ON REVERSE DEGREE DISTANCE OF UNICYCLIC GRAPHS

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ABSTRACT. The reverse degree distance of a connected graph $G$ is defined in discrete mathematical chemistry as

$$rD'(G) = 2(n - 1)m - \sum_{u \in V(G)} d_G(u)D_G(u),$$

where $n$, $m$ and $d$ are the number of vertices, the number of edges and the diameter of $G$, respectively, $d_G(u)$ is the degree of vertex $u$, $D_G(u)$ is the sum of distances between vertex $u$ and all other vertices of $G$, and $V(G)$ is the vertex set of $G$. We determine the unicyclic graphs of given girth, number of pendant vertices and maximum degree, respectively, with maximum reverse degree distances. We also determine the unicyclic graphs of given number of vertices, girth and diameter with minimum degree distance.

1. Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, let $d_G(u)$ be the degree of $u$ in $G$ and $D_G(u)$ be the sum of the distances between $u$ and all other vertices of $G$. Obviously, $D_G(u) = \sum_{v \in V(G)} d_G(u, v)$, where $d_G(u, v)$ is the distance between the

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vertices \( u \) and \( v \) in \( G \) for \( u, v \in V(G) \). The degree distance of \( G \) is defined as \([6, 10, 11]\)

\[
D'(G) = \sum_{u \in V(G)} d_G(u)D_G(u).
\]

It is a useful molecular descriptor \([21]\). Earlier as noted in \([15, 19]\), this graph invariant appeared to be part of the molecular topological index (or Schultz index) \([18]\), which may be expressed as \(D'(G) + \sum_{u \in V(G)} d_G(u)^2\), see \([11, 14, 16, 26]\), where the latter part \(\sum_{u \in V(G)} d_G(u)^2\) is known as the first Zagreb index \([12, 13, 17]\). Thus the degree distance is also called the true Schultz index in chemical literature \([4]\).

I. Tomescu \([23]\) showed that the star is the unique graph with minimum degree distance in the class of connected graphs with \( n \) vertices. Further work on the minimum degree distance (especially for unicyclic and bicyclic graphs) may be found in A.I. Tomescu \([22]\), I. Tomescu \([24]\) and Bucicovschi and Cioabă \([2]\). Dankelmann et al. \([3]\) gave asymptotically sharp upper bounds for the degree distance. Among others, the authors \([8]\) studied the ordering of unicyclic graphs with large degree distances, and bicyclic graphs were also considered in \([9]\).

Recall that the Wiener index \([25]\) of the graph \( G \) is defined as

\[
W(G) = \frac{1}{2} \sum_{u \in V(G)} D_G(u).
\]

Gutman \([11]\) showed that if \( G \) is a tree with \( n \) vertices, then

\[
D'(G) = 4W(G) - n(n - 1).
\]

Thus there is no need to study the degree distance for trees because this is equivalent to the study of the Wiener index, see, e.g., \([5, 20]\).

The reverse degree distance of the graph \( G \) is defined as \([27]\)

\[
rD'(G) = 2(n - 1)md - D'(G)
\]

where \( n, m \) and \( d \) are the number of vertices, the number of edges and the diameter of \( G \), respectively. Some basic properties of the reverse degree distance have been established by Zhou and Trinajstić \([27]\), and in particular, it was shown that the reverse degree distance satisfies the basic requirement to be a branching index usable in chemistry.

Recall that, earlier, Balaban et al. \([1]\) introduced the concept of reverse Wiener index, which is defined to be \( \Lambda(G) = \frac{n(n-1)d}{2} - W(G) \). Let
\( \Lambda'(G) = \frac{(n-1)^2d}{2} - W(G) \), which is a revised version of the reverse Wiener index of \( G \) [27]. If \( G \) is a tree, then from the result of Gutman [11] mentioned above, we have

\[ D'(G) = 4\Lambda'(G) + n(n - 1). \]

Thus for trees the study of the reverse degree distance is equivalent to the study of the revised reverse Wiener index, see [27].

In continuation to the study of the reverse degree distance, a natural starting point is the reverse degree distances of unicyclic graphs. In this paper, we determine the graphs with maximum reverse degree distance in the class of unicyclic graphs (connected graphs with a unique cycle) with given girth (cycle length), number of pendant vertices (vertices of degree one), and maximum degree, respectively. Additionally, we also determine the graphs with minimum degree distance in the class of unicyclic graphs with given number of vertices, girth and diameter.

2. Preliminaries

In this section we give some lemmas that will be used in the next sections.

**Lemma 2.1.** Let \( G \) be a graph of the form in Fig. 1, where \( M \) and \( N \) are vertex-disjoint connected graphs, \( T \) is a tree on \( k \geq 2 \) vertices such that \( M \) and \( T \) have only one common vertex \( u \), and \( T \) and \( N \) have only one common vertex \( v \). Let \( G^* \) be the graph obtained from \( M \) and \( N \) by identifying vertices \( u \) and \( v \) which is denoted by \( u, \) and attaching \( k - 1 \) pendant vertices to \( u. \)

(i) If \( V(N) = \{v\} \) and \( G \not\cong G^* \), then \( D'(G) > D'(G^*). \)

(ii) If \( |V(M)|, |V(N)| \geq 3 \), then \( D'(G) > D'(G^*). \)

![Fig. 1. The graphs G and G* in Lemma 2.1.](image-url)
Proof. For vertex-disjoint connected graphs $Q_1$ and $Q_2$ with $|V(Q_1)|$, $|V(Q_2)| \geq 2$, and $s \in V(Q_1)$, $t \in V(Q_2)$, let $H$ be the graph obtained from $Q_1$ and $Q_2$ by joining $s$ and $t$ by an edge, and $H_1$ be the graph obtained by identifying vertices $s$ and $t$ which is denoted by $s$, and attaching a pendant vertex $w$ to $s$.

Let $d_x = d_H(x)$ for $x \in V(H)$. It is easily seen that

\[
(d_s + d_t - 1)D_{H_1}(s) + d_sD_H(s) - d_tD_H(t) \\
= d_s[D_{H_1}(s) - D_H(s)] + d_t[D_{H_1}(s) - D_H(t)] + [D_{H_1}(w) - D_{H_1}(s)] \\
= -d_s(|V(Q_2)| - 1) - d_t(|V(Q_1)| - 1) + (|V(Q_1)| + |V(Q_2)| - 2) \\
= -(d_s - 1)(|V(Q_2)| - 1) - (d_t - 1)(|V(Q_1)| - 1).
\]

Then

\[
D'(H_1) - D'(H) \\
= -(|V(Q_2)| - 1) \sum_{x \in V(Q_1) \backslash \{s\}} d_x - (|V(Q_1)| - 1) \sum_{x \in V(Q_2) \backslash \{t\}} d_x \\
+ (d_s + d_t - 1)D_{H_1}(s) + 1 \cdot D_{H_1}(w) - d_tD_H(t) \\
= -(|V(Q_2)| - 1) \sum_{x \in V(Q_1) \backslash \{s\}} d_x - (|V(Q_1)| - 1) \sum_{x \in V(Q_2) \backslash \{t\}} d_x \\
-(d_s - 1)(|V(Q_2)| - 1) - (d_t - 1)(|V(Q_1)| - 1) < 0,
\]

and thus $D'(H_1) < D'(H)$.

