valued case important problems in measure theory, such as non-atomicity or pure atomicity (Aumann and Shapley [2]), that have applications in coincidence and rigidity phenomena (Chişescu [3]).

2. Non-Additive Set Multifunctions

Let $T$ be an abstract nonvoid set, $\mathcal{C}$ a ring of subsets of $T$ and $\mathcal{P}(T)$ the family of all subsets of $T$.
In the sequel, \((X, +, 0)\) will be a commutative semigroup with unity 0 and \(\mathcal{P}_0(X)\) the family of non-empty subsets of \(X\). On \(\mathcal{P}_0(X)\) we consider an order relation denoted by “\(\preceq\)”.

We write \(E < F\) if \(E \preceq F\) and \(E \neq F\), for every \(E, F \in \mathcal{P}_0(X)\). The notation \(F \succeq E\) (\(F \succ E\) respectively) will often be used in the place of \(E \preceq F\) (\(E \prec F\) respectively). We shall write \((\mathcal{P}_0(X), \preceq)\). For every \(E, F \in \mathcal{P}_0(X)\), let \(E + F = \{x + y | x \in E, y \in F\}\).

**Example 2.1.** I. The usual set inclusion “\(\subseteq\)” is an order relation on \(\mathcal{P}_0(X)\).

If \(X\) is a normed space, then \(\mathcal{P}_e(X)\) is the family of non-empty closed subsets of \(X\) and \(\mathcal{P}_c(X)\) is the family of non-empty closed bounded subsets of \(X\).

The set of all real numbers is denoted by \(\mathbb{R}\). We denote \(\mathbb{N}^* = \mathbb{N}\setminus\{0\}\), where \(\mathbb{N}\) is the set of all positive integers.

**Definition 2.2.** [5] A function \(|\cdot| : \mathcal{P}_0(X) \rightarrow [0, +\infty]\) is called a set-norm on \(\mathcal{P}_0(X)\) if it satisfies the conditions:

(i) \(|E| = 0 \Leftrightarrow E = \{0\}, \forall E \in \mathcal{P}_0(X)\).

(ii) \(|E + F| \leq |E| + |F|, \forall E, F \in \mathcal{P}_0(X)\).

**Definition 2.3.** [5] A set-norm \(|\cdot|\) on \((\mathcal{P}_0(X), \preceq)\) is called monotone if for every sets \(E, F \in \mathcal{P}_0(X)\), \(E \preceq F \Rightarrow |E| \leq |F|\). We denote \((\mathcal{P}_0(X), \preceq, |\cdot|)\) when \((\mathcal{P}_0(X), \preceq)\) is endowed with a monotone set-norm \(|\cdot|\).

**Example 2.4.** Let \((X, \|\cdot\|)\) be a real normed space and \(|E|_s = \sup_{x \in E} \|x\|\), for every \(E \in \mathcal{P}_0(X)\). Then the function \(|\cdot|_s\) is a monotone set-norm on \((\mathcal{P}_0(X), \subseteq)\) and we denote this by \((\mathcal{P}_0(X), \subseteq, |\cdot|_s)\).

**Definition 2.5.** A set multifunction \(\mu : \mathcal{C} \rightarrow \mathcal{P}_0(X)\) is called:

(i) a multimeasure if \(\mu(\emptyset) = \{0\}\) and \(\mu(A \cup B) = \mu(A) + \mu(B)\), for every \(A, B \in \mathcal{C}\), with \(A \cap B = \emptyset\).

(ii) null-additive if for every \(A, B \in \mathcal{C}\),

\[\mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(A)\].

(iii) null-null-additive if for every \(A, B \in \mathcal{C}\),

\[\mu(A) = \mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(\emptyset)\].

**Definition 2.6.** Let \(\mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \preceq)\) be a set multifunction. \(\mu\) is said to be:

(i) monotone if \(\mu(A) \preceq \mu(B)\), for every \(A, B \in \mathcal{C}\), with \(A \subseteq B\).

(ii) fuzzy if \(\mu\) is monotone and \(\mu(\emptyset) = \{0\}\).

(iii) subadditive if \(\mu(A \cup B) \preceq \mu(A) + \mu(B)\), for every \(A, B \in \mathcal{C}\).

(iv) a multisubmeasure if \(\mu\) is fuzzy and subadditive.

**Remark 2.7.** I. If \(X\) is a normed space and \(\mu\) is \(\mathcal{P}_e(X)\)-valued, then in the definition of a multi(sub)measure, it usually appears “+” instead of “\(\preceq\)”, because the sum of two closed sets is not always closed.

II. The following implications hold:

(i) If \(\mu\) is a multisubmeasure, then \(\mu\) is null-additive.
(ii) If μ is null-additive, then μ is null-null-additive.

III. The concepts in Definitions 2.5 and 2.6 do not reduce to the usual single-valued case. The difficulty arises here since we have to consider an order relation on \( \mathcal{P}_0(X) \) and many classical measure theory proof methods fail. For instance, if \( \mu : (\mathcal{P}_0(\mathbb{R}), \subseteq) \) is single-valued and monotone, then \( \mu \) reduces in fact to a constant function \( \mu(A) = \{\mu(0)\}, \forall A \in \mathcal{C} \).

