$L$–ORDERED FUZZIFYING CONVERGENCE SPACES

W. WU AND J. FANG

Abstract. Based on a complete Heyting algebra, we modify the definition of lattice-valued fuzzifying convergence space using fuzzy inclusion order and construct in this way a Cartesian-closed category, called the category of $L$–ordered fuzzifying convergence spaces, in which the category of $L$–fuzzifying topological spaces can be embedded. In addition, two new categories are introduced, which are called the category of principal $L$–ordered fuzzifying convergence spaces and that of topological $L$–ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of $L$–fuzzifying neighborhood spaces and that of $L$–fuzzifying topological spaces respectively.

1. Introduction

Convergence structures are more general than topological structures. If a convergence structure additionally satisfies proper conditions, it is equivalent to a topological structure. Lowen [12] constructed convergence systems using prefilters, through which Min [13] proposed fuzzy limit structures. Xu [14] proved that topological $L$–fuzzifying convergence structures and $L$–fuzzifying topologies [17] are equivalent, where classical filters play a crucial role. By stratified $L$–filters [7], Jäger [8] introduced stratified $L$–fuzzy convergence spaces in the many-valued case. The category of these spaces was developed to a significant extent in the recent years [1,2,4,5,9-11,14,15].

In 2009, Yao [16] defined $L$–fuzzifying convergence spaces, and showed the category of $L$–fuzzifying topological spaces [17] could be embedded in the category of $L$–fuzzifying convergence spaces as a reflective subcategory and the latter is Cartesian-closed. $L$–fuzzifying convergence spaces were based on $L$–filters of ordinary subsets.

This paper can be seen as a further step towards [16]. It proposes a new lattice-valued fuzzifying convergence structure, called $L$–ordered fuzzifying convergence structure, which is compatible with the fuzzy inclusion order of $L$–filters of ordinary subsets. The category of $L$–fuzzifying topological spaces [17] can be embedded in the resulting category. As a matter of fact, it is easier for a bigger category to be Cartesian-closed, and it makes sense to establish a smaller Cartesian-closed category. Note that the category of $L$–ordered fuzzifying convergence spaces is

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“smaller” than that of \( L \)-fuzzifying convergence spaces [16], and it is Cartesian-closed. In addition, two new categories are introduced, which are called the category of principal \( L \)-ordered fuzzifying convergence spaces and that of topological \( L \)-ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of \( L \)-fuzzifying neighborhood spaces and that of \( L \)-fuzzifying topological spaces respectively.

2. Preliminaries

Let \((L, \lor, \land)\) be a complete lattice. If the finite meets are distributive over arbitrary joins, i.e. for all \(a, b_i \in L, (i \in J)\)

\[
a \land (\bigvee_{i \in J} b_i) = \bigvee_{i \in J} (a \land b_i),
\]

\(L\) is called a complete Heyting algebra. For \(L\), we define an implication operator \(\rightarrow: L \times L \to L\) as follows:

\[
\forall a, b \in L, a \to b = \bigvee \{ c \in L | a \land c \leq b \}.
\]

Then it is the right adjoint for \(\land\), i.e.,

\[
\forall a, b, c \in L, a \land c \leq b \iff c \leq a \to b.
\]

**Theorem 2.1.** [7] Let \(L\) be a complete Heyting algebra. For all \(a, b, c, d, a_i, b_i \in L, (i \in J)\), the following holds:

1. \((H1)\) \(a \leq (b \to c) \iff a \land b \leq c, \) and \(a \leq b \iff (a \to b) = 1,\)
2. \((H2)\) \(a \to (\bigwedge_{i \in J} b_i) = \bigwedge_{i \in J} (a \to b_i), \) \( (\bigvee_{i \in J} b_i) \to a = \bigvee_{i \in J} (b_i \to a),\)
3. \((H3)\) \((b \to c) \leq (a \to b) \to (a \to c), \) \((a \to c) \land (b \to d) \leq (a \land b) \to (c \land d),\)
4. \((H4)\) \(a \to b \geq b, \) \(a \leq (a \to b) \to b,\)
5. \((H5)\) \(a \land b = a \land (a \to b), \) therefore, \(b = 1 \to b,\)
6. \((H6)\) \(a \to (b \to c) = (a \land b) \to c,\)
7. \((H7)\) \(\bigwedge_{i \in J} (a_i \to b_i) \leq (\bigwedge_{i \in J} a_i) \to (\bigwedge_{i \in J} b_i).\)

In what follows, we consider \(X\) a nonempty set and \(L\) a complete Heyting algebra unless otherwise stated.

For a given set \(X, L^X\) denotes the set of all \(L\)-subsets on \(X\). Define a binary mapping \(S(-, -): L^X \times L^X \to L\) by \(S(U, V) = \bigwedge_{x \in X} (U(x) \to V(x))\) for each pair \((U, V) \in L^X \times L^X.\)

**Definition 2.2.** [6] A map \(\mathcal{F}: 2^X \to L\) is called an \(L\)-filter of ordinary subsets of \(X\) if it satisfies \(\forall x \in X, A, B \in 2^X,\)

1. \((F1)\) \(\mathcal{F}(\emptyset) = 0, \mathcal{F}(X) = 1,\)
2. \((F2)\) \(A \subseteq B \Rightarrow \mathcal{F}(A) \leq \mathcal{F}(B),\)
3. \((F3)\) \(\mathcal{F}(A \cap B) \geq \mathcal{F}(A) \land \mathcal{F}(B).\)
The family of all $L$–filters of ordinary subsets on $X$ will be denoted by $\mathcal{F}_L(X)$. An order on $\mathcal{F}_L(X)$ is defined as follows: $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), \mathcal{F} \leq \mathcal{G} \iff \forall U \in 2^X, \mathcal{F}(U) \leq \mathcal{G}(U)$.