Now (i) and (ii) follow by applying to $G$ the transformation from $H$ to $H_1$ repeatedly. \(\square\)

Lemma 2.2. Let $G_0$ be a connected graph with at least three vertices and let $u$ and $v$ be two distinct vertices of $G_0$. Let $G_{s,t}$ be the graph obtained from $G_0$ by attaching $s$ and $t$ pendant vertices to $u$ and $v$, respectively. If $s, t \geq 1$, then $D'(G_{s,t}) > \min\{D'(G_{s+t,0}), D'(G_{0,s+t})\}$.

Proof. Let $d_x = d_{G_0}(x)$ and $d(x,y) = d_{G_0}(x,y)$ for $x, y \in V(G_0)$. It is easily seen that

\[
[(d_u + s + t)D_{G_{s,t,0}}(u) - (d_u + s)D_{G_{s,t}}(u)] \\
+ [d_vD_{G_{s,t,0}}(v) - (d_v + t)D_{G_{s,t}}(v)] \\
= (d_u + s)[D_{G_{s,t,0}}(u) - D_{G_{s,t}}(u)] + t[D_{G_{s+t,0}}(u) - D_{G_{s,t}}(v)] \\
+ d_v[D_{G_{s+t,0}}(v) - D_{G_{s,t}}(v)] \\
= -t \cdot d(u,v) \cdot (d_u + s)
\]
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\[ t \left[ -s \cdot d(u, v) + \sum_{x \in V(G_0) \setminus \{u, v\}} (d(x, u) - d(x, v)) \right] + t \cdot d(u, v) \cdot d_v \]

\[ = t \left[ (d_v - d_u - 2s)d(u, v) + \sum_{x \in V(G_0) \setminus \{u, v\}} (d(x, u) - d(x, v)) \right] \]

and thus

\[ D'(G_{s+t,0}) - D'(G_{s,t}) \]

\[ = t \sum_{x \in V(G_0) \setminus \{u, v\}} d_x (d(x, u) - d(x, v)) - st \cdot d(u, v) + t \left[ -s \cdot d(u, v) + \sum_{x \in V(G_0) \setminus \{u, v\}} (d(x, u) - d(x, v)) \right] \]

\[ + \left[ (d_u + s + t)D_{G_{s+t,0}}(u) - (d_u + s)D_{G_{s,t}}(u) \right] + \left[ d_v D_{G_{s+t,0}}(v) - (d_v + t)D_{G_{s,t}}(v) \right] \]

\[ = t \sum_{x \in V(G_0) \setminus \{u, v\}} (d_x + 1)(d(x, u) - d(x, v)) - 2st \cdot d(u, v) \]

\[ + t \left[ (d_v - d_u - 2s)d(u, v) + \sum_{x \in V(G_0) \setminus \{u, v\}} (d(x, u) - d(x, v)) \right] \]

\[ = t \left[ (d_v - d_u - 2s)d(u, v) + \sum_{x \in V(G_0) \setminus \{u, v\}} (d(x, u) - d(x, v)) \right]. \]

Similarly, we have

\[ D'(G_{0,s+t}) - D'(G_{s,t}) \]

\[ = s \left[ (d_u - d_v - 4t)d(u, v) + \sum_{x \in V(G_0) \setminus \{u, v\}} (d_x + 2)(d(x, v) - d(x, u)) \right]. \]

If \( D'(G_{s+t,0}) \geq D'(G_{s,t}) \), then

\[ \sum_{x \in V(G_0) \setminus \{u, v\}} (d_x + 2)(d(x, v) - d(x, u)) \leq (d_v - d_u - 4s)d(u, v) \]

and thus

\[ D'(G_{0,s+t}) - D'(G_{s,t}) \]
\[
\begin{align*}
&\leq s \left[ (d_u - d_v - 4t)d(u, v) + (d_v - d_u - 4s)d(u, v) \right] \\
&= -4s(s + t)d(u, v) < 0.
\end{align*}
\]

The result follows. \(\square\)

Now we give a technique to compare the degree distances of two connected graphs.

Let \(G\) and \(H\) be two connected graphs. Let \(V_1(G) = \{x \in V(G) : d_G(x) = 2\}\) and \(V_2(G) = V(G) \setminus V_1(G)\). Let \(d_x = d_G(x)\) for \(x \in V(G)\), and \(d^*_x = d_H(x)\) for \(x \in V(H)\). Then
\[
\begin{align*}
D'(H) - D'(G) &= 2 \sum_{x \in V_1(G)} D_H(x) + \sum_{x \in V_2(G)} d^*_x D_H(x) - 2 \sum_{x \in V_1(G)} D_G(x) \\
&\quad - \sum_{x \in V_2(G)} d_x D_G(x) \\
&= 4[W(H) - W(G)] + \sum_{x \in V_2(G)} (d^*_x - 2)D_H(x) \\
&\quad - \sum_{x \in V_2(G)} (d_x - 2)D_G(x).
\end{align*}
\]

Let \(P_n\) and \(C_n\) be the path and the cycle on \(n\) vertices, respectively.

Let \(G\) be the unicyclic graph obtained from a cycle \(C_{m} = v_0v_1\ldots v_{m-1}v_0\) by attaching a path \(P_a\) and a path \(P_b\) to \(v_i\) and \(v_j\), respectively, where \(a \geq 1\) and \(b \geq 2\). Label the vertices of the path \(P_b\) attached to \(v_j\) as \(u_1, u_2, \ldots, u_b\) consecutively, where \(u_1\) is adjacent to \(v_j\) in \(G\). For integer \(h \geq 1\), let \(G^{(1)}_{v_i, h}\) be the graph obtained from \(G\) by attaching \(h\) pendant vertices to \(u_i\), where \(1 \leq t \leq b - 1\), and let \(G^{(2)}_{v_i, h}\) be the graph obtained from \(G\) by attaching \(h\) pendant vertices to \(v_i\), where \(0 \leq t \leq m - 1\).

**Lemma 2.3.** Let \(c = d_G(v_i, v_j)\), \(t_1 = d_G(v_i, v_t)\) and \(t_2 = d_G(v_j, v_t)\), where \(G = G^{(2)}_{v_t, h}\). If \(t \neq i\), then
\[
D' \left( G^{(2)}_{v_i, h} \right) - D' \left( G^{(2)}_{v_t, h} \right) = 4h[b(c - t_2) - at_1].
\]

**Proof.** Denote by \(u^*\) the pendant vertex of the path attached to \(v_i\) in \(G^{(2)}_{v_t, h}\), and \(v\) a pendant vertex attached to \(v_j\) in \(G^{(2)}_{v_t, h}\) and \(v_i\) in \(G^{(2)}_{v_i, h}\), respectively. Let \(G_1 = G^{(2)}_{v_i, h}\) and \(G_2 = G^{(2)}_{v_t, h}\). Note that \(D_{G_1}(v_i) -
Adding the edges of $E$ and if $i \neq j$ and $t \neq j$, then

$$D' \left( \tilde{G}_{v_i,h}^{(2)} \right) - D' \left( \tilde{G}_{v_j,h}^{(2)} \right) = 4[W(G_1) - W(G_2)] - [D_{G_1}(v_i) - D_{G_2}(v_i)]$$
$$+ [D_{G_1}(v_j) - D_{G_2}(v_j)] - [D_{G_1}(u_b) - D_{G_2}(u_b)]$$
$$- h[D_{G_1}(v) - D_{G_2}(v)] + (h + 1)D_{G_1}(v_i) - hD_{G_2}(v_i)$$
$$- D_{G_2}(v_i)$$
$$= 4[W(G_1) - W(G_2)] - h[D_{G_1}(v) - D_{G_2}(v)] + h[D_{G_1}(v_i) - D_{G_2}(v_i)]$$
$$= 4[W(G_1) - W(G_2)]$$
$$= 4h[b(c - t_2) - at_1],$$

and if $i = j$ or $t = j$, then by similar argument, the result holds also. \(\square\)