Moreover, \( \mathcal{P}_0(X) \) (and also \( \mathcal{P}_c(X) \)) is not a linear space since \( \mathcal{P}_0(X) \) is not a group with respect to the addition "+" defined by \( M + N = \{x + y | x \in M, y \in N\} \), for every \( M, N \in \mathcal{P}_0(X) \).

**Definition 2.8.** A set multifunction \( \mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|) \) is said to be:

(i) **set-norm exhaustive** (shortly, sn-exhaustive) if \( \lim_{n \to \infty} |\mu(A_n)| = 0 \), for every pairwise disjoint sequence of sets \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \).

(ii) **set-norm continuous** (shortly, sn-continuous) if \( \lim_{n \to \infty} |\mu(A_n)| = 0 \), for every \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \), such that \( A_n \searrow \emptyset \) (i.e. \( A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}^* \wedge \bigcap_{n=1}^{\infty} A_n = \emptyset \)).

(iii) **strongly-set-norm continuous** (shortly, strongly sn-continuous) if \( \lim_{n \to \infty} |\mu(A_n)| = 0 \) for every sequence \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \) such that \( A_{n+1} \subseteq A_n, \forall n \in \mathbb{N}^* \) and \( \mu(\bigcap_{n=1}^{\infty} A_n) = \{0\} \).

(iv) **null-continuous** if \( \mu(\bigcup_{n=1}^{\infty} A_n) = \{0\} \) for every sequence \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \) such that \( A_n \subseteq A_{n+1} \) and \( \mu(A_n) = \{0\}, \forall n \in \mathbb{N}^* \).

We now establish some relationships among the set multifunctions introduced in Definition 2.8.

We recall that \( \mathcal{C} \) is a \( \sigma \)-ring if the following conditions hold:

(i) \( A \setminus B \in \mathcal{C} \), for every \( A, B \in \mathcal{C} \).

(ii) \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{C} \), for every \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \).

**Theorem 2.9.** If \( \mathcal{C} \) is a \( \sigma \)-ring and \( \mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|) \) is fuzzy and sn-continuous, then \( \mu \) is sn-exhaustive.

**Proof.** Let \( (A_n)_{n \in \mathbb{N}^*} \) be a sequence of mutually disjoint sets of \( \mathcal{C} \) and let \( B_n = \bigcup_{k=n}^{\infty} A_k \), for all \( n \in \mathbb{N}^* \). Then \( B_n \in \mathcal{C} \), for every \( n \in \mathbb{N}^* \) and \( B_n \searrow \emptyset \). Since \( \mu \) is sn-continuous, it results \( |\mu(B_n)| \to 0 \), which implies \( |\mu(A_n)| \to 0 \). So \( \mu \) is sn-exhaustive. \( \square \)

**Theorem 2.10.** Suppose \( \mathcal{C} \) is a \( \sigma \)-ring and \( \mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|) \) is fuzzy. If \( \mu \) is null-null-additive and strongly-sn-continuous, then \( \mu \) is null-null-additive.

**Proof.** Let \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \) such that \( A_n \subseteq A_{n+1} \) and \( \mu(A_n) = \{0\}, \forall n \in \mathbb{N}^* \). Denote \( A = \bigcup_{n=1}^{\infty} A_n \). We recurrently define a subsequence \( (A_n)_{n \in \mathbb{N}^*} \) of \( (A_n)_{n \in \mathbb{N}^*} \) as follows.
Let \( n_1 = 1 \). For every \( k \in \mathbb{N}^* \), since \( \mu(A_{n_k}) = \{0\} \) and \( A_{n_k} \cup (A\setminus A_{n_k}) \setminus A_{n_k} \), when \( n \to \infty \), by the fact that \( \mu \) is strongly-sn-continuous, we can choose \( n_{k+1} \) so that \( n_{k+1} > n_k \) and

\[
|\mu(A_{n_k} \cup (A\setminus A_{n_{k+1}}))| < \frac{1}{k}.
\]

Denote \( B = \bigcup_{k=1}^{\infty} (A_{n_{2k}} \setminus A_{n_{2k-1}}) \) and \( C = A\setminus B = A_{n_1} \cup \bigcup_{k=1}^{\infty} (A_{n_{2k+1}} \setminus A_{n_{2k}}) \). For all \( k \in \mathbb{N}^* \), since \( B \subseteq A_{n_{2k}} \cup (A\setminus A_{n_{2k+1}}) \), it results:

\[
|\mu(B)| \leq |\mu(A_{n_{2k}} \cup (A\setminus A_{n_{2k+1}}))| < \frac{1}{2k}.
\]

It follows \( \mu(B) = \{0\} \). Analogously, for every \( k \in \mathbb{N}^* \), since \( C \subseteq A_{n_{2k-1}} \cup (A\setminus A_{n_{2k}}) \), it follows that:

\[
|\mu(C)| \leq |\mu(A_{n_{2k-1}} \cup (A\setminus A_{n_{2k}}))| < \frac{1}{2k},
\]

which implies that \( \mu(C) = \{0\} \). Since \( \mu \) is null-null-additive, we obtain \( \mu(A) = \mu(B \cup C) = \{0\} \). So, \( \mu \) is null-continuous.

**Example 2.11.** I. If \( \mathcal{C} \) is finite, then every set multifunction \( \mu : \mathcal{C} \to (\mathcal{P}(\mathbb{N}), \subseteq, |\cdot|) \), with \( \mu(\emptyset) = \{0\} \), is sn-exhaustive and sn-continuous.