For every $x \in X$, $[x] \in \mathcal{F}_L(X)$ is defined by $\forall A \in 2^X$,

$$[x](A) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\mathcal{F}$ be a filter of ordinary subsets on $X$ and $f : X \to Y$ be a mapping. Then the mapping $f^\leftarrow(\mathcal{F}) : 2^Y \to L$, where $\forall B \in 2^Y, f^\leftarrow(\mathcal{F})(B) = \mathcal{F}(f^\leftarrow(B))$, is an $L$–filter of ordinary subsets on $Y$ and is called the image of $\mathcal{F}$ under $f$.

For every $\mathcal{F} \in \mathcal{F}_L(X), \mathcal{G} \in \mathcal{F}_L(Y), \mathcal{F} \times \mathcal{G} \in \mathcal{F}_L(X \times Y)$ is defined as follows: $\forall C \in 2^{X \times Y}, (\mathcal{F} \times \mathcal{G})(C) = \bigvee_{A \times B \subseteq C} \mathcal{F}(A) \land \mathcal{G}(B)$.

**Definition 2.3.** [18] An $L$–fuzzifying neighborhood structure on a set $X$ is a family of functions $N = \{N_x : 2^X \to L \mid x \in X\}$ with the following conditions: For all $x \in X, U, V \in 2^X$,

1. $N_x(X) = 1$,
2. $N_x(U) > 0$ implies $x \in U$,
3. $N_x(U \cap V) = N_x(U) \land N_x(V)$.

The pair $(X, N)$ is called an $L$–fuzzifying neighborhood space, and it will be called topological if it satisfies moreover: For all $x \in X, U \in 2^X$,

4. $N_x(U) = \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N_y(V)$.

A continuous function between $L$–fuzzifying neighborhood spaces $(X, N^1)$ and $(Y, N^2)$ is a map $f : X \to Y$ such that for all $x \in X, U \in 2^X, N^1_x(f^\leftarrow(U)) \geq N^2_x(U)$.


**Definition 2.4.** [17] An $L$–fuzzifying topology on $X$ is a function $\tau : 2^X \to L$ which satisfies

1. $\tau(\emptyset) = \tau(X) = 1$,
2. $\tau(\bigcap A) \geq \tau(A) \land \tau(B)$,
3. $\tau\left(\bigcup_{j \in J} A_j\right) \geq \bigwedge_{j \in J} \tau(A_j)$.

For an $L$–fuzzifying topology $\tau$ on $X$, the pair $(X, \tau)$ is called an $L$–fuzzifying topological space. A map $f : X \to Y$ is called continuous with respect to the given $L$–fuzzifying topological spaces $(X, \tau_1)$ and $(Y, \tau_2)$ if $\forall B \in 2^Y, \tau_2(\bigcap f^\leftarrow(B)) \geq \tau_2(B)$.

The category of $L$–fuzzifying topological spaces with continuous maps as morphisms will be denoted by $L$–FYS.

It was proved in [20] that for any completely distributive lattice $L$, topological $L$–fuzzifying neighborhood systems and $L$–fuzzifying topologies are conceptually equivalent with transferring process $N_x(U) = \bigvee_{x \in V \subseteq U} \tau(V)$ and $\tau(U) = \bigwedge_{x \in U} N_x(U)$.
Theorem 2.5. [19] Let \( \varphi : (X, \tau_1) \to (Y, \tau_2) \) be a mapping. If \( L \) is a completely distributive lattice, then \( \varphi \) is continuous iff \( N^L_{\varphi^{-1}}(U) \geq N^L_{\varphi(x)}(U), \forall x \in X, U \in 2^Y \).

3. \( L \)-ordered Fuzzifying Convergence Structure

In [16], the author developed lattice-valued convergence structure \( \text{lim} : \mathcal{F}_L(X) \to L^X \) as follows:

**Definition 3.1.** [16] A mapping \( \text{lim} : \mathcal{F}_L(X) \to L^X \), subject to the conditions

(LY1) \( \forall x \in X, \lim[x](x) = 1 \),

(LY2) \( \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), \mathcal{F} \leq \mathcal{G} \Rightarrow \forall x \in X, \lim \mathcal{F}(x) \leq \lim \mathcal{G}(x) \),

is called an \( L \)-fuzzifying convergence structure on \( X \), and \( (X, \text{lim}) \) an \( L \)-fuzzifying convergence space.

The set of all \( L \)-fuzzifying convergence structures on \( X \) is denoted by \( \lim_L(X) \). An order on \( \lim_L(X) \) can be defined by \( \lim_1 \leq \lim_2 \) iff for all \( \mathcal{F} \in \mathcal{F}_L(X), x \in X, \lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x) \).

In Definition 3.1, the \( L \)-filters in the axiom (LY2) are in nature \( L \)-sets on \( 2^X \). We use the method in [3] and define an \( L \)-partial order \( S_F(-, -) \) on \( \mathcal{F}_L(X) \) as follows: \( S_F(-, -) : \mathcal{F}_L(X) \times \mathcal{F}_L(X) \to L \)

\[ \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), S_F(\mathcal{F}, \mathcal{G}) = \bigwedge_{U \in 2^X} (\mathcal{F}(U) \to \mathcal{G}(U)) \]

In this case, we can redefine the axiom (LY2) in Definition 3.1, proposing the following new lattice-valued convergence structure.

**Definition 3.2.** An \( L \)-fuzzifying convergence structure \( \text{lim} : \mathcal{F}_L(X) \to L^X \), satisfying the following condition:

(OLY2) \( \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), S_F(\mathcal{F}, \mathcal{G}) \leq S(\lim \mathcal{F}, \lim \mathcal{G}) \),

is called an \( L \)-ordered fuzzifying convergence structure, and the pair \( (X, \text{lim}) \) an \( L \)-ordered fuzzifying convergence space.

