SINGULAR VALUE INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES

L. ZOU* AND Y. JIANG

(Communicated by Behzad Djafari-Rouhani)

Abstract. In this note, we obtain some singular values inequalities for positive semidefinite matrices by using block matrix technique. Our results are similar to some inequalities shown by Bhatia and Kittaneh in [Linear Algebra Appl. 308 (2000) 203-211] and [Linear Algebra Appl. 428 (2008) 2177-2191].

Keywords: Singular values, Positive semidefinite matrices, Block matrix technique.

MSC(2010): Primary: 15A60; Secondary: 15A18, 15A42.

1. Introduction

Let $M_n$ be the space of $n \times n$ complex matrices. Let $\| \|$ denote any unitarily invariant norm on $M_n$. We shall always denote the singular values of $A$ by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. Let $A, B \in M_n$ be Hermitian, the order relation $A \geq B$ means, as usual, that $A - B$ is positive semidefinite. We use the direct sum notation $A \oplus B$ for the block-diagonal operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $M_n \oplus M_n$.

Bhatia and Kittaneh ([5], p.206) proved that if $A, B \in M_n$ are positive semidefinite, then

$$2s_j \left( A^{1/2} (A + B) B^{1/2} \right) \leq s_j \left( (A + B)^2 \right)$$

Received: 17 April 2013, Accepted: 4 May 2013.

*Corresponding author.
for all $j = 1, \ldots, n$. Bhatia and Kittaneh ([6], p.2186) generalized the inequality (1.1) to the following form:

\begin{equation}
2s_j \left( A^{1/2}(A + B)^r B^{1/2} \right) \leq s_j \left( (A + B)^{r+1} \right), \ r \geq 0
\end{equation}

for all $j = 1, \ldots, n$.

In Section 2, we first present an inequality for singular values, which is similar to the inequalities (1.2). After that, we generalize the inequality (1.1), which is a special case of the inequality (1.2). Section 3 contains some remarks.

2. Main results

In this section, we first present an inequality, which is similar to the inequality (1.2). To do this, we need the following lemmas.

**Lemma 2.1.** ([2], Theorem 1) Let $f(t)$ be an operator monotone function and $A, B \in M_n$ be positive semidefinite. Then

\[
\left( \frac{A + B}{2} \right)^{1/2} (f(A) + f(B)) \left( \frac{A + B}{2} \right)^{1/2} \leq A f(A) + B f(B).
\]

**Lemma 2.2.** ([8], Theorem 1) Let $A, B, X \in M_n$ such that

\[
\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \succeq 0.
\]

Then

\[s_j(X) \leq \frac{1}{2} s_j \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}\]

for all $j = 1, \ldots, n$.

**Theorem 2.3.** Let $A, B \in M_n$ be positive semidefinite and suppose that

\[
K = \left( A^{1/(q+1)} + B^{1/(q+1)} \right)^{1/2}, \ 0 \leq q \leq 1.
\]

Then, we have

\[s_j \left( A^{q/2(q+1)} K (A + B)^r K B^{q/2(q+1)} \right) \leq s_j \left( (A + B)^{1+r} \right), \ r \geq 0
\]

for all $j = 1, \ldots, n$.

**Proof.** It is known that the function $f(t) = t^q$, $0 \leq q \leq 1$ is operator monotone on $(0, \infty)$. So, by Lemma 2.1, we have