II. Let \( T = \mathbb{R} \), \( \mathcal{C} = \{A \subseteq T \mid A \text{ is finite}\} \) and \( \mu : \mathcal{C} \to (\mathcal{P}(\mathbb{R}), \subseteq, |\cdot|) \) be defined by \( \mu(A) = [0, \nu(A)] \), where \( \nu(A) = \begin{cases} 0, A = \emptyset \\ 1 + \text{card}A, A \neq \emptyset \end{cases} \), \( \forall A \in \mathcal{C} \) and \( \text{card}A \) is the number of elements in \( A \). Then \( \mu \) is sn-continuous, but not sn-exhaustive.

III. Let \( T = \mathbb{N} \), \( \mathcal{C} = \mathcal{P}(\mathbb{N}) \) and \( \mu : \mathcal{C} \to (\mathcal{P}(\mathbb{R}), \subseteq, |\cdot|) \) be defined by

\[
\mu(A) = \begin{cases} \{0\}, A \neq \mathbb{N} \\ \{0, 1\} \cup [3, 7], A = \mathbb{N}. \end{cases}
\]

\( \mu \) is not null-null-additive, because there exist \( A = \{0\} \) and \( B = \mathbb{N}^* \) so that \( \mu(A) = \mu(B) = \{0\} \), but \( \mu(A \cup B) = \mu(\mathbb{N}) \neq \{0\} \).

\( \mu \) is not null-continuous since there is \( (A_n)_{n \in \mathbb{N}} \), \( A_n = \{0, 1, 2, \ldots, n\} \), for every \( n \in \mathbb{N} \), such that \( A_n \subseteq A_{n+1} \) and \( \mu(A_n) = \{0\} \), for all \( n \in \mathbb{N} \), but \( \mu(\bigcup_{n=0}^{\infty} A_n) = \mu(\mathbb{N}) \neq \{0\} \).

### 3. Atoms and Pseudo-Atoms

We now give some properties regarding atoms and pseudo-atoms of set multifunctions taking values in the family of nonvoid subsets of a commutative semigroup with unity.

In the sequel, \( (X, +, 0) \) is a commutative semigroup with unity 0 and "\( \leq \)" is an order relation on \( \mathcal{P}_0(X) \).

**Definition 3.1.** Let \( \mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq) \) be a set multifunction.

(i) A set \( A \in \mathcal{C} \) is said to be an atom of \( \mu \) if \( \mu(A) > \mu(\emptyset) \) and for every \( B \in \mathcal{C} \), with \( B \subseteq A \), we have \( \mu(B) = \mu(\emptyset) \) or \( \mu(A\setminus B) = \mu(\emptyset) \).
(ii) A set \( A \in \mathcal{C} \) is called a pseudo-atom of \( \mu \) if \( \mu(A) > \mu(\emptyset) \) and for every \( B \in \mathcal{C} \), with \( B \subseteq A \), we have \( \mu(B) = \mu(\emptyset) \) or \( \mu(B) = \mu(A) \).

(iii) \( \mu \) is called non-atomic (non-pseudo-atomic respectively) if it has no atoms (pseudo-atoms respectively).

**Remark 3.2.** If \( \mu \) is fuzzy, then \( \mu \) is non-atomic (non-pseudo-atomic respectively) if and only if for every \( A \in \mathcal{C} \) with \( \mu(A) > \{0\}; \) there exists \( B \in \mathcal{C} \), so that \( B \subseteq A \), \( \mu(B) > \{0\} \) and \( \mu(A,B) > \{0\} \) (\( \mu(B) < \mu(A) \) respectively).

**Proposition 3.3.** Suppose \( \mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \leq) \) is fuzzy and null-additive (or a multimeasure). Then every atom of \( \mu \) is a pseudo-atom of \( \mu \).

**Proof.** (i) Suppose \( \mu \) is fuzzy and null-additive and let \( A \in \mathcal{C} \) be an atom of \( \mu \). Let \( B \in \mathcal{C}, B \subseteq A \) so that \( \mu(B) \neq \{0\} \). Since \( A \) is an atom of \( \mu \), it results \( \mu(A \setminus B) = \{0\} \). Since \( \mu \) is null-additive, it follows \( \mu(A) = \mu(B \cup (A \setminus B)) = \mu(B) \) which proves that \( A \) is a pseudo-atom of \( \mu \).

(ii) Suppose \( \mu \) is a multimeasure and let \( A \in \mathcal{C} \) be an atom of \( \mu \). Let \( B \in \mathcal{C}, B \subseteq A \) so that \( \mu(B) \neq \{0\} \). Since \( A \) is an atom of \( \mu \), it results in \( \mu(A \setminus B) = \{0\} \).

Since \( \mu \) is a multimeasure, we have:

\[
\mu(A) = \mu(B \cup (A \setminus B)) = \mu(B) + \mu(A \setminus B) = \mu(B) + \{0\} = \mu(B).
\]

So \( A \) is pseudo-atom of \( \mu \).

**Remark 3.4.** The converse of Proposition 3.3 is not valid (see Example 3.7). As we shall see in the sequel, if \( X \) is a normed space, then this converse is true. To prove this we need the following lemma.