A function \( \varphi : (X, \lim^X) \to (Y, \lim^Y), (X, \lim^X), (Y, \lim^Y) \) \( L \)-ordered fuzzifying convergence spaces, is called continuous iff for all \( \mathcal{F} \in \mathcal{F}_L(X), x \in X, \lim^X \mathcal{F}(x) \leq \lim^Y \varphi^{-1}(\mathcal{F})(\varphi(x)) \).

We do not go into details here, but only remark that (OLY2) implies (LY2).

The next example shows there exists an \( L \)-fuzzifying convergence structure \( \text{lim} \) which is not an \( L \)-ordered fuzzifying convergence structure.

**Example 3.3.** Let \( X = \{x, y\}, L = \{0, 1\} \) be a chain. Define a map \( \text{lim} : \mathcal{F}_L(X) \to L^X \), \( \forall \mathcal{F} \in \mathcal{F}_L(X), z \in X \),

\[ \text{lim} \mathcal{F}(z) = \begin{cases} 1, & \mathcal{F} \geq [z], \\ 0, & \text{otherwise}. \end{cases} \]

It is obvious that \( \text{lim} \) is an \( L \)-fuzzifying convergence structure. Define a mapping \( \mathcal{F}^+ : 2^X \to L \) as follows: \( \forall A \in 2^X \),

\[ \mathcal{F}^+(A) = \begin{cases} 1, & A \geq [z], \\ 0, & \text{otherwise}. \end{cases} \]
It can be verified that $\mathcal{F}^*$ is an $L$–filter of ordinary subsets on $X$. Then

$$S_F([x], \mathcal{F}^*) = \bigwedge_{A \in 2^X} ([x](A) \to \mathcal{F}^*(A))$$

$$= ([x](\emptyset) \to \mathcal{F}^*(\emptyset)) \land ([x]([\{x\}]) \to \mathcal{F}^*(\{x\}))$$

$$\land ([x](\{y\}) \to \mathcal{F}^*(\{y\})) \land ([x](X) \to \mathcal{F}^*(X))$$

$$= 1 \land \alpha \land 1 \land 1$$

$$= \alpha$$

And

$$S(\lim [x], \lim \mathcal{F}^*) = \bigwedge_{x \in X} (\lim [x](z) \to \lim \mathcal{F}^*(z))$$

$$= (\lim [x](x) \to \lim \mathcal{F}^*(x)) \land (\lim [x](y) \to \lim \mathcal{F}^*(y))$$

$$= (1 \to 0) \land (0 \to 0)$$

$$= 0$$

We can see that $S_F([x], \mathcal{F}^*) \neq S(\lim [x], \lim \mathcal{F}^*)$, hence $\lim$ is not an $L$–ordered fuzzifying convergence structure.

**Example 3.4.** Let $(X, \tau) \in L$–FYS and define a mapping $\lim_{\tau} : \mathcal{F}_L(X) \to L^X$, $\forall \mathcal{F} \in \mathcal{F}_L(X)$, $x \in X, \lim_{\tau} \mathcal{F}(x) = S_F(N^\tau_x, \mathcal{F})$. Here, the $L$–fuzzifying neighborhood system $N^\tau_x$ of $x \in X$ is defined by $N^\tau_x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$. Then $\lim_{\tau}$ is an $L$–ordered fuzzifying convergence structure.

From Example 3.4, we see that an $L$–fuzzifying topology can induce an $L$–ordered fuzzifying convergence structure. The following theorem shows that the induced $L$–ordered fuzzifying convergence structure from the $L$–fuzzifying topology can determine the induced $L$–fuzzifying neighborhood structure from the $L$–fuzzifying topology. This idea has been presented in [8].

**Theorem 3.5.** Let $(X, \tau) \in L$–FYS. Then the following holds:

$$N^\tau_x(U) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_{\tau} \mathcal{F}(x) \to \mathcal{F}(U)), \forall x \in X, U \in 2^X.$$


The set of all $L$–ordered fuzzifying convergence structures on $X$ is denoted by $\lim_{\text{top}}(X)$. An order on $\lim_{\text{top}}(X)$ can be defined by $\lim_1 \leq \lim_2$ iff for all $\mathcal{F} \in \mathcal{F}_L(X)$, $x \in X$, $\lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x)$. For $\lim_{\text{top}}(X)$ here, we immediately
obtain that there are a maximum element and a minimum element in \((\lim_{\text{loy}}(X), \leq)\), denoted by \(\lim_{\text{sm}}\) and \(\lim_{\text{m}}\) respectively: 
\[
\forall F \in F, L(X), x \in X, \lim_{\text{sm}} F = 1_X; \lim_{\text{m}} F(x) = S_F([x], \mathcal{F}).
\]
The supremum element of a family of \(L\)-ordered fuzzifying convergence structures \((\lim_j)_{j \in J} \subseteq \lim_{\text{loy}}(X)\) is defined by \(\sup_{j \in J} \lim_j F(x) = \bigwedge_{j \in J} \lim_j F(x), \forall F \in F, L(X), x \in X\). Obviously, \(\sup_{j \in J} \in \lim_{\text{loy}}(X)\). Therefore, the following proposition holds.

**Proposition 3.6.** \((\lim_{\text{loy}}(X), \leq)\) is a complete lattice.

We will next address the result that the category of \(L\)-ordered fuzzifying convergence spaces is a topological category. To this end, we note the following proposition.