**Lemma 2.4.** Let $n_1 = a + m - 1$ and $n_2 = b - t$ in $G_{u,t,h}^{(1)}$. Then

$$D' \left( \tilde{G}_{v_i,h}^{(2)} \right) - D' \left( \tilde{G}_{u,t,h}^{(1)} \right) = 2ht[2(n_2 - n_1) - 1].$$

**Proof.** Denote by $u^*$ the pendant vertex of the path attached to $v_i$ in $G_{u,t,h}^{(1)}$, and $v$ a pendant vertex attached to $v_j$ in $G_{v_j,h}^{(2)}$ and $u_t$ in $G_{u,t,h}^{(1)}$, respectively. Let $G_1 = G_{v_j,h}^{(2)}$ and $G_2 = G_{u,t,h}^{(1)}$. Note that $D_{G_1}(v_i) - D_{G_2}(v_i) = D_{G_1}(u^*) - D_{G_2}(u^*)$ and $D_{G_1}(v_j) - D_{G_2}(v_j) = D_{G_1}(u_t) - D_{G_2}(v).$ If $i \neq j$, then

$$D' \left( \tilde{G}_{v_j,h}^{(2)} \right) - D' \left( \tilde{G}_{u,t,h}^{(1)} \right) = 4[W(G_1) - W(G_2)] - [D_{G_1}(v_i) - D_{G_2}(v_i)]$$
$$- [D_{G_1}(u*) - D_{G_2}(u*)] - [D_{G_1}(u_t) - D_{G_2}(u_t)]$$
$$- h[D_{G_1}(v) - D_{G_2}(v)] + (h + 1)D_{G_1}(v_i) - hD_{G_2}(v_i)$$
$$- D_{G_2}(v_i)$$
$$= 4[W(G_1) - W(G_2)] - [D_{G_1}(u*) - D_{G_2}(u*)] + [D_{G_1}(v_j) - D_{G_2}(v_j)]$$
$$= 4ht(n_2 - n_1) - ht - ht$$
$$= 2ht[2(n_2 - n_1) - 1],$$

and if $i = j$, then by similar argument, the result holds also. \(\square\)

As usual, $G - E_1$ means the graph obtained from $G$ by deleting the edges of $E_1 \subseteq E(G)$, and $G + E_2$ means the graph obtained from $G$ by adding the edges of $E_2 \subseteq E(\overline{G})$, where $\overline{G}$ is the complement of $G$. 

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3. Minimum Degree Distance of Unicyclic Graphs with Given Girth and Diameter

In this section we determine the unicyclic graphs with minimum degree distance when the number of vertices, girth and diameter are given.

Let \( n, m \) and \( d \) be integers with \( 3 \leq m \leq n-1 \) and \( 2 \leq d \leq n - \left\lfloor \frac{m+1}{2} \right\rfloor \).

For \( a \geq b \geq 0 \) and \( a \geq 1 \), let \( U_{n,m,d}^{k}(a, b) \) be the unicyclic graph obtained from the cycle \( C_{m} = v_{0}v_{1} \ldots v_{m-1}v_{0} \) by attaching a path \( P_{a} \) to \( v_{0} \) and a path \( P_{b} \) to \( v_{\left\lfloor \frac{m}{2} \right\rfloor} \) respectively (if \( b = 0 \), then by attaching only a path \( P_{a} \) to \( v_{0} \)), where \( a + b = d - \left\lfloor \frac{m}{2} \right\rfloor \), and attaching \( n - d - \left\lfloor \frac{m+1}{2} \right\rfloor \) pendant vertices to \( v_{k} \), where \( 0 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor \). Let \( U_{n,m,d}(a, b) = U_{n,m,d}^{0}(a, b) \).

For \( U_{n,m,d}(a, b) \), let \( u_{0} \) be the pendant vertex on the path attached to \( v_{0} \), let \( u_{1} \) be the pendant vertex on the path attached to \( v_{\left\lfloor \frac{m}{2} \right\rfloor} \) if \( b \geq 1 \), and \( u_{1} = v_{\left\lfloor \frac{m}{2} \right\rfloor} \) if \( b = 0 \), and let \( u \) be any of the pendant vertices attached to \( v_{0} \).

Let \( \alpha = \alpha(n, m, d) = \frac{(n-d-\left\lfloor \frac{m+1}{2} \right\rfloor)\left\lfloor \frac{m}{2} \right\rfloor}{n-d-\frac{m}{2}} \). Let \( \gamma \) and \( \theta \) be integers such that \( \gamma + \theta = d - \left\lfloor \frac{m}{2} \right\rfloor \) and \( \gamma - \theta \) is an integer as large as possible but no more than \( \alpha + 1 \). Let \( U_{n,m,d} = U_{n,m,d}(\gamma, \theta) = U_{n,m,d}^{\theta}(\gamma, \theta) \).

**Lemma 3.1.** Let \( n, m \) and \( d \) be fixed integers with \( 3 \leq m \leq n-2 \) and \( 3 \leq d \leq n - \left\lfloor \frac{m+1}{2} \right\rfloor \). Then \( D'(U_{n,m,d}(a, b)) \) with \( a \geq b \) and \( a+b = d - \left\lfloor \frac{m}{2} \right\rfloor \) is minimum if and only if \( (a, b) = (\gamma, \theta) \), \( (\gamma - 1, \theta + 1) \) if \( \alpha \geq 1 \) is an integer with different parity as \( d - \left\lfloor \frac{m}{2} \right\rfloor \), and \( (a, b) = (\gamma, \theta) \) otherwise.

**Proof.** Let \( h = n-d-\left\lfloor \frac{m+1}{2} \right\rfloor \). Let \( w \) be the neighbor of \( u_{0} \) in \( U_{n,m,d}(a, b) \). Note that for \( a-b \geq 2 \), \( U_{n,m,d}(a-1, b+1) \cong U_{n,m,d}(a, b) - \{u_{0}w\} + \{u_{0}u_{1}\} \). Let \( G_{1} = U_{n,m,d}(a-1, b+1) \) and \( G_{2} = U_{n,m,d}(a, b) \). If \( a \geq b \geq 1 \), then

\[
D'(U_{n,m,d}(a-1, b+1)) - D'(U_{n,m,d}(a, b)) = 4[\Delta(G_{1}) - \Delta(G_{2})] - h[D_{G_{1}}(u) - D_{G_{2}}(u)]
\]

\[
- [D_{G_{1}}(u_{0}) - D_{G_{2}}(u_{0})] + [D_{G_{1}}(v_{\left\lfloor \frac{m}{2} \right\rfloor}) - D_{G_{2}}(v_{\left\lfloor \frac{m}{2} \right\rfloor})]
\]

\[
= 4\left[ (1 - a + b) \left( h + \left\lfloor \frac{m-1}{2} \right\rfloor + 1 \right) + h \left\lfloor \frac{m}{2} \right\rfloor \right],
\]

and if \( d - \left\lfloor \frac{m}{2} \right\rfloor \) and \( b = 0 \), then

\[
D'(U_{n,m,d}(a-1, b+1)) - D'(U_{n,m,d}(a, b)) = 4[\Delta(G_{1}) - \Delta(G_{2})] - h[D_{G_{1}}(u) - D_{G_{2}}(u)]
\]

\[
- [D_{G_{1}}(u_{0}) - D_{G_{2}}(u_{0})] + (h + 1)[D_{G_{1}}(v_{0}) - D_{G_{2}}(v_{0})]
\]
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Let \( U(n, m, d) \) be the set of unicyclic graphs with \( n \) vertices, girth \( m \) and diameter \( d \), where \( 2 \leq d \leq n - \left\lceil \frac{m+1}{2} \right\rceil \) and \( 3 \leq m \leq n - 2 \). If \( G \in U(n, m, d) \), then \( m = 3 \) and \( G \cong U_n(1, 0) \).

Let \( G \) be a unicyclic graph with \( n \) vertices and let \( C_m = v_0v_1 \ldots v_{m-1}v_0 \) be its unique cycle. Then \( G - E(C_m) \) consists of \( m \) trees \( T_0, T_1, \ldots, T_{m-1} \), where \( v_i \in V(T_i) \) for \( i = 0, 1, \ldots, m - 1 \). If the degree of \( v_i \) is at least three, then the components obtained from \( T_i \) by deleting the vertex \( v_i \) (and its incident edges) are called the branches of \( G \) at \( v_i \), each containing a neighbor of \( v_i \) in \( T_i \).

**Lemma 3.2.** Let \( n, m \) and \( d \) be integers with \( 3 \leq m \leq n - 2 \) and \( 3 \leq d \leq n - \left\lceil \frac{m+1}{2} \right\rceil \), and let \( \beta = \frac{1}{2}(d - \left\lceil \frac{m}{2} \right\rceil) \). If \( G \) is a graph with minimum degree distance in \( U(n, m, d) \), then \( G \cong U_{n,m,d}(a, b) = U^0_{n,m,d}(a, b) \) with \( a \geq b \) or \( G \cong U^k_{n,m,d}(\beta, \beta) \) for \( k = 1, 2, \ldots, \left\lceil \frac{m}{4} \right\rceil \).

**Proof.** Let \( C_m = v_0v_1 \ldots v_{m-1}v_0 \) be the unique cycle of \( G \), and let \( P(G) = u_0u_1 \ldots u_d \) be a diametrical path of \( G \). Let \( d(x, y) = d_G(x, y) \) for \( x, y \in V(G) \).

Suppose that \( P(G) \) has no common vertices with \( C_m \). Let \( u_s \) and \( u_t \) be the vertices such that \( d(u_s, v_t) = \min\{d(u, v) : u \in V(P(G)), v \in V(C_m)\} \). Using Lemma 2.1 (ii) by setting \( u = u_s, v = v_t, M \) to be the subgraph of \( G \) consisting of the path \( P(G) \) and the trees attached to \( u_i \) for all \( 1 \leq i \leq d - 1 \) and \( i \neq s, N \) to be the subgraph of \( G \) by deleting all branches at \( v_t \), we can obtain a graph \( G^* \) for which \( P(G^*) \) (\( = P(G) \)) and the cycle \( C_m \) have exactly one common vertex and \( D'(G^*) < D'(G) \), a contradiction. Thus \( P(G) \) and \( C_m \) have at least one common vertex. We may choose \( P(G) \) such that \( P(G) \) and the cycle \( C_m \) have common vertices as many as possible and \( u_0 \) is a pendant vertex.

Let \( u_a \) and \( u_l \) be the common vertices of \( P(G) \) and \( C_m \) such that they have the smallest and largest subscript, respectively, among vertices in \( P(G) \), where \( 0 < a \leq l \leq d \). Let \( u_a = v_i \) and \( u_l = v_j \). By Lemma 2.1 (i),
all vertices outside $C_m$ except those in $T_i$ and $T_j$ are pendant vertices attached to vertices that are nearest to them in $C_m$, all vertices in $T_i$ and $T_j$ except those in $P(G)$ are pendant vertices attached to vertices that are nearest to them in $P(G)$.

Suppose that $P(G)$ and $C_m$ have only one common vertex, i.e., $i = j$, $a = l$ and $l < d$. By the choice of $P(G)$, we have $a \geq 2$. By Lemma 2.2, all pendant vertices in $G$ except $u_0$ and $u_d$ are actually attached to some vertex, say $s$, of $G$.

Suppose that $s \in \{v_0, v_1, \ldots, v_{m-1}\} \setminus \{v_i\}$, say $s = v_q$. Denote by $v_{q_1}, v_{q_2}, \ldots, v_{q_l}$ the pendant neighbors of $v_q$. Let $H = G - \{v_qv_{q_1}, v_qv_{q_2}, \ldots, v_qv_{q_l}\} + \{v_iv_{q_1}, v_iv_{q_2}, \ldots, v_iv_{q_l}\}$. Then $H \in U(n, m, d)$. By Lemma 2.3, we have

$$D'(H) - D'(G) = -4dt \cdot d(v_q, v_i) < 0,$$

a contradiction. Thus $s \in \{u_1, u_2, \ldots, u_{d-1}\}$. Suppose without loss of generality that $s = \{u_{a}, u_{a+1}, \ldots, u_{d-1}\}$. Let $H^* = G - \{u_{a-2}u_{a-1}\} + \{u_{a-2}v_i\}$. Then $H^* \in U(n, m, d)$. Note that the (diametrical) path $P(H^*) = u_0u_1 \ldots u_{a-2}v_iu_{a+1}u_{a+2} \ldots u_d$ has more than one common vertex with the cycle $C_m$ and the same length as $P(G)$. Then

$$D'(H^*) - D'(G) = 4[W(H^*) - W(G)] - [D_{H^*}(u_0) - D_G(u_0)]$$

$$+ D_{H^*}(v_{i-1}) - D_{H^*}(u_{a-1})$$

$$= -4(m - 2)(a - 1) + (m - 2) - 2a - m + 4$$

$$= -2(a - 1)(2m - 3) < 0,$$

and thus $D'(H^*) < D'(G)$, a contradiction. It follows that $P(G)$ and $C_m$ have at least two common vertices, i.e., $a < l$.

By Lemma 2.2, all pendant vertices in $G$ except $u_0$ and $u_d$ are actually attached to some vertex, say $x$, in $G$. Thus $x$ has exactly $h = n - m - a - (d - l)$ pendant neighbors outside $P(G)$. Let $b = d - l$. Assume that $a \geq b$.

Suppose that $l < d$ and $x \in \{u_{l+1}, u_{l+2}, \ldots, u_{d-1}\}$, say $x = u_q$, where $l < q \leq d - 1$. Let $u_{q_1}, u_{q_2}, \ldots, u_{q_h}$ be the pendant neighbors of $u_q$ outside $P(G)$. Let $G_1 = G - \{u_qu_{q_1}, u_qu_{q_2}, \ldots, u_qu_{q_h}\} + \{u_qu_{q_1}, u_qu_{q_2}, \ldots, u_qu_{q_h}\}$. Then $G_1 \in U(n, m, d)$. Using Lemma 2.4 by setting $t = q - l$, $n_1 = a + m - 1$ and $n_2 = b - t$, and noting that $n_1 > n_2$ since $a \geq b$, we have

$$D'(G_1) - D'(G) = 2h(q - l)[2(n_2 - n_1) - 1] < 0,$$

and then $D'(G_1) < D'(G)$, a contradiction. Thus $x \notin \{u_{l+1}, u_{l+2}, \ldots, u_{d-1}\}$ if $l < d$. Moreover, if $a = b$, then by similar argument, $x \notin \{u_1, u_2, \ldots, u_{a-1}\}$, and thus $x \in \{v_0, v_1, \ldots, v_{m-1}\}$.
Case 1. $a > b$.

First we prove that $x \in \{u_1, u_2, \ldots, u_a\}$. Suppose to the contrary that $x = v_s$ with $0 \leq s \leq m - 1$ and $s \neq i$. Denote by $v_s, v_{s+1}, \ldots, v_{s_1}$ the pendant neighbors of $v_s$. Suppose that $d(v_i, v_j) = c$, $d(v_i, v_s) = t_1$ and $d(v_j, v_s) = t_2$, then $c \leq t_1 + t_2$. Let $G_2 = G - \{v_s, v_{s+1}, \ldots, v_{s_i}\}$. Then $G_2 \in \mathbb{U}(n, m, d)$. Note that $b = 0$ if $l = d$. By Lemma 2.3, we have

$$D'(G_2) - D'(G) = 4h[bc - at_2] \leq 4h(bt_1 - at_1)$$

and then $D'(G_2) < D'(G)$, a contradiction. Thus $x \in \{u_1, u_2, \ldots, u_b\}$, say $x = u_p$ with $1 \leq p \leq a$.

Next we prove that $d(v_i, v_j) = \left\lfloor \frac{m}{2} \right\rfloor$. If $l = d$, then it is obvious. Suppose that $l < d$ and $c = d(v_i, v_j) < \left\lfloor \frac{m}{2} \right\rfloor$. Let $v$ be the neighbor of $v_j$ on $C_m$ with $d(v, v_1) = c + 1$ ($v = v_{j+1}$ if $v_i, v_{j+1}, \ldots, v_{j-1}, v_j$ is a shortest path from $v_i$ to $v_j$). By the choice of $P(G)$, we have $b + c > \left\lfloor \frac{m}{2} \right\rfloor$, and then $b > 1$. Let $G_3 = G - \{v_ju_{j+1}\} + \{v_{j+1}u_{j+1}\} - \{u_{d-1}u_d\} + \{v_iu_d\}$. Then $G_3 \in \mathbb{U}(n, m, d)$. If $1 \leq p \leq a - 1$, then

$$D'(G_3) - D'(G) = 4[W(G_3) - W(G)] = D_{G_3}(u_0) - D_G(u_0) - D_{G_3}(u_d) - D_G(u_d)$$

$$+ 2D_{G_3}(v_i) + D_{G_3}(v) - D_{G_3}(u_{d-1}) - D_G(v_i) - D_G(v_j)$$

$$= -2(2m - 3)(b - 1) - 4c(n - m - 2b + 1) < 0,$$

if $p = a$, then by similar calculation,

$$D'(G_3) - D'(G) = -2(2m - 3)(b - 1) - 4c(n - m - 2b + 1) < 0.$$

It follows that $D'(G_3) < D'(G)$ for $1 \leq p \leq a$, a contradiction. Thus $d(v_i, v_j) = \left\lfloor \frac{m}{2} \right\rfloor$ and $h = n - d - \left\lfloor \frac{m+1}{2} \right\rfloor$.

Now we prove that $p = a$. Suppose to the contrary that $p \leq a - 1$. Let $u_{p_1}, u_{p_2}, \ldots, u_{p_b}$ be the pendant neighbors of $u_p$ outside $P(G)$. Suppose first that $b + m > a$. Then $p < b + m - 1$. Let $G_4 = G - \{u_p, u_{p_1}, u_{p_2}, \ldots, u_{p_b}\} + \{u_{a}u_{p_1}, u_{a}u_{p_2}, \ldots, u_{a}u_{p_b}\}$. Then $G_4 \in \mathbb{U}(n, m, d)$. Using Lemma 2.4 by setting $t = a - p$, $n_1 = b + m - 1$ and $n_2 = p$, we have

$$D'(G_4) - D'(G) = 2h(a - p)[2p - 2(b + m - 1) - 1] < 0,$$

and thus $D'(G_4) < D'(G)$, a contradiction. Now suppose that $b + m \leq a$. Then $b - (h - 1) < a$. Let $G_4 = G - \{u_p, u_{p_1}, u_{p_2}, \ldots, u_{p_b}\} + \{u_{a}u_{p_1}, u_{a}u_{p_2}, \ldots, u_{a}u_{p_b}\}$. Then $b - (h - 1) < a$. Let $G_4 = G - \{u_p, u_{p_1}, u_{p_2}, \ldots, u_{p_b}\} + \{u_{a}u_{p_1}, u_{a}u_{p_2}, \ldots, u_{a}u_{p_b}\}$. Then $b - (h - 1) < a$. Let $G_4 = G - \{u_p, u_{p_1}, u_{p_2}, \ldots, u_{p_b}\} + \{u_{a}u_{p_1}, u_{a}u_{p_2}, \ldots, u_{a}u_{p_b}\}$. Then $b - (h - 1) < a$. Let $G_4 = G - \{u_p, u_{p_1}, u_{p_2}, \ldots, u_{p_b}\} + \{u_{a}u_{p_1}, u_{a}u_{p_2}, \ldots, u_{a}u_{p_b}\}$. Then $b - (h - 1) < a$. Let $G_4 = G - \{u_p, u_{p_1}, u_{p_2}, \ldots, u_{p_b}\} + \{u_{a}u_{p_1}, u_{a}u_{p_2}, \ldots, u_{a}u_{p_b}\}$. Then $b - (h - 1) < a$.
If \( l < d \) and \( 1 \leq p \leq a - 2 \), then

\[
D'(G_4) = D'(G) = 4[692 Du and Zhou]
\]

\[
= 4[W(G_4) - W(G)] + [D_{G_4}(u_0) - D_G(u_0)] + [D_{G_4}(u_1) - D_G(u_1)]
\]

\[
- [D_{G_4}(u_0) - D_G(u_0)] - h[D_{G_4}(u_{p+1}) - D_G(u_{p+1})]
\]

\[
+ hD_{G_4}(u_{p+1}) - D_{G_4}(u_1) - hD_G(u_1) + D_G(u_d)
\]

\[
= 2 \left( 2 \left( \frac{m-1}{2} \right) + 1 \right) (b - a + 1 - h) < 0.
\]

By similar calculation, \( D'(G_4) - D'(G) < 0 \) holds also if \( l = d \) or \( p = a - 1 \).

Then in any case we have \( D'(G_4) < D'(G) \), a contradiction. Thus \( p = a \).

Now we have proved that \( G \cong U_{n,m,d}(a,b) \), where \( a > b \) and \( a + b = d - \lfloor \frac{m}{2} \rfloor \).

**Case 2.** \( a = b \).

Note that \( x \in \{v_0, v_1, \ldots, v_{m-1}\} \), say \( x = v_0 \). Assume that \( v_i v_{i+1} \ldots v_{j-1} v_j \) is a shortest path from \( v_i \) to \( v_j \) in \( G \). Obviously, \( m \geq 2(j - i) \).

If \( m = 2(j-i) \), then by symmetry, we may assume that \( i \leq s \leq j \). Suppose that \( m > 2(j-i) \) and \( s \not\in \{i, i+1, \ldots, j-1, j\} \). Denote by \( v_{s_1}, v_{s_2}, \ldots, v_{s_h} \) the pendant neighbors of \( v_s \). Let \( d(v_i, v_j) = j - i = c \), \( d(v_i, v_s) = t_1 \) and \( d(v_j, v_s) = t_2 \). Then \( c < t_1 + t_2 \). Let \( G_5 = G - \{v_{s_1} v_{s_2}, \ldots, v_{s_{h-1}} \} \) and \( G_6 \in U(n,m,d) \). By Lemma 2.3, we have

\[
D'(G_5) - D'(G) = 4h[b(c - t_2) - at_1] = 4ha(c - t_1 - t_2) < 0,
\]

and then \( D'(G_5) < D'(G) \), a contradiction. Thus \( i \leq s \leq j \).

Suppose that \( c = d(v_i, v_j) < \lfloor \frac{m}{2} \rfloor \). Note that \( d(v_i, v_{j+1}) = c + 1 \). By the choice of \( P(G) \), we have \( b + c > \lfloor \frac{m}{2} \rfloor \), and then \( b > 1 \). Let \( G_6 = G - \{v_j u_d \} \) and \( G_6 \in U(n,m,d) \). If \( i + 1 \leq s \leq j - 1 \), then

\[
D'(G_6) = D'(G) = 4[W(G_6) - W(G)] - [D_{G_6}(u_0) - D_G(u_0)] - [D_{G_6}(u_d) - D_G(u_d)]
\]

\[
- h[D_{G_6}(v_{s_1}) - D_G(v_{s_1})] + [D_{G_6}(v_s) - D_G(v_s)]
\]

\[
+ (h + 1)D_{G_6}(v_s) - D_{G_6}(v_{s-1}) + D_{G_6}(v_{s+1}) - hD_G(v_s) - D_G(v_s)
\]

\[
= -2(2m - 3)(b - 1) - 4(j - s)(h + 1) < 0.
\]

By similar calculation, \( D'(G_6) - D'(G) < 0 \) holds also if \( s = i \) or \( j \). In any case, we have \( D'(G_6) < D'(G) \), a contradiction. Thus \( d(v_i, v_j) = \lfloor \frac{m}{2} \rfloor \) and \( h = n - d - \lfloor \frac{m+1}{2} \rfloor \). By Lemma 2.3, we have \( U^k(n,m,d) \) for
\( k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \) have equal degree distance, and thus \( G \cong U_{n,m,d}^k(\beta, \beta) \) for \( k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \).

By combining Cases 1 and 2, we have \( G \cong U_{n,m,d}(a, b) = U_{n,m,d}^0(a, b) \) with \( a \geq b \) or \( G \cong U_{n,m,d}^k(\beta, \beta) \) for \( k = 1, 2, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \).

**Theorem 3.3.** Let \( n, m \) and \( d \) be integers with \( 3 \leq m \leq n - 2 \) and \( 3 \leq d \leq n - \left\lfloor \frac{m+1}{2} \right\rfloor \), and let
\[
\alpha = \left( \frac{n-d-\left\lfloor \frac{m+1}{2} \right\rfloor}{n-d-\frac{m}{2}} \right) \left\lfloor \frac{m}{2} \right\rfloor, \quad \beta = \frac{1}{2}(d-\left\lfloor \frac{m}{2} \right\rfloor).
\]

(i) If \( 0 < \alpha < 1 \) and \( d - \left\lfloor \frac{m}{2} \right\rfloor \) is even, then \( U_{n,m,d}^k(\beta, \beta) \) for \( k = 0, 1, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \) are the unique graphs in \( \langle n, m, d \rangle \) with minimum degree distance.

(ii) If \( \alpha = 1 \) and \( d - \left\lfloor \frac{m}{2} \right\rfloor \) is even, then \( U_{n,m,d} = U_{n,m,d}(\beta+1, \beta-1) \) and \( U_{n,m,d}^k(\beta, \beta) \) for \( k = 0, 1, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \) are the unique graphs in \( \langle n, m, d \rangle \) with minimum degree distance.

(iii) If \( \alpha > 1 \) is an integer with different parity as \( d - \left\lfloor \frac{m}{2} \right\rfloor \), then \( U_{n,m,d} \) and \( U_{n,m,d}(\gamma - 1, \theta + 1) \) are the unique graphs in \( \langle n, m, d \rangle \) with minimum degree distance.

(iv) If \( \alpha = 0 \), or \( 0 < \alpha < 1 \) and \( d - \left\lfloor \frac{m}{2} \right\rfloor \) is odd, or \( \alpha > 1 \) is not an integer or is an integer with the same parity as \( d - \left\lfloor \frac{m}{2} \right\rfloor \), then \( U_{n,m,d} \) is the unique graph in \( \langle n, m, d \rangle \) with minimum degree distance.

**Proof.** Suppose that \( G \) is a graph in \( \langle n, m, d \rangle \) with minimum degree distance. By Lemma 3.2, we have \( G \cong U_{n,m,d}(a, b) = U_{n,m,d}^0(a, b) \) with \( a \geq b \) or \( G \cong U_{n,m,d}^k(\beta, \beta) \) for \( k = 1, 2, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \). If \( G \cong U_{n,m,d}(a, b) \), then we have by Lemma 3.1 that \( G \cong U_{n,m,d} \) or \( U_{n,m,d}(\gamma - 1, \theta + 1) \) if \( \alpha \geq 1 \) is an integer with different parity as \( d - \left\lfloor \frac{m}{2} \right\rfloor \), and \( G \cong U_{n,m,d} \) otherwise.

If \( d - \left\lfloor \frac{m}{2} \right\rfloor \) is odd, then \( G \cong U_{n,m,d} \) or \( U_{n,m,d}(\gamma - 1, \theta + 1) \) if \( \alpha \geq 1 \) is an even integer, and \( G \cong U_{n,m,d} \) otherwise.

Suppose in the following that \( d - \left\lfloor \frac{m}{2} \right\rfloor \) is even. Then \( G \cong U_{n,m,d}^k(\beta, \beta) \) for \( k = 1, 2, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \), or \( G \cong U_{n,m,d} \) or \( U_{n,m,d}(\gamma - 1, \theta + 1) \) if \( \alpha \geq 1 \) is an odd integer, and \( G \cong U_{n,m,d} \) otherwise.

If \( \alpha = 0 \), then \( G \cong U_{n,m,d} \).

If \( 0 < \alpha < 1 \), then \( G \cong U_{n,m,d}^k(\beta, \beta) \) for \( k = 0, 1, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \).

Suppose that \( \alpha = 1 \). Then \( G \cong U_{n,m,d} \) or \( U_{n,m,d}(\gamma - 1, \theta + 1) \), or \( G \cong U_{n,m,d}^k(\beta, \beta) \) for \( k = 1, 2, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \). Since \( (\gamma - 1, \theta + 1) = (\beta, \beta) \), we have \( G \cong U_{n,m,d} \) or \( U_{n,m,d}^k(\beta, \beta) \) for \( k = 0, 1, \ldots, \left\lfloor \frac{m}{4} \right\rfloor \).
Suppose that $\alpha > 1$ is an odd integer. Then $G \cong U_{n,m,d}, U_{n,m,d}(\gamma - 1, \theta + 1)$, or $G \cong U_{n,m,d}^k(\beta, \beta)$ for $k = 1, 2, \ldots, \lfloor \frac{m}{4} \rfloor$. By Lemmas 2.3 and 3.1, we have

$$D'(U_{n,m,d}^k(\beta, \beta)) = D'(U_{n,m,d}^0(\beta, \beta)) > D'(U_{n,m,d}(\gamma, \theta)) = D'(U_{n,m,d}(\gamma - 1, \theta + 1))$$

for $k = 1, 2, \ldots, \lfloor \frac{m}{4} \rfloor$. Thus $G \cong U_{n,m,d}$ or $U_{n,m,d}(\gamma - 1, \theta + 1)$.

Suppose that $\alpha > 1$ is not an odd integer. Then $G \cong U_{n,m,d}$, or $G \cong U_{n,m,d}^k(\beta, \beta)$ for $k = 1, 2, \ldots, \lfloor \frac{m}{4} \rfloor$. By Lemmas 2.3 and 3.1, we have

$$D'(U_{n,m,d}^k(\beta, \beta)) = D'(U_{n,m,d}^0(\beta, \beta)) > D'(U_{n,m,d}(\gamma, \theta))$$

for $k = 1, 2, \ldots, \lfloor \frac{m}{4} \rfloor$. Thus $G \cong U_{n,m,d}$. \hfill $\Box$

**Corollary 3.4.** Let $G \in \mathcal{V}(n,m,d)$ with $3 \leq m \leq n - 2$ and $3 \leq d \leq n - \lfloor \frac{m+1}{2} \rfloor$. Then

$$D'(G) \geq D'(U_{n,m,d}).$$

4. **Maximum Reverse Degree Distance of Unicyclic Graphs**

In this section we determine the unicyclic graphs on $n$ vertices with maximum reverse degree distances when girth, number of pendant vertices and maximum degree are given, respectively.

**Lemma 4.1.** For $3 \leq m \leq n - 2$ and $2 \leq d < n - \lfloor \frac{m+1}{2} \rfloor$,

$$D'(U_{n,m,d}) > D'(U_{n,m,d+1}).$$

**Proof.** Let $h = n - d - \lfloor \frac{m}{2} \rfloor$. Let $u_2$ be a pendant neighbor of $v_0$ different from $u$ in $U_{n,m,d}$ if $h \geq 2$. Recall that $U_{n,m,d} = U_{n,m,d}(\gamma, \theta)$. Note that we may obtain $U_{n,m,d+1}(\gamma + 1, \theta)$ from $U_{n,m,d}(\gamma, \theta)$ by deleting the edge $uv_0$ and adding the edge $uv_0$. Let $G_1 = U_{n,m,d+1}(\gamma + 1, \theta)$ and $G_2 = U_{n,m,d}(\gamma, \theta)$. If $\theta \geq 1$ and $h \geq 2$, then $D_{G_1}(v_{\lfloor \frac{m}{2} \rfloor}) - D_{G_2}(v_{\lfloor \frac{m}{2} \rfloor}) = D_{G_1}(u_1) - D_{G_2}(u_1)$, and thus

$$D'(U_{n,m,d+1}(\gamma + 1, \theta)) - D'(U_{n,m,d}(\gamma, \theta)) = 4[W(G_1) - W(G_2)] - (h - 1)[D_{G_1}(u_2) - D_{G_2}(u_2)]$$

$$= 4\gamma(n - \gamma - 2) - (h - 1)\gamma - \gamma(n - \gamma - 2) + \gamma(n - \gamma + h - 1)$$

$$= -4\gamma^2 + 2(2n - 3)\gamma.$$
If $\theta = 0$ or $h = 1$, then by similar calculation, we also have $D'(U_{n,m,d+1}(\gamma + 1, \theta)) - D'(U_{n,m,d}(\gamma, \theta)) = -4\gamma^2 + 2(2n - 3)\gamma$. Thus

\[
\begin{align*}
'\Delta'(U_{n,m,d+1}(\gamma + 1, \theta)) - '\Delta'(U_{n,m,d}(\gamma, \theta)) & = 2(n - 1)n - [D'(U_{n,m,d+1}(\gamma + 1, \theta)) - D'(U_{n,m,d}(\gamma, \theta))] \\
& = 4\gamma^2 - 2(2n - 3)\gamma + 2n^2 - 2n \\
& \geq 4 \left(\frac{2n - 3}{4}\right)^2 - 2(2n - 3) \cdot \frac{2n - 3}{4} + 2n^2 - 2n \\
& = n^2 + n - \frac{9}{4} > 0.
\end{align*}
\]

By Corollary 3.4, we have $'\Delta'(U_{n,m,d+1}(\gamma + 1, \theta)) \leq '\Delta'(U_{n,m,d+1})$. Then the result follows clearly.

**Theorem 4.2.** Let $G$ be a unicyclic graph with $n$ vertices and girth $m$, where $3 \leq m \leq n - 2$. Then

\[
'\Delta'(G) \leq '\Delta'(U_{n,m,n-\lfloor \frac{m+1}{2} \rfloor})
\]

with equality if and only if $G \cong U_{n,m,n-\lfloor \frac{m+1}{2} \rfloor}$.

**Proof.** Let $d$ be the diameter of $G$. Then $2 \leq d \leq n - \lfloor \frac{m+1}{2} \rfloor$. If $d = n - \lfloor \frac{m+1}{2} \rfloor$, then $\alpha = 0$, and by Theorem 3.3 (iv),

\[
'\Delta'(G) \leq '\Delta'(U_{n,m,n-\lfloor \frac{m+1}{2} \rfloor})
\]

with equality if and only if $G \cong U_{n,m,n-\lfloor \frac{m+1}{2} \rfloor}$. If $d = 2$, then $G \cong U_{n,3,2}(1,0)$, similar to the proof of Lemma 4.1, we have $'\Delta'(U_{n,3,2}(1,0)) < '\Delta'(U_{n,3,3})$. If $3 \leq d < n - \lfloor \frac{m+1}{2} \rfloor$, then by Corollary 3.4 and Lemma 4.1,

\[
'\Delta'(G) \leq '\Delta'(U_{n,m,d}) < '\Delta'(U_{n,m,n-\lfloor \frac{m+1}{2} \rfloor}).
\]

Then the result follows easily. \qed

**Lemma 4.3.** [7] For $3 \leq m \leq n - 1$ and $3 \leq d \leq n - \lfloor \frac{m+1}{2} \rfloor$, let $h = n - d - \lfloor \frac{m+1}{2} \rfloor$. Then

\[
W(U_{n,m,d}(a,b)) = \left( a + b + \frac{m}{2} \right) \left[ \frac{m^2}{4} \right] + \left( a + 1 \right) \left( \frac{1}{3} \right) + \left( b + 1 \right) \left( \frac{1}{3} \right) \\
+ m \left[ \left( a + 1 \right) \left( \frac{1}{2} \right) + \left( b + 1 \right) \left( \frac{1}{2} \right) \right] + \frac{1}{2} ab \left( 2 \left[ \frac{m}{2} \right] + a + b + 2 \right)
\]
\[ + h \left[ \left\lfloor \frac{m^2}{4} \right\rfloor + m + \frac{1}{2} a (a + 3) + \frac{1}{2} b \left( 2 \left\lfloor \frac{m}{2} \right\rfloor + b + 3 \right) \right] + h (h - 1), \]

where \( a, b \) are integers with \( a + b = d - \left\lfloor \frac{m}{2} \right\rfloor \), \( a \geq b \geq 0 \) and \( a \geq 1 \).

By simple calculation, we have

**Lemma 4.4.** For \( G = U_{n,m,d}(a, b) \) with \( 3 \leq m \leq n - 1 \) and \( 3 \leq d \leq n - \left\lfloor \frac{m+1}{2} \right\rfloor \), let \( h = n - d - \left\lfloor \frac{m+1}{2} \right\rfloor \). Then

\[
D_G(v_0) = \left\lfloor \frac{m^2}{4} \right\rfloor + \frac{1}{2} a (a + 1) + \frac{1}{2} b \left( b + 1 + 2 \left\lfloor \frac{m}{2} \right\rfloor \right) + h,
\]

\[
D_G\left( v_{\lfloor \frac{m}{2} \rfloor} \right) = \left\lfloor \frac{m^2}{4} \right\rfloor + \frac{1}{2} a \left( a + 1 + 2 \left\lfloor \frac{m}{2} \right\rfloor \right) + \frac{1}{2} b (b + 1)
+ h \left( 1 + \left\lfloor \frac{m}{2} \right\rfloor \right),
\]

\[
D_G(u) = \left\lfloor \frac{m^2}{4} \right\rfloor + m + \frac{1}{2} a (a + 3) + \frac{1}{2} b \left( 2 \left\lfloor \frac{m}{2} \right\rfloor + b + 3 \right)
+ 2(h - 1),
\]

\[
D_G(u_0) = \left\lfloor \frac{m^2}{4} \right\rfloor + a \left\lfloor \frac{1}{2} (a - 1) + m \right\rfloor + \frac{1}{2} b \left( 2a + 2 \left\lfloor \frac{m}{2} \right\rfloor + b + 1 \right)
+ h (a + 1),
\]

\[
D_G(u_1) = \left\lfloor \frac{m^2}{4} \right\rfloor + b \left\lfloor \frac{1}{2} (b - 1) + m \right\rfloor + \frac{1}{2} a \left( 2b + 2 \left\lfloor \frac{m}{2} \right\rfloor + a + 1 \right)
+ h \left( b + \left\lfloor \frac{m}{2} \right\rfloor + 1 \right).
\]

**Lemma 4.5.** Let \( n \) and \( m \) be integers with \( 5 \leq m \leq n - 1 \). Let \( d = n - m - \left\lfloor \frac{m+1}{2} \right\rfloor \) and \( a = n - m \). Then

\[ D'(U_{n,m,d}(a, 0)) < D'(U_{n,m-2,d+1}(a + 2, 0)). \]

**Proof:** Let \( G_1 = U_{n,m-2,d+1}(a + 2, 0) \) and \( G_2 = U_{n,m,d}(a, 0) \). Note that \( h = n - d - \left\lfloor \frac{m+1}{2} \right\rfloor = 0 \). By Lemmas 4.3 and 4.4, we have

\[
D'(U_{n,m-2,d+1}(a + 2, 0)) - D'(U_{n,m,d}(a, 0)) = 4[W(G_1) - W(G_2)] - [D_{G_1}(u_0) - D_{G_2}(u_0)]
+ [D_{G_1}(v_0) - D_{G_2}(v_0)]
= 4 \left[ -\frac{3}{2} m^2 + \left( n + \frac{9}{2} \right) m + \left\lfloor \frac{m^2}{4} \right\rfloor - \left( \frac{m^2}{4} \right) \right]
+ (2n - 3m + 4).
\]
=\ -6m^2 + 2(2n + 7)m + 4 \left\lfloor \frac{m^2}{4} \right\rfloor - 6n - 10.

Thus
\[
\begin{align*}
\mathcal{D}'(U_{n,m-2,d+1}(a+2,0)) - \mathcal{D}'(U_{n,m,d}(a,0)) &= 2(n-1)n - [\mathcal{D}'(U_{n,m-2,d+1}(a+2,0)) - \mathcal{D}'(U_{n,m,d}(a,0))] \\
&= 6m^2 - 2(2n + 7)m - 4 \left\lfloor \frac{m^2}{4} \right\rfloor + 2n^2 + 4n + 10
\end{align*}
\]
\[
= \begin{cases} 
5m^2 - 2(2n + 7)m + 2n^2 + 4n + 10 & \text{if } m \text{ is even} \\
5m^2 - 2(2n + 7)m + 2n^2 + 4n + 11 & \text{if } m \text{ is odd} 
\end{cases}
\geq 5m^2 - 2(2n + 7)m + 2n^2 + 4n + 10
\]
\[
\geq 5 \cdot \left(\frac{2n + 7}{5}\right)^2 - 2(2n + 7) \cdot \frac{2n + 7}{5} + 2n^2 + 4n + 10
\]
\[
= \frac{1}{5}(6n^2 - 8n + 1) > 0.
\]

The result follows.

\[\Box\]

**Lemma 4.6.** Let \(n, m\) and \(d\) be integers with \(5 \leq m \leq n - 2, 3 \leq d \leq n - \left\lceil \frac{m + 1}{2} \right\rceil\), and let \(a\) and \(b\) be integers with \(a + b = d - \left\lceil \frac{m}{2} \right\rceil, a \geq b \geq 0\) and \(a \geq 1\). Then
\[
\mathcal{D}'(U_{n,m,d}(a,b)) < \mathcal{D}'(U_{n,m-2,d+1}(a+1,b+1)).
\]

**Proof.** Let \(h = n - d - \left\lceil \frac{m+1}{2} \right\rceil\). Let \(G_1 = U_{n,m-2,d+1}(a+1,b+1)\) and \(G_2 = U_{n,m,d}(a,b)\). If \(b \geq 1\), then by Lemmas 4.3 and 4.4, we have
\[
\begin{align*}
\mathcal{D}'(U_{n,m-2,d+1}(a+1,b+1)) - \mathcal{D}'(U_{n,m,d}(a,b)) &= 4[W(G_1) - W(G_2)] \\
&= (h+1)[D_{G_1}(v_0) - D_{G_2}(v_0)] + D_{G_1} \left(v_{\left\lfloor \frac{m+1}{2} \right\rceil-1} \right) - D_{G_2} \left(v_{\left\lfloor \frac{m}{2} \right\rceil} \right) \\
&- h[D_{G_1}(u) - D_{G_2}(u)] - [D_{G_1}(u_0) - D_{G_2}(u_0)] \\
&- [D_{G_1}(u_1) - D_{G_2}(u_1)] \\
&= 4 \left( h \left(2 + a - \left\lfloor \frac{m + 1}{2} \right\rfloor \right) - 2 + ab + \left\lfloor \frac{m