**Lemma 3.5.** Let \( (X, \|\cdot\|) \) be a normed space and \( A, B \in \mathcal{P}_0(X) \) so that \( A + B = B \) and \( B \) is bounded. Then \( A = \{0\} \).

**Proof.** Since \( B \) is bounded, there is \( M > 0 \) so that \( \|y\| \leq M \), for every \( y \in B \). Let \( a \in A, b \in B \). It results in \( a + b \in A + B = B \). Then \( 2a + b = a + (a + b) \in A + B = B \). By induction we obtain that \( na + b \in B \), for every \( n \in \mathbb{N} \). It follows \( \|na + b\| \leq M \). Consequently, we have \( \|a\| \leq \frac{2M}{n} \), for every \( n \in \mathbb{N}^* \), which proves that \( a = 0 \). So \( A = \{0\} \).

**Proposition 3.6.** If \( X \) is a normed space and \( \mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \leq) \) is a multimeasure, then \( A \in \mathcal{C} \) is a pseudo-atom of \( \mu \) if and only if \( A \) is an atom of \( \mu \).

**Proof.** I. Suppose \( A \) is an atom of \( \mu \) and let \( B \in \mathcal{C}, B \subseteq A \) such that \( \mu(B) \neq \{0\} \).

Since \( A \) is an atom of \( \mu \), it results in \( \mu(A \setminus B) = \{0\} \). Since \( \mu \) is a multimeasure, we have:

\[
\mu(A) = \mu(A \setminus B) \cup B = \mu(A \setminus B) + \mu(B) = \{0\} + \mu(B) = \mu(B),
\]

which proves that \( A \) is a pseudo-atom of \( \mu \).

II. Suppose \( A \) is a pseudo-atom of \( \mu \) and let \( B \in \mathcal{C}, B \subseteq A \) such that \( \mu(B) \neq \{0\} \).

Since \( A \) is a pseudo-atom of \( \mu \), we have \( \mu(B) = \mu(A) \). Since \( \mu \) is a multimeasure, it results in:

\[
\mu(A) = \mu((A \setminus B) \cup B) = \mu(A \setminus B) + \mu(B) = \mu(A \setminus B) + \mu(A).
\]
According to Lemma 3.5, it follows $\mu(A \setminus B) = \{0\}$. Consequently, $A$ is an atom of $\mu$. 

Example 3.7. I. Let $T = \{a,b,c\}$, $C = \mathcal{P}(T)$, $\mu : C \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in C$ by $\mu(A) = \begin{cases} [0,1] & \text{if } A \neq \emptyset, \\ \{0\} & \text{if } A = \emptyset. \end{cases}$ Then $\mu$ is null-additive, $A = \{a, b\}$ is a pseudo-atom of $\mu$, but not an atom of $\mu$.

II. Let $T = \{a, b\}$, $C = \mathcal{P}(T)$, $\mu : C \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in C$ by $\mu(A) = \begin{cases} [0,2] & \text{if } A = T, \\ [0,1] & \text{if } A = \{b\}, \\ \{0\} & \text{if } A = \emptyset \text{ or } A = \{a\}. \end{cases}$ Then $\mu$ is not null-additive, $T$ is an atom of $\mu$, but not a pseudo-atom of $\mu$.

III. Let $T = 2\mathbb{N} = \{0, 2, 4, \ldots\}$, $C = \mathcal{P}(T)$ and $\mu : C \to (\mathcal{P}_0(X), \subseteq)$ be defined for every $A \in C$ by $\mu(A) = \begin{cases} \{0\} & \text{if } A = \emptyset, \\ \frac{1}{2}A \cup \{0\} & \text{if } A \neq \emptyset. \end{cases}$ where $\frac{1}{2}A = \{\frac{x}{2} \mid x \in A\}$. Then $\mu$ is a multisubmeasure.

If $A \in C$ has card $A = 1$ and $A \neq \{0\}$ or $A \in C$, $A = \{0, 2n\}$, $n \in \mathbb{N}^*$, then $A$ is an atom of $\mu$ (and a pseudo-atom of $\mu$ too).

If $A \in C$ has card $A \geq 2$ and there exist $a, b \in A$ such that $a \neq b$ and $ab \neq 0$, then $A$ is not a pseudo-atom of $\mu$ (and not an atom of $\mu$).

IV. Let $T = \mathbb{N}$, $C = \mathcal{P}(\mathbb{N})$ and $\mu : C \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in C$ by $\mu(A) = \begin{cases} \{0\} & \text{if } A \text{ is finite,} \\ \{0\} \cup (m_A, +\infty), & \text{if } A \text{ is infinite and} \\ n_A = \min A. \end{cases}$ Then $\mu$ is monotone and non-pseudo-atomic.

From definitions we obtain the following properties of pseudo-atoms.

Proposition 3.8. Let $\mu : C \to (\mathcal{P}_0(X), \subseteq)$ be a fuzzy set multifunction.

I. If $A \in C$ is a pseudo-atom of $\mu$ and $B \in C, B \subseteq A$ is so that $\mu(B) > \{0\}$, then $B$ is a pseudo-atom of $\mu$ and $\mu(B) = \mu(A)$.