**Proposition 3.7.** The category \(L\)-OFYC is a full reflective subcategory in the category \(L\)-FYCS.

**Proof.** Let \((X, \overline{\lim}) \in L\)-FYCS and \(E_{\overline{\lim}} = \{ \lim | (X, \lim) \in L\)-OFYC , \(\lim \leq \overline{\lim}\}\). Note that \(E_{\overline{\lim}}\) is not empty because it always contains \(\lim_{\text{sm}}\). Then with Proposition 3.6, we can construct an \(L\)-ordered fuzzifying convergence structure \(\lim_{\overline{\lim}} : F_L(X) \to L^X\) as follows: For all \(F \in F_L(X), x \in X, \lim_{\overline{\lim}} F(x) = \bigwedge_{\lim \in E_{\overline{\lim}}} \lim F(x)\). From this, we have

1. \(i_{\overline{\lim}} : (X, \overline{\lim}) \to (X, \lim_{\overline{\lim}})\) is trivially continuous;

2. For an \(L\)-ordered fuzzifying convergence space \((Y, \lim_Y)\), if \(f : (X, \overline{\lim}) \to (Y, \lim_Y)\) is a continuous mapping, then \(f : (X, \lim_{\overline{\lim}}) \to (Y, \lim_Y)\) is also continuous. We leave the above check to the reader.

From the above facts, we immediately obtain that \(L\)-OFYC is a full reflective subcategory in \(L\)-FYCS. \(\square\)

In [16] Yao proved that the category \(L\)-FYCS is topological. By Proposition 3.7, we have the following main result.

**Theorem 3.8.** The category of \(L\)-ordered fuzzifying convergence spaces \(L\)-OFYC is topological.

4. The Relations Between Categories of \(L\)-FYS and \(L\)-OFYC

This section is motivated by reference [8]. In this section, we will resolve the embedding of \(L\)-FYS into \(L\)-OFYC. By Example 3.4 and Theorem 3.5, we see that \(L\)-ordered convergence structures can be induced from \(L\)-fuzzifying topologies. Moreover, they are unique. In order to show that \(L\)-FYS can be embedded in the category of \(L\)-OFYC, the following theorem is necessary.

**Theorem 4.1.** Let \(L\) be a completely distributive lattice. Then the map \(f : (X, \tau_1) \to (Y, \tau_2)\) between two \(L\)-fuzzifying topological spaces is continuous iff \(f : (X, \lim_{\tau_1}) \to (Y, \lim_{\tau_2})\) is continuous.
Proof. Suppose that \( f : (X, \tau_1) \to (Y, \tau_2) \) is continuous, by Theorem 2.5, we have for all \( \mathcal{F} \in \mathcal{F}_L(X), x \in X, \)
\[
\lim_{\tau_2} \varphi^\Rightarrow (\mathcal{F})(\varphi(x)) = \bigwedge_{V \in 2^Y} (N_{\tau_2}^\varphi(V) \to \varphi^\Rightarrow (\mathcal{F})(V)) \geq \bigwedge_{V \in 2^Y} (N_{\tau_1}^\varphi (\varphi^+(V)) \to \mathcal{F}(\varphi^+(V))) \geq \bigwedge_{U \in 2^X} (N_{\tau_1}^\varphi(U) \to \mathcal{F}(U)) = \lim_{\tau_1} \mathcal{F}(x).
\]
Hence, \( f : (X, \lim_{\tau_1}) \to (Y, \lim_{\tau_2}) \) is continuous.

Conversely, if \( f : (X, \lim_{\tau_1}) \to (Y, \lim_{\tau_2}) \) is continuous, by Theorem 3.5, we have \( \forall x \in X, U \in 2^Y, \)
\[
N_{\tau_1}^\varphi (\varphi^+(U)) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_{\tau_1} \mathcal{F}(x) \to \mathcal{F}(\varphi^+(U))) \geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(Y)} (\lim_{\tau_2} (\mathcal{F}(\varphi(x))) \to (\varphi^+(\mathcal{F}))(U)) \geq \bigwedge_{\mathcal{G} \in \mathcal{G}_L(Y)} (\mathcal{G}(\varphi(x)) \to \mathcal{G}(U)) = N_{\tau_2}^\varphi(U).
\]
Therefore, by Theorem 2.5, \( f : (X, \tau_1) \to (Y, \tau_2) \) is continuous. \( \square \)

As a consequence of the above theorems, we have the following result.

**Theorem 4.2.** Let \( L \) be a completely distributive lattice. \( L-\text{FYS} \) can be embedded in the category of \( L-\text{OFYC} \).

In Theorem 3.8 we know that \( L-\text{OFYC} \) is topological. So, in order to show that it is Cartesian-closed, the following results are necessary. Similar to the definition of product spaces in \( L-\text{FYCS} \), it can be shown that there are also product spaces in \( L-\text{OFYC} \). We refer the reader to [16]. Here, we only present the main results. Note that for two \( L-\text{ordered} \) fuzzifying convergence spaces \( (X, \lim_X), (Y, \lim_Y) \), let \( [X \to Y] \) denote the set of all continuous maps from \( (X, \lim_X) \) to \((Y, \lim_Y)\).

**Lemma 4.3.** [16] Let \( g : X \to Y \) and \( \mathcal{G} \in \mathcal{F}_L(X) \), then \( g^\Rightarrow (\mathcal{G}) \leq ev^\Rightarrow ([g] \times \mathcal{G}) \), where \( ev : [X \to Y] \times X \to Y \) is the evaluation map.