\[
\frac{1}{2} (A + B)^{1/2} (A^q + B^q) (A + B)^{1/2} \leq A^{1+q} + B^{1+q}.
\]
It is known that if \( X \leq Y \), then \( ZXZ^* \leq ZY \). It follows that
\[
\frac{1}{2} (A^{1+q} + B^{1+q})^{r/2} L (A^q + B^q) L (A^{1+q} + B^{1+q})^{r/2} \leq (A^{1+q} + B^{1+q})^{1+r},
\]
where \( L = (A + B)^{1/2} \). Note that the matrices \( XY \) and \( YX \) have the same eigenvalues. Thus, by this last inequality, we obtain
\[
(2.1) \quad \frac{1}{2} \lambda_j \left( (A^q + B^q) L (A^{1+q} + B^{1+q})^r L \right) \leq \lambda_j \left( (A^{1+q} + B^{1+q})^{1+r} \right).
\]
Let
\[
X = (A + B)^{1/2} (A^{1+q} + B^{1+q})^{r/2}.
\]
The inequality (2.1) is equivalent to
\[
(2.2) \quad \frac{1}{2} \lambda_j \left( (A^q + B^q) XX^* \right) \leq \lambda_j \left( (A^{1+q} + B^{1+q})^{1+r} \right).
\]
Except for trivial zeros, the eigenvalues of \((A^q + B^q) XX^*\) are the same as the following matrix
\[
\begin{bmatrix}
A^{q/2} & B^{q/2} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A^{q/2} & 0 \\
0 & B^{q/2}
\end{bmatrix}
\begin{bmatrix}
XX^* & 0 \\
0 & 0
\end{bmatrix},
\]
and in turn, these are the same as the eigenvalues of
\[
\begin{bmatrix}
A^{q/2} & 0 \\
0 & B^{q/2}
\end{bmatrix}
\begin{bmatrix}
XX^* & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A^{q/2} & B^{q/2} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A^{q/2} & 0 \\
0 & B^{q/2}
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A^{q/2} & B^{q/2} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
XX^* & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A^{q/2} & 0 \\
0 & B^{q/2}
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\geq 0.
\]
So, by Lemma 2.2 and the inequality (2.2), we have
\[
s_j \left( A^{q/2} XX^* B^{q/2} \right) \leq \lambda_j \left( (A^{1+q} + B^{1+q})^{1+r} \right).
\]
That is,
\[
s_j \left( A^{q/2(q+1)} K (A + B)^{r} KB^{q/2(q+1)} \right) \leq s_j \left( (A + B)^{1+r} \right)
\]
on replacing \( A \) by \( A^{1/(q+1)} \) and \( B \) by \( B^{1/(q+1)} \). This completes the proof. \( \square \)

Next, we generalize the inequality (1.1).
Theorem 2.4. Let $A, B \in M_n$ be positive semidefinite and suppose that
\[ K = \left(A^{1/(q+1)} + B^{1/(q+1)}\right)^{1/2}, \quad 0 \leq q \leq 1. \]

Then, we have
\[ s_j \left( A^{1/2} K \left( A^{q/(1+q)} + B^{q/(1+q)} \right) K B^{1/2} \right) \leq s_j \left( (A + B)^2 \right) \]
for all $j = 1, \ldots, n$.

Proof. By the inequality (2.1) with $r = 1$, we have
\[ \frac{1}{2} \lambda_j \left( (A^q + B^q) (A + B)^{1/2} (A^{1+q} + B^{1+q}) (A + B)^{1/2} \right) \leq \lambda_j \left( \left( A^{1+q} + B^{1+q} \right)^2 \right). \]

Let
\[ X = (A + B)^{1/2} A^{(1+q)/2}, \quad Y = (A + B)^{1/2} B^{(1+q)/2}. \]

Then
\[ \frac{1}{2} \lambda_j \left( (A^q + B^q) (XX^* + YY^*) \right) \leq \lambda_j \left( \left( A^{1+q} + B^{1+q} \right)^2 \right). \]

Except for trivial zeros, the eigenvalues of $(A^q + B^q) (XX^* + YY^*)$ are the same as those of
\[ M = \begin{bmatrix} A^q + B^q & 0 & X Y^* & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

Meanwhile, we know that the eigenvalues of $M$ are the same as those of
\[ \begin{bmatrix} X^* & 0 & X Y \end{bmatrix} \begin{bmatrix} A^q + B^q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \end{bmatrix}, \]
\[ \begin{bmatrix} Y^* & 0 & X Y^* \end{bmatrix} \begin{bmatrix} A^q + B^q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y \end{bmatrix}. \]