II. If $A, B \in C$ are pseudo-atoms of $\mu$ and $\mu(A) \neq \mu(B)$, then $\mu(A \cap B) = \{0\}$.

III. Moreover, suppose $\mu$ is null-null-additive and let $A, B \in C$ be pseudo-atoms of $\mu$. Then the following statements hold:

(i) If $\mu(A \setminus B) = \{0\}$, then $A \setminus B$ and $B \setminus A$ are pseudo-atoms of $\mu$ and $\mu(A \setminus B) = \mu(A)$, $\mu(B \setminus A) = \mu(B)$.

(ii) If $\mu(A \cap B) > \{0\}$ and $\mu(A \setminus B) = \mu(B \setminus A) = \{0\}$, then $A \cap B$ is a pseudo-atom of $\mu$ and $\mu(A \Delta B) = \{0\}$ (where $A \Delta B = (A \setminus B) \cup (B \setminus A)$).
Proof. I. Since $\mu(B) \neq \{0\}$ and $A$ is a pseudo-atom of $\mu$, it results in $\mu(B) = \mu(A)$. Let $C \in C$, $C \subseteq B$ and suppose $\mu(C) \neq \{0\}$. Since $C \subseteq A$ and $A$ is a pseudo-atom, it follows $\mu(C) = \mu(A)$. This shows that $\mu(C) = \mu(B)$. So $B$ is a pseudo-atom of $\mu$.

II. Let $a, B \in C$ be pseudo-atoms of $\mu$ such that $\mu(A) \neq \mu(B)$. Suppose, by contrary, that $\mu(A \cap B) > \{0\}$. Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $A, B$ are pseudo-atoms of $\mu$, according to Proposition 3.8-I, we have $\mu(A \cap B) = \mu(A)$ and $\mu(A \cap B) = \mu(B)$. It follows $\mu(A) = \mu(B)$, false!

III. (i) Suppose $\mu(A \setminus B) = \{0\}$. Since $\mu(A \cap B) = \{0\}$ and $\mu$ is null-null-additive, it results $\mu((A \cup B) \cup (A \setminus B)) = \mu(A) = \{0\}$, that is false because $A$ is a pseudo-atom. So $\mu(A \setminus B) > \{0\}$. By Proposition 3.8-I, it follows that $A \setminus B$ is a pseudo-atom and $\mu(A \setminus B) = \mu(A)$. In the same way it follows that $B \setminus A$ is a pseudo-atom and $\mu(B \setminus A) = \mu(B)$.

(ii) Since $A \cap B \subseteq A$ and $\mu(A \cap B) > \{0\}$, by Proposition 3.8-I, it results that $A \cap B$ is a pseudo-atom of $\mu$. Since $\mu$ is null-null-additive, we have $\mu(A \cap B \cap C) = \{0\}$. □

Proposition 3.9. Suppose $\mu : C \rightarrow (P_0(X), \leq)$ is a null-additive fuzzy set multifunction and $A \in C$ is an atom (pseudo-atom respectively). If $E \in C$ is so that $\mu(E) = \{0\}$, then $B = A \cup E$ is also an atom (pseudo-atom respectively).

Proof. Suppose $A$ is a pseudo-atom of $\mu$ and let $C \in C$, $C \subseteq B = A \cup E$. We can set $C = (C \cap A) \cup (C \cap E)$. By the monotonicity of $\mu$, we have $\mu(C \cap E) = \{0\}$. Since $\mu$ is null-additive, the following relation holds:

$$\mu(C \cap A) = \mu(C).$$

But $A$ is a pseudo-atom of $\mu$ and $C \cap A \subseteq A$. Then we have $\mu(C \cap A) = \{0\}$ or $\mu(C \cap A) = \mu(A)$.

(i) If $\mu(C \cap A) = \{0\}$, by (1) it results $\mu(C) = \{0\}$.

(ii) If $\mu(C \cap A) = \mu(A)$, by (1) it results $\mu(A) = \mu(C)$. Since $\mu(E) = \{0\}$ and $\mu$ is null-additive, we have $\mu(B) \neq \mu(A)$. So $\mu(C) = \mu(B)$.

This shows that $B$ is a pseudo-atom of $\mu$.

The case when $A$ is an atom of $\mu$ analogously follows. □

Proposition 3.10. Suppose $\mu : C \rightarrow (P_0(X), \leq, | \cdot |)$ is a null-additive fuzzy set multifunction and let $A \in C$ be an atom of $\mu$. If $\{B_i\}_{i=1}^n \subseteq C$ is a partition of $A$, then there exists a unique $i_0 \in \{1, \ldots, n\}$ such that $\mu(B_{i_0}) = \mu(A)$ and $\mu(B_i) = \{0\}$, for $i \in \{1, \ldots, n\}, i \neq i_0$.

Proof. We have two cases:

I. $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\}$. From the null-additivity of $\mu$, it results $\mu(A) = \{0\}$, which is false.

II. There exists $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$.

Suppose without loss in generality that $\mu(B_1) > \{0\}$ and $\mu(B_2) > \{0\}$. Since $A$ is an atom of $\mu$, it follows $\mu(A \setminus B_1) = \{0\}$. Since $B_2 \subseteq A \setminus B_1$ and $\mu$ is fuzzy, $\mu(B_2) = \{0\}$, which is false.
It results that there exists a unique $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$. Since $A$ is an atom of $\mu$, it follows $\mu(A \setminus B_{i_0}) = \{0\}$. From the null-additivity of $\mu$ we obtain $\mu(A) = \mu(B_{i_0})$. Since $B_i \subseteq A \setminus B_{i_0}$, for every $i \in \{1, \ldots, n\} \setminus \{i_0\}$ and $\mu$ is fuzzy, it follows that $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\} \setminus \{i_0\}$ and the proof is finished. \hfill $\square$

**Definition 3.11.** For a set multifunction $\mu : C \to (\mathcal{P}_0(X), \leq, |\cdot|)$, the following set function (called the variation of $\mu$) is introduced:

$$\bar{\mu} : \mathcal{P}(T) \to [0, +\infty], \bar{\mu}(E) = \sup \{ \sum_{i=1}^{n} |\mu(A_i)| ; A_i \subseteq E, A_i \in \mathcal{C}, A_i \cap A_j = \emptyset, i \neq j, i, j \in \{1, \ldots, n\}, n \in \mathbb{N}^* \}, \text{for every } E \in \mathcal{P}(T).$$

**Remark 3.12.** Let $\mu : C \to \mathcal{P}_0(X)$ be a set multifunction. Then the following statements hold:

I. $|\mu(A)| \leq \bar{\mu}(A)$, for every $A \in \mathcal{C}$.

II. $\bar{\mu}(A) = 0 \Rightarrow |\mu(A)| = 0$, for every $A \in \mathcal{C}$.

III. Moreover, if $\mu : C \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is fuzzy, then we also have:

$$|\mu(A)| = 0 \Rightarrow \bar{\mu}(A) = 0, \forall A \in \mathcal{C}. \quad (2)$$

Indeed, let $\{B_i\}_{i=1}^{n} \subseteq \mathcal{C}$ be a partition of $A$. Since $\mu$ is fuzzy, $\sum_{i=1}^{n} |\mu(B_i)| = 0$ and so, $\bar{\mu}(A) = 0$.

If $\mu$ is not fuzzy, then (2) may be false as we can see in Example 3.13-I.

**Example 3.13.** I. Let $T = \{a, b\}$, $C = \mathcal{P}(T)$ and $\mu : C \to (\mathcal{P}_0(\mathbb{R}), \leq, |\cdot|_A)$ be defined by:

$$\mu(A) = \begin{cases} \{0\}, & A = \emptyset \text{ or } A = T \\ [0, 1] \cup \{2, 3\}, & A = \{a\} \text{ or } A = \{b\}. \end{cases}$$

We have $|\mu(T)| = 0$ and $\bar{\mu}(T) = 6$.

II. Let $T = \{0, 2, 3\}$, $C = \mathcal{P}(T)$, $X = \{f : [0, +\infty) \to [0, +\infty) \}$ and $|E| = \sup f \|f\|_u$, for every $E \in \mathcal{P}_0(X)$, where $\|f\|_u = \sup_{x \in E} |f(x)|$. Let $\mu : C \to (\mathcal{P}_0(X), |\cdot|)$ be defined by:

$$\mu(A) = \{X_{[0, n]} | n \in A\}, \forall A \in \mathcal{C},$$

where $X_{[0, a]}$ is the characteristic function of $[0, a]$.

In this setting we have $|\mu(T)| = 3 < 6 = \bar{\mu}(T)$.

**Proposition 3.14.** If $\mu : C \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is a fuzzy set multifunction and $A \in \mathcal{C}$ is an atom of $\mu$, then $\bar{\mu}(A) = |\mu(A)|$.

**Proof.** According to the Remark 3.12-I, we only have to prove:

$$\bar{\mu}(A) \leq |\mu(A)|. \quad (3)$$
Let \( \{B_i\}_{i=1}^n \subset C \) be an arbitrary partition of \( A \), where \( n \in \mathbb{N}^* \). We have two cases:

1. \( \mu(B_i) = \{0\} \), for every \( i \in \{1, \ldots, n\} \). Then \( \sum_{i=1}^n |\mu(B_i)| = 0 \leq |\mu(A)| \).

2. There exists \( i_0 \in \{1, \ldots, n\} \) so that \( \mu(B_{i_0}) > \{0\} \).

Suppose without loss of generality that \( \mu(B_1) > \{0\} \) and \( \mu(B_2) > \{0\} \). Since \( A \) is an atom of \( \mu \), it follows \( \mu(A \setminus B_1) = \{0\} \). Since \( B_2 \subseteq A \setminus B_1 \) and \( \mu \) is fuzzy, it results in \( \mu(B_2) = \{0\} \), which is false. It results there is a unique \( i_0 \in \{1, \ldots, n\} \) so that \( \mu(B_{i_0}) > \{0\} \). Since \( B_i \subseteq A \setminus B_{i_0} \), for every \( i \in \{1, \ldots, n\} \setminus \{i_0\} \) and \( \mu \) is fuzzy, it follows that \( \mu(B_i) = \{0\} \), for every \( i \in \{1, \ldots, n\} \setminus \{i_0\} \). So \( \sum_{i=1}^n |\mu(B_i)| \leq |\mu(A)| \).

Since \( \{B_i\}_{i=1}^n \) is an arbitrary partition of \( A \), it results (3). \( \square \)

**Proposition 3.15.** Suppose \( \mu : C \rightarrow (\mathcal{P}_0(X), \leq, |\cdot|) \) is a sn-exhaustive fuzzy set multifunction. Then for every \( E \in \mathcal{P}(T) \) and every \( \varepsilon > 0 \), there is \( A \in C \) so that \( A \subseteq E \) and \( |\mu(B \setminus A)| < \varepsilon \), for all \( B \in C, A \subseteq B \subseteq E \).

**Proof.** Suppose on the contrary that there exist \( E_0 \in \mathcal{P}(T) \) and \( \varepsilon > 0 \) so that for all \( A \in C, \) with \( A \subseteq E_0 \), there is \( B_0 \in C \) so that \( A \subseteq B_0 \subseteq E_0 \) and \( |\mu(B_0 \setminus A)| \geq \varepsilon \). We construct by recurrence a sequence of mutual disjoint sets \( \{L_n\} \subset C \) such that \( L_n \subseteq E_0 \) for every \( n \in \mathbb{N} \) and \( |\mu(L_n)| \geq \varepsilon \).

Suppose we obtained \( L_1, L_2, \ldots, L_n \) and let \( K = \bigcup_{i=1}^n L_i \). Obviously, \( K \in C \) and \( K \subseteq E_0 \). Then there is \( B \in C \) so that \( K \subseteq B \subseteq E_0 \) and \( |\mu(B \setminus K)| \geq \varepsilon \). If we set \( L_{n+1} = B \setminus K \), then we have \( L_{n+1} \in C, L_{n+1} \subseteq E_0, |\mu(L_{n+1})| \geq \varepsilon \) and \( L_{n+1} \cap L_i = \emptyset \), for every \( i \in \{1, \ldots, n\} \). Since \( \mu \) is sn-exhaustive, we have \( \lim_{n \rightarrow \infty} |\mu(L_n)| = 0 \), that is a contradiction. \( \square \)

**Definition 3.16.** Suppose \( |\cdot| \) is a monotone set-norm on \( (\mathcal{P}_0(X), \leq) \) and let \( \mathcal{R} \) be a non-empty subset of \( \mathcal{P}_0(X) \).

(i) A net \((Y_i) \in \mathcal{P}_0(X) \) is called **sn-Cauchy** if the net \((|Y_i|)\) is Cauchy (i.e., for every \( \varepsilon > 0 \), there is \( i_0 \) such that \(|Y_i - Y_j| < \varepsilon \), for every \( i, j \geq i_0 \)).

(ii) A net \((Y_i) \in \mathcal{R} \) is called **sn-convergent** in \( \mathcal{R} \) if there is a unique \( Y_0 \in \mathcal{R} \) so that \( \lim_{i \rightarrow \infty} Y_i = Y_0 \). We denote this by \( \lim_{i \rightarrow \infty} Y_i = Y_0 \).

(iii) \( \mathcal{R} \) is called **sn-complete** if every sn-Cauchy net of \( \mathcal{R} \) is convergent in \( \mathcal{R} \).

**Example 3.17.** Let \( \mathcal{R} = \{(0, x) | x \in [0, +\infty)\} \). Then \( \mathcal{R} \) is a sn-complete subspace of \( (\mathcal{P}_0(X), \leq, |\cdot|) \).

**Theorem 3.18.** Suppose \( |\cdot| \) is a monotone set-norm on \( (\mathcal{P}_0(X), \leq) \), let \( \mathcal{R} \) be a sn-complete subset of \( (\mathcal{P}_0(X), \leq, |\cdot|) \) and let \( \mu : C \rightarrow \mathcal{R} \) be an sn-exhaustive subadditive fuzzy set multifunction. Then the following statements hold:

(i) For every \( E \in \mathcal{P}(T) \), there exists \( \lim_{A \subseteq E} \mu(A) = \mu^*(E) \in \mathcal{R} \), where the net \( (\mu(A))_{A \subseteq E} \) is directed by the usual inclusion \( \subseteq \) of sets and the limit is in the sense of Definition 3.16-(ii). We obtain a set multifunction \( \mu^* : \mathcal{P}(T) \rightarrow \mathcal{R} \).
(ii) \(|\mu^*(A)| = |\mu(A)|, \forall A \in \mathcal{C}\).

(iii) \(\forall E_1, E_2 \in \mathcal{P}(T), E_1 \subseteq E_2 \Rightarrow |\mu^*(E_1)| \leq |\mu^*(E_2)|\).

(iv) \(\mu^*\) is sn-exhaustive.

(v) If \(\mu\) is non-atomic, then \(\mu^*\) is non-atomic.

Proof. (i) According to Proposition 3.15, for every \(E \in \mathcal{P}(T)\) and \(\varepsilon > 0\), there exists \(A_0 \in \mathcal{C}\) so that \(A_0 \subseteq E\) and for every \(A \in \mathcal{C}\) with \(A_0 \subseteq A \subseteq E\), we have

\[0 \leq |\mu(A)| - |\mu(A_0)| \leq |\mu(A \setminus A_0)| < \varepsilon.\]

So \((\mu(A))_{A \subseteq E}^{\mathcal{A}^E}\) is sn-Cauchy in \(\mathcal{R}\) and since \(\mathcal{R}\) is sn-complete, the net \((\mu(A))_{A \subseteq E}^{\mathcal{A}^E}\) is sn-convergent in \(\mathcal{R}\).

(ii) Let \(A \in \mathcal{C}\). For every \(\varepsilon > 0\) there is \(B_0 \in \mathcal{C}\) so that \(B_0 \subseteq A\) and for every \(B \in \mathcal{C}\) with \(B_0 \subseteq B \subseteq A\) we have \(|\mu(B)| - |\mu^*(A)| < \varepsilon\). Particularly, \(\|\mu(A)\| - |\mu^*(A)| < \varepsilon\) for every \(\varepsilon > 0\). Hence \(|\mu^*(A)| = |\mu(A)|\).

For \(A = \emptyset\) it results in \(|\mu^*(\emptyset)| = |\mu(\emptyset)| = 0\) which implies that \(\mu^*(\emptyset) = \{0\}\).

(iii) Let \(E_1, E_2 \in \mathcal{P}(T)\) be so that \(E_1 \subseteq E_2\). Consider an arbitrary \(\varepsilon > 0\). Then there exists \(A_1 \subseteq \mathcal{C}\) so that \(A_1 \subseteq E_1\) and for every \(A \in \mathcal{C}\), with \(A \subseteq A_1 \subseteq E_1\), we have \(|\|\mu(A)| - |\mu^*(E_1)|\| < \frac{\varepsilon}{2}\). Particularly we have

\[|\mu^*(E_1)| - |\mu(A_1)| < \frac{\varepsilon}{2}. \tag{4}\]

Analogously there exists \(A_2 \subseteq \mathcal{C}\) so that \(A_2 \subseteq E_2\) and for every \(A \in \mathcal{C}\), with \(A_2 \subseteq A \subseteq E_2\) we have

\[|\mu^*(E_2)| - |\mu(A)| < \frac{\varepsilon}{2}. \tag{5}\]

Let \(A_0 = A_1 \cup A_2 \in \mathcal{C}\). Then \(A_2 \subseteq A_0 \subseteq E_2\) and we have

\[|\mu^*(E_2)| - |\mu(A_0)| < \frac{\varepsilon}{2}. \tag{5}\]

Now, from (4) and (5) we obtain:

\[|\mu^*(E_1)| < |\mu(A_1)| + \frac{\varepsilon}{2} \leq |\mu(A_0)| + \frac{\varepsilon}{2} < |\mu^*(E_2)| + \varepsilon\]

for every \(\varepsilon > 0\). Hence \(|\mu^*(E_1)| \leq |\mu^*(E_2)|\).

(iv) Let \((E_n) \subseteq \mathcal{P}(T)\) be so that \(E_n \cap E_m = \emptyset\), for every \(m \neq n\). For every \(n \in \mathbb{N}\) and \(\varepsilon > 0\), there exists \(A_n \in \mathcal{C}\) such that \(A_n \subseteq E_n\) and \(||\mu(A_n)| - |\mu^*(E_n)|| < \frac{\varepsilon}{2}\).

Since \(A_n \cap A_m = \emptyset\), for every \(n \neq m\) and \(\mu\) is sn-exhaustive, it results that there is \(n_0 \in \mathbb{N}\) so that \(|\mu(A_n)| < \frac{\varepsilon}{2}\), for every \(n \geq n_0\). So we have:

\[|\mu^*(E_n)| \leq ||\mu^*(E_n)\| - |\mu(A_n)|\| + |\mu(A_n)| < \varepsilon, \ \forall n \geq n_0,\]

which proves that \(\mu^*\) is sn-exhaustive.

(v) On the contrary suppose there is \(E \in \mathcal{P}(T)\) an atom of \(\mu^*\). Then \(\mu^*(E) > \{0\}\) and so \(|\mu^*(E)| > 0\). It results that there is \(A \in \mathcal{C}\) so that \(A \subseteq E\) and \(|\mu(A)| > 0\). Since \(\mu(A) > \{0\}\) and \(\mu\) is non-atomic, there exists \(B \in \mathcal{C}\) such that \(B \subseteq A\), \(\mu(B) > \{0\}\) and \(\mu(A \setminus B) > \{0\}\). Since \(B \subseteq E\) and \(E\) is an atom of \(\mu^*\), it follows that \(\mu^*(B) = \{0\}\) or \(\mu^*(E \setminus B) = \{0\}\).
I. If \( \mu^*(B) = \{0\} \), then by (ii) we have \( |\mu(B)| = |\mu^*(B)| = 0 \) and so \( \mu(B) = \{0\} \) which is false.

II. If \( \mu^*(E \setminus B) = \{0\} \), then by (ii) and (iii) we have:
\[
0 < |\mu(A \setminus B)| = |\mu^*(A \setminus B)| \leq |\mu^*(E \setminus B)| = 0
\]
that is a contradiction.
Consequently, \( \mu^* \) is non-atomic. \( \square \)

4. Conclusion

In this paper we presented properties of set-norm exhaustive set multifunctions and also some properties of atoms and pseudo-atoms of set-multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. It would be interesting to see what non-atomicity becomes in the absence of fuzzyness of \( \mu \).

Acknowledgements. The authors are grateful to the Referees for valuable suggestions in improvement of this paper.

REFERENCES


Anca Croitoru, Faculty of Mathematics, "A. I. Cuza" University, Bd. Carol I, no 11, Iași-700506, Romania
E-mail address: croitoru@uaic.ro

Alina Gavrilut, Faculty of Mathematics, "A. I. Cuza" University, Bd. Carol I, no 11, Iași-700506, Romania
E-mail address: gavrilut@uaic.ro

*Corresponding author