**Theorem 4.4.** Let \( (X, \lim_X), (Y, \lim_Y) \) be \( L-\text{ordered} \) fuzzifying convergence spaces, then \( \lim_{[X \to Y]} : F_L([X \to Y]) \to L^{[X \to Y]}, \forall \mathcal{F} \in F_L([X \to Y]), \forall f \in [X \to Y], \lim_{[X \to Y]} (\mathcal{F}(f) = \bigwedge_{(\mathcal{F}, x) \in \mathcal{F}_L(X) \times X} (\lim_X \mathcal{G}(x) \to \lim_Y ev^\Rightarrow (\mathcal{F} \times \mathcal{G})(f(x))) \) is an \( L-\text{ordered} \) fuzzifying convergence structure on \([X \to Y]\).
Proof. For (LY1), \( \forall g \in [X \rightarrow Y] \),

\[
\lim_{[X \rightarrow Y]} [g](g) = \bigwedge_{(g,x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{E}(x) \rightarrow \lim_Y (\mathcal{E}^\Rightarrow ([g] \times \mathcal{E}))(g(x)) \geq \bigwedge_{(g,x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{E}(x) \rightarrow \lim_Y (\mathcal{E}^\Rightarrow (\mathcal{E}))(g(x)) = 1.
\]

For (OLY2), \( \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L([X \rightarrow Y]) \),

\[
S(\lim_{[X \rightarrow Y]} \mathcal{F}, \lim_{[X \rightarrow Y]} \mathcal{G}) = \bigwedge_{g \in [X \rightarrow Y]} \left( \bigwedge_{(\mathcal{E}, x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{E}(x) \rightarrow \lim_Y (\mathcal{E}^\Rightarrow (\mathcal{F} \times \mathcal{E}))(g(x)) \right) = \bigwedge_{(\mathcal{E}, x) \in \mathcal{F}_L(X) \times X} \lim_X \mathcal{E}(x) \rightarrow \lim_Y (\mathcal{E}^\Rightarrow (\mathcal{F} \times \mathcal{E}))(g(x)) \geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \bigwedge_{\mathcal{G} \in \mathcal{F}_L(X)} S(\lim_Y (\mathcal{F} \times \mathcal{E}), \lim_Y (\mathcal{G} \times \mathcal{E})) \geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} S_F (\mathcal{E}^\Rightarrow (\mathcal{F} \times \mathcal{E}), \mathcal{E}^\Rightarrow (\mathcal{G} \times \mathcal{E})) \forall \mathcal{H} \in \mathcal{F}_L(X),
\]

\[
S_F (\mathcal{E}^\Rightarrow (\mathcal{F} \times \mathcal{E}), \mathcal{E}^\Rightarrow (\mathcal{G} \times \mathcal{E})) = \bigwedge_{U \in 2^Y} \left( (\mathcal{F} \times \mathcal{H})(\mathcal{E}^\Rightarrow (U)) \rightarrow (\mathcal{G} \times \mathcal{H})(\mathcal{E}^\Rightarrow (U)) \right) \geq \bigwedge_{A \times B \in \mathcal{E}^\Rightarrow (U)} \left( (\mathcal{F} \times \mathcal{H})(A) \land \mathcal{H}(B) \rightarrow (\mathcal{G} \times \mathcal{H})(A) \land \mathcal{H}(B) \right) \geq \bigwedge_{A \times B \in \mathcal{E}^\Rightarrow (U)} \left( \mathcal{F}(A) \rightarrow \mathcal{G}(A) \right) \geq \bigwedge_{C \in [X \rightarrow Y]} (\mathcal{F}(C) \rightarrow \mathcal{G}(C)) = S_F (\mathcal{F}, \mathcal{G}).
\]

Therefore, the above completes the proof. In other words, \( \lim_{[X \rightarrow Y]} \) is an \( L \)-ordered fuzzifying convergence structure on \( [X \rightarrow Y] \). \( \square \)
Remark 4.5. The evaluation map $ev : [X \rightarrow Y] \times X \rightarrow Y$ mentioned above is continuous. Let $f : X \times Y \rightarrow Z$ be a map, $\forall x \in X$, define a map $f_x : Y \rightarrow Z$, $\forall y \in Y$. $f_x(y) = f(x,y)$, $f^* : X \rightarrow Z^Y, \forall x \in X$, $f^*(x) = f_x$, and $\varphi : Z^{(X \rightarrow Y)} \rightarrow (Z^Y)^X, \forall f \in Z^{(X \rightarrow Y)}$, $\varphi(f) = f^*$. Then it can be proved that

1. If $f : (X, \text{lim}_X) \times (Y, \text{lim}_Y) \rightarrow (Z, \text{lim}_Z)$ is continuous, then for each $x \in X$, $f_x : (Y, \text{lim}_Y) \rightarrow (Z, \text{lim}_Z)$ is continuous.
2. For all $\mathcal{F} \in \mathcal{F}_L(X)$, $\mathcal{G} \in \mathcal{F}_L(Y)$, $ev^\mathcal{F} (\varphi(f))^\mathcal{G} (\mathcal{F} \times \mathcal{G}) = f^\mathcal{G} (\mathcal{F} \times \mathcal{G})$.
3. If $f : X \times Y \rightarrow Z$ is continuous, then $\varphi(f) : X \rightarrow [Y \rightarrow Z]$ is continuous. (We refer to [16] for a detail proof of the above results.)

We collect our findings in the following theorem.

Theorem 4.6. $L$–OFYC is a Cartesian-closed category.

5. The Relations Between $L$–fuzzifying Neighborhood Spaces and Principle $L$–ordered Fuzzifying Convergence Spaces

In this section, we define a subcategory of the category of $L$–ordered fuzzifying convergence spaces: the category of principle $L$–ordered fuzzifying convergence spaces and show that the new category and that of $L$–fuzzifying neighborhood spaces are isomorphic. Furthermore, each fibre on a fixed set of the category of $L$–fuzzifying neighborhood spaces and that of the category of principal $L$–ordered fuzzifying convergence spaces are isomorphic. At the end of the section, we propose that the category of principle $L$–ordered fuzzifying convergence spaces is a reflective subcategory of $L$–OFYC and it is a topological category. Again, this section is mostly motivated by reference [8].

Proposition 5.1. Let $(X, \text{lim}) \in L$–OFYC. The structure $\{N^x_{\text{lim}} : 2^X \rightarrow L\}_{x \in X}$ defined by: For $x \in X$, $\forall U \in 2^X$, $N^x_{\text{lim}}(U) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\text{lim} \mathcal{F}(x) \rightarrow \mathcal{F}(U))$ is an $L$–fuzzifying neighborhood structure. We call it the induced $L$–fuzzifying neighborhood structure of $(X, \text{lim})$.

Theorem 3.5 suggests for $(X, \text{lim}) \in L$–OFYC the following definition.

Definition 5.2. Let $\text{lim}$ be an $L$–ordered fuzzifying convergence structure. If in addition the following condition (LYP) holds,

(LYP) $\forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \text{lim} \mathcal{F}(x) = SF(N^x_{\text{lim}}, \mathcal{F})$,

then $\text{lim}$ is called a principal $L$–ordered fuzzifying convergence structure, and the pair $(X, \text{lim})$ is called a principle $L$–ordered fuzzifying convergence space.

The full subcategory of $L$–OFYC consisting of all principle $L$–ordered fuzzifying convergence spaces is denoted by $L$–POFYC.

If an $L$–ordered fuzzifying convergence spaces satisfies (LYP), then a nice characterization of principle $L$–ordered convergence spaces in terms of $L$–fuzzifying neighborhood spaces is possible. We first need three theorems for preparation.
Theorem 5.3. Let \((X, N)\) be an \(L\)-fuzzifying neighborhood space. Then there exists a principle \(L\)-ordered fuzzifying convergence structure \(\lim\) on \(X\) satisfying 
\[
\forall x \in X, N^x_{\lim} = N^x.
\]

Proof. For the \(L\)-fuzzifying neighborhood space \((X, N)\), define \(\lim_N : \mathcal{F}_L(X) \to L^X\)
\[
\forall \mathcal{F} \in \mathcal{F}_L(X), \quad x \in X, \quad \lim_N \mathcal{F}(x) = \bigwedge_{A \in 2^X} (N^x(A) \to \mathcal{F}(A)) = S_F(N^x, \mathcal{F}).
\]

It is then readily checked that for \((X, \lim_N)\), the axiom (LY1), (OLY2), (LYP) hold. The properties of the residual implication of Theorem 2.1 are used.

(OLY2): In fact,
\[
S(\lim_N \mathcal{F}, \lim_N \mathcal{G}) = \bigwedge_{x \in X} (S_F(N^x, \mathcal{F}) \to S_F(N^x, \mathcal{G}))
\]
\[
= \bigwedge_{x \in X} \left( \bigwedge_{A \in 2^X} (N^x(A) \to \mathcal{F}(A)) \to \bigwedge_{B \in 2^X} (N^x(B) \to \mathcal{G}(B)) \right)
\]
\[
= \bigwedge_{x \in X} \left( \bigwedge_{A \in 2^X} \left( \bigwedge_{B \in 2^X} (N^x(B) \to \mathcal{F}(B)) \to (N^x(B) \to \mathcal{G}(B)) \right) \to \mathcal{F}(A) \to \mathcal{G}(A) \right)
\]
\[
\leq \bigwedge_{x \in X} \left( \bigwedge_{A \in 2^X} (N^x(A) \to \mathcal{F}(A)) \right)
\]
\[
= S_F(\mathcal{F}, \mathcal{G}).
\]

(LYP): For all \(\mathcal{F} \in \mathcal{F}_L(X)\), we prove \(\lim_N \mathcal{F}(x) = S_F(N^x_{\lim_N}, \mathcal{F})\). By the definition of \(\lim_N \), \(\lim_N \mathcal{F}(x) = S_F(N^x, \mathcal{F})\). It remains to verify that \(N^x_{\lim_N} = N^x\).

On one hand, for all \(A \in 2^X\)
\[
N^x_{\lim_N}(A) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_N \mathcal{F}(x) \to \mathcal{F}(A))
\]
\[
\leq \lim_N N^x(A) \to N^x(A)
\]
\[
= N^x(A).
\]

On the other hand,
\[
N^x_{\lim_N}(A) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (\lim_N \mathcal{F}(x) \to \mathcal{F}(A))
\]
\[
= \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \left( \bigwedge_{B \in 2^X} (N^x(B) \to \mathcal{F}(B)) \to \mathcal{F}(A) \right)
\]
\[
\geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} (N^x(A) \to \mathcal{F}(A) \to \mathcal{F}(A))
\]
\[
\geq N^x(A).
\]
From this, the result follows by a standard argument. □

In view of the above theorem, if \( N \) is an \( L \)-fuzzifying neighborhood structure, then there exists a principle \( L \)-ordered fuzzifying convergence structure \( \lim N \) on \( X \). Moreover, \( N_{\lim N} \) is also an \( L \)-fuzzifying neighborhood structure and \( N_{\lim N} = N \). Conversely, we have the following theorem.

**Theorem 5.4.** If \( \lim \) is a principle \( L \)-ordered fuzzifying convergence structure on \( X \), then \( \lim N_{\lim} = \lim \).

**Proof.** For all \( \mathcal{F} \in F_L(X) \), \( x \in X \), by (LYP), we have,

\[
\lim_{N_{\lim}} \mathcal{F}(x) = \bigwedge_{A \in 2^X} \left( N_{\lim}(A) \rightarrow \mathcal{F}(A) \right) = S_F(N_{\lim}, \mathcal{F}) = \lim \mathcal{F}(x). \]

With respect to Theorem 5.3 and Theorem 5.4, we have a one-one correspondence between the objects of \( L-\text{NGH} \) and \( L-\text{POFYC} \). The following theorem is about the relation between morphisms of them.

**Theorem 5.5.** Let \((X, \lim^X),(Y, \lim^Y)\) be principle \( L \)-ordered fuzzifying convergence spaces, \((X, N_1), (Y, N_2)\) be \( L \)-fuzzifying neighborhood spaces, then we have

1. If \( f : (X, \lim^X) \rightarrow (Y, \lim^Y) \) is continuous, then \( f : (X, N_{\lim}) \rightarrow (Y, N_{\lim}) \) is also continuous;
2. If \( f : (X, N_1) \rightarrow (Y, N_2) \) is continuous, then \( f : (X, \lim^X) \rightarrow (Y, \lim^Y) \) is also continuous.

**Proof.** (1) By the fact that \( f : (X, \lim^X) \rightarrow (Y, \lim^Y) \) is continuous, we have \( \forall x \in X, U \in 2^Y \),

\[
N_{\lim^X}^x(f^+(U)) = \bigwedge_{\mathcal{F} \in F_L(X)} \left( \lim^X \mathcal{F}(x) \rightarrow \mathcal{F}(f^+(U)) \right)
\]

\[
\geq \bigwedge_{\mathcal{F} \in F_L(X)} \left( \lim^Y f^+(\mathcal{F})(f(x)) \rightarrow f^+(\mathcal{F})(U) \right)
\]

\[
\geq \bigwedge_{\mathcal{F} \in F_L(Y)} \left( \lim^Y \mathcal{F}(f(x)) \rightarrow \mathcal{F}(U) \right)
\]

\[
= N_{\lim^Y}^x(U),
\]

as desired.

(2) Conversely, by the fact that \( f : (X, N_1) \rightarrow (Y, N_2) \) is continuous, we have \( \forall \mathcal{F} \in F_L(X), x \in X \),

\[
\lim_{N_{\lim^Y}}^x \mathcal{F}(f(x)) = \bigwedge_{B \in 2^Y} \left( N_2^f(x)(B) \rightarrow f^+(\mathcal{F})(B) \right)
\]

\[
\geq \bigwedge_{B \in 2^Y} \left( N_1^f(f^+(B)) \rightarrow f^+(\mathcal{F})(B) \right)
\]

\[
\geq \bigwedge_{A \in 2^X} \left( N_1^x(A) \rightarrow \mathcal{F}(A) \right)
\]

\[
= \lim_{N_{\lim^X}}^x \mathcal{F}(x),
\]
as desired.

By Theorems 5.3, 5.4 and 5.5, we actually have proved the following comprehensive theorem.

**Theorem 5.6.** \(L-\text{NGH} \) is isomorphic to \(L-\text{POFYC} \).

Let \( X \) be a set. A fibre on \( X \) of the category of \( L-\) fuzzifying neighborhood spaces is denoted by \( \text{PrFN}_L(X) \). An order \( \leq \) on \( \text{PrFN}_L(X) \) can be defined by

\[
N^1 \leq N^2 \iff \forall x \in X, A \in 2^X, N^1(A) \leq N^2(A).
\]

A fibre on \( X \) of the category of principle \( L-\) ordered fuzzifying convergence spaces is denoted by \( \text{PFYC}_L(X) \). An order \( \leq \) on \( \text{PFYC}_L(X) \) can be defined as follows:

\[
\lim_1 \leq \lim_2 \iff \forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x).
\]

Then we have the following result.

**Theorem 5.7.** \((\text{PFYC}_L(X), \leq)\) and \((\text{PrFN}_L(X), \leq)\) are isomorphic.

**Proof.** Define a mapping: \( h : \text{PFYC}_L(X) \to \text{PrFN}_L(X) \), \( \forall \lim \in \text{PFYC}_L(X) \), \( h(\lim) = N_{\lim} \), and a mapping: \( k : \text{PrFN}_L(X) \to \text{PFYC}_L(X) \), \( \forall N \in \text{PrFN}_L(X) \), \( k(N) = \lim_N \). It has been verified in Theorems 5.3 and 5.4 that \( h \circ k = id_{\text{PrFN}_L(X)} \), \( k \circ h = id_{\text{PFYC}_L(X)} \). So \( h \) and \( k \) are both bijective. Furthermore, \( k = h^{-1} \).

1. For all \( \lim_1, \lim_2 \in \text{PFYC}_L(X) \), if \( \lim_1 \leq \lim_2 \), then for all \( x \in X, A \in 2^X \),

\[
N^\lim_1(A) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \left( \lim_1 \mathcal{F}(x) \to \mathcal{F}(A) \right)
\]

\[
\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \left( \lim_2 \mathcal{F}(x) \to \mathcal{F}(A) \right)
\]

\[
= N^\lim_2(A).
\]

So, \( N_{\lim_1} \leq N_{\lim_2} \), i.e. \( h(\lim_1) \leq h(\lim_2) \). Therefore, \( h \) is an order preserving map.

2. For all \( N_1, N_2 \in \text{PrFN}_L(X) \), if \( N_1 \leq N_2 \), then for all \( \mathcal{F} \in \mathcal{F}_L(X), x \in X \),

\[
\lim_{N_1} \mathcal{F}(x) = \bigwedge_{A \in 2^X} \left( N^\lim_1(A) \to \mathcal{F}(A) \right)
\]

\[
\geq \bigwedge_{A \in 2^X} \left( N^\lim_2(A) \to \mathcal{F}(A) \right)
\]

\[
= \lim_{N_2} \mathcal{F}(x).
\]

Hence, \( \lim_{N_1} \leq \lim_{N_2} \), i.e. \( h^{-1}(N_1) \leq h^{-1}(N_2) \). So \( h^{-1} \) is also an order preserving mapping.

From the above proof, we conclude that \((\text{PFYC}_L(X), \leq)\) and \((\text{PrFN}_L(X), \leq)\) are isomorphic.

At the end of this section, we propose the following results.
Theorem 5.8. The category $L$–POFYC is a reflective subcategory of $L$–OFYC.

Proof. Let $(X, \lim) \in L$–OFYC and $E_{\lim} = \{ \lim \mid (X, \lim) \in L$–POFYC, $\lim \leq \lim \}$. Note that $E_{\lim}$ is not empty because it always contains $\lim_{sm}$. Then we can construct a principal $L$–ordered fuzzifying convergence structure $\lim^* : F_L(X) \to L^X$ as follows: For all $F \in F_L(X)$, $x \in X$, $\lim^* F(x) = \bigwedge_{\lim \in E_{\lim}} \lim F(x)$. From this, we have

1. $id_X : (X, \lim) \to (X, \lim^*)$ is trivially continuous;
2. For a principal $L$–ordered fuzzifying convergence space $(Y, \lim^Y)$, if $f : (X, \lim) \to (Y, \lim^Y)$ is a continuous mapping, then $f : (X, \lim^*) \to (Y, \lim^Y)$ is also continuous.

From the above facts, we immediately obtain that $L$–POFYC is a full reflective subcategory in $L$–OFYC. □

Corollary 5.9. The category $L$–POFYC is topological.

6. The Relations Between $L$–fuzzifying Topological Spaces and Topological $L$–ordered Fuzzifying Convergence Spaces

In this section, we define another important subcategory of $L$–OFYC: the category of topological $L$–ordered fuzzifying convergence spaces. We will find out that the category mentioned above is isomorphic to $L$–FYS and to $L$–TNGH in case of a completely distributive lattice $L$. Furthermore, each fibre on $X$ of the category of topological $L$–ordered fuzzifying convergence spaces is isomorphic to that of $L$–fuzzifying topological spaces and that of topological $L$–fuzzifying neighborhood spaces.

Definition 6.1. Let $(X, \lim) \in L$–POFYC, if in addition the mapping $\lim : F_L(X) \to L^X$ satisfies the following axiom:

$$(LYT) \forall U \in 2^X, \quad N^x_{\lim}(U) \leq \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N^y_{\lim}(V),$$

then $\lim$ is called a topological $L$–ordered fuzzifying convergence structure, and $(X, \lim)$ is a topological $L$–ordered fuzzifying convergence space. The full subcategory of $L$–OFYC consisting of all topological $L$–ordered fuzzifying convergence spaces is denoted by $L$–TOFYC.

If $\lim$ is a topological $L$–ordered fuzzifying convergence structure, then a nice characterization of $L$–fuzzifying topologies is possible. We need two lemmas for preparation.

Lemma 6.2. Let $(X, N)$ be a topological $L$–fuzzifying neighborhood space, then $(X, \lim_N)$ is a topological $L$–ordered fuzzifying convergence space.

Proof. As for $(X, N)$ is a topological $L$–fuzzifying neighborhood space, $(X, N)$ is an $L$–fuzzifying neighborhood space. By Theorem 5.3, we see $N = N_{\lim_N}$. For $N$ is a topological $L$–fuzzifying neighborhood structure, we know for all $x \in \ldots$
Lemma 6.3. Let \((X, \lim) \in L^{-\text{TOFYC}}\), then there exists an \(L\)-fuzzifying topology \(\tau\) on \(X\) such that \(\lim \tau = \lim\).

Proof. Firstly, by Definition 6.1, \((X, \lim) \in L^{-\text{TOFYC}}\) implies that \(N_{\lim}\) satisfies (N1) – (N4).

Secondly, let \(\tau : 2^X \rightarrow L, \forall A \in 2^X, \tau(A) = \bigwedge_{x \in A} N_{\lim}^x(A)\). It can be easily proved that \(\tau\) is an \(L\)-fuzzifying topology on \(X\). Moreover, \(N_{\tau} = N_{\lim}\). In fact, for all \(x \in X, A \in 2^X\), we have by the axiom (LYT),

\[
N_{\tau}^x(A) = \bigvee_{x \in B \subseteq A} \tau(B) = \bigvee_{x \in B \subseteq A} \bigwedge_{y \in B} N_{\lim}^y(B) = N_{\lim}^x(A). \quad \text{(LYT)}
\]

With this and (LMP), we obtain for all \(F \in \mathcal{F}_L(X), x \in X, \lim F(x) = S_F(N_{\tau}^x, F) = S_F(N_{\lim}^x, F) = \lim F(x)\).

Therefore, \(\lim \tau = \lim\). □

Lemmas 6.2, 6.3 together with the relations between topological \(L\)-fuzzifying neighborhood spaces and \(L\)-fuzzifying topological spaces in case that \(L\) is a completely distributive lattice show the following result.

Theorem 6.4. If \(L\) is a completely distributive lattice, then \(L^{-\text{FYS}}, L^{-\text{TOFYC}}\) and \(L^{-\text{TNGH}}\) are isomorphic to each other.

We denote a fibre on \(X\) of the category of \(L\)-fuzzifying topological spaces by \(\mathbf{FY}_L(X)\), and an order “\(\leq\)” on it is defined as follows: \(\forall \tau_1, \tau_2 \in \mathbf{FY}_L(X), \tau_1 \leq \tau_2 \Leftrightarrow \forall A \in 2^X, \tau_1(A) \leq \tau_2(A)\).

Denote a fibre on \(X\) of the category of \(L\)-fuzzifying neighborhood spaces by \(\mathbf{FN}_L(X)\) and a fibre on \(X\) of the category of \(L\)-ordered fuzzifying convergence spaces by \(\mathbf{TFY}_C(X)\). In the same way as in the proof of Theorem 5.7, we obtain the following theorem trivially, and leave the straightforward proof for the interested reader.

Theorem 6.5. \((\mathbf{FN}_L(X), \leq), (\mathbf{FY}_L(X), \leq), (\mathbf{TFY}_C(X), \leq)\) are isomorphic.

References


WENCHAO WU*, DEPARTMENT OF MATHEMATICS, OCEAN UNIVERSITY OF CHINA, 266100 QINGDAO, P. R. CHINA
E-mail address: wuwenchao107@163.com

JINMING FANG, DEPARTMENT OF MATHEMATICS, OCEAN UNIVERSITY OF CHINA, 266100 QINGDAO, P. R. CHINA
E-mail address: jinming-fang2163.com

*CORRESPONDING AUTHOR