So, by Lemma 2.2, we have
\[ s_j \left( (A^q + B^q) (XX^* + YY^*) \right) \leq \frac{1}{2} \lambda_j \left( (A^q + B^q) (XX^* + YY^*) \right). \]

It follows from (2.3) and (2.4) that
\[ s_j \left( A^{(1+q)/2} (A + B)^{1/2} (A^q + B^q) (A + B)^{1/2} B^{(1+q)/2} \right) \leq \lambda_j \left( \left( A^{1+q} + B^{1+q} \right)^2 \right), \]
which is equivalent to
\[ s_j \left( A^{1/2} K \left( A^{q/(1+q)} + B^{q/(1+q)} \right) K B^{1/2} \right) \leq s_j (A + B)^2 \]
for all \( j = 1, \ldots, n \). This completes the proof. \( \square \)

3. Remarks

**Remark 3.1.** Putting \( q = 0 \) in Theorem 2.3, we obtain the inequality (1.1). Putting \( q = 1 \) in Theorem 2.4, we have
\[ s_j \left( A^{1/2} \left( A^{1/2} + B^{1/2} \right)^2 B^{1/2} \right) \leq s_j (A + B)^2, \quad j = 1, \ldots, n. \]
For more information on singular value inequalities for positive semidefinite matrices the reader is referred to \([3, 4, 10]\).

**Remark 3.2.** Let \( A, X, B \in M_n \) such that \( A \) and \( B \) are positive semi-definite. The inequality (1.1) is equivalent to
\[ 2s_j \left( A^{3/2} B^{1/2} + A^{1/2} B^{3/2} \right) \leq s_j (A + B)^2, \quad j = 1, \ldots, n. \]
Recently, Drury ([7], Theorem 1) proved that
\[ 4s_j (AB) \leq s_j ((A + B)^2), \quad j = 1, \ldots, n, \]
which is a question posed by Bhatia and Kittaneh ([5], p.204). Having the inequalities (3.1) and (3.2), it is natural to raise the following question: For \( \frac{1}{2} \leq v \leq \frac{3}{2} \), is it true that
\[ 2s_j \left( A^v B^{2-v} + A^{2-v} B^v \right) \leq s_j (A + B)^2, \quad j = 1, \ldots, n? \]
An inequality weaker than (3.3) is
\[ \| A^v B^{2-v} + A^{2-v} B^v \| \leq \frac{1}{2} \| (A + B)^2 \|. \]
This is true. In fact, it is known ([3], p.265) that the function
\[ g(r) = \| A^r B^{1-r} + A^{1-r} B^r \| \]
is convex on \([0, 1]\). Replacing \( A \) by \( A^2 \), \( B \) by \( B^2 \), and \( 2r \) by \( v \), we know that the function
\[ f(v) = \| A^v B^{2-v} + A^{2-v} B^v \| \]
is convex on \([0, 2]\). It follows that this function is also convex on \([\frac{1}{2}, \frac{3}{2}]\).
So, by the convexity of the function \( f(v) \), we obtain the inequality (3.4).
Zhan ([9], Theorem 6) has proved that if $\frac{1}{2} \leq v \leq \frac{3}{2}$ and $-2 < t \leq 2$, then

$$\|A^v X B^{2-v} + A^{2-v} X B^v\| \leq \frac{2}{t+2} \|A^2 X + t A X B + X B^2\|$$

The special case $X = I, t = 2$

$$\|A^v B^{2-v} + A^{2-v} B^v\| \leq \frac{1}{2} \|A^2 + 2 A B + B^2\|$$

is an inequality weaker than (3.4).

**Remark 3.3.** Let $A, B \in M_n$ be positive semidefinite and suppose that $0 \leq v \leq 2$. An inequality weaker than (3.3) is

(3.5) \[ s_j \left( A^v B^{2-v} + A^{2-v} B^v \right) \leq s_j \left( A^2 + B^2 \right), \quad j = 1, \ldots, n. \]

This is a conjecture by Zhan ([10], Conjecture 3.4), settled in the affirmative by Audenaert ([2], Theorem 2). Bhatia and Kittaneh ([6], p.2182) stated that the following inequalities

(3.6) \[ s_j \left( A^v B^{2-v} + B^{2-v} A^v \right) \leq s_j \left( A^2 + B^2 \right), \quad j = 1, \ldots, n, \]

(3.7) \[ s_j \left( A^v B^{2-v} + B^v A^{2-v} \right) \leq s_j \left( A^2 + B^2 \right), \quad j = 1, \ldots, n, \]

are not always true. The inequalities (3.6) and (3.7) are similar to the inequality (3.5). The inequality (3.6) is not always true even for scalars $a$ and $b$ unless $v = 1$. By Theorem 2.7 of [1], we have

$$s_j \left( A^v B^{2-v} + B^{2-v} A^v \right) \leq \frac{1}{7v^2} \left( (A^v + B^{2-v}) \oplus (A^v - B^{2-v})^2 \right) \leq s_j \left( (A^{2v} + B^{1-2v}) \oplus (A^{2v} + B^{4-2v}) \right)$$

for all $j = 1, \ldots, n$. This inequality is similar to the inequality (3.6) and has been obtained by Bhatia and Kittaneh ([6], Proposition 6.2). For $v = 1$, the inequality (3.7) is

$$s_j \left( A B + B A \right) \leq s_j \left( A^2 + B^2 \right), \quad j = 1, \ldots, n.$$

This is not possible by the example

$$A = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In fact, we have

$$s_2 \left( A B + B A \right) = 4.1421 \geq 1.9600 = s_2 \left( A^2 + B^2 \right).$$

Meanwhile, we also know that the inequality (3.7) does not hold for $v = 0.5$ and $v = 1.5$ with the same matrices above. On the other hand, if $A B = B A$, then by the the inequality (3.5), we know that the
inequality (3.7) holds for all $0 \leq v \leq 2$.

**Remark 3.4.** Let $A, B$ be positive semidefinite and $0 \leq v \leq 2$. Here, we give an inequality for singular values, which is similar to the inequality (3.7). It is known that

$$\begin{bmatrix}
A^{2v} & A^v B^{2-v} \\
B^{2-v} A^v & B^{4-2v}
\end{bmatrix} = \begin{bmatrix}
A^v & 0 \\
0 & B^{2-v}
\end{bmatrix} \begin{bmatrix}
A^v & B^{2-v} \\
0 & 0
\end{bmatrix} \geq 0$$

and

$$\begin{bmatrix}
B^{2v} & B^v A^{2-v} \\
A^{2-v} B^v & A^{4-2v}
\end{bmatrix} = \begin{bmatrix}
B^v & 0 \\
0 & A^{2-v}
\end{bmatrix} \begin{bmatrix}
B^v & A^{2-v} \\
0 & 0
\end{bmatrix} \geq 0.$$ 

It follows from (3.8) and (3.9) that

$$\begin{bmatrix}
A^{2v} + B^{2v} & A^v B^{2-v} + B^v A^{2-v} \\
B^{2-v} A^v + A^{2-v} B^v & A^{4-2v} + B^{4-2v}
\end{bmatrix} \geq 0.$$ 

So, by Theorem 2.1 of [1], we have

$$s_j \left( A^v B^{2-v} + B^v A^{2-v} \right) \leq s_j \left( (A^{2v} + B^{2v}) \oplus (A^{4-2v} + B^{4-2v}) \right)$$

for all $j = 1, \ldots, n$.

**Acknowledgments**

The authors wish to express their heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript. This research was supported by the Fundamental Research Funds for the Central Universities (No. SEUjc2013014).

**References**


(Limin Zou) School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, P.R. China

E-mail address: limin-zou@163.com

(youyi Jiang) School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, P.R. China

E-mail address: yyy.j123456@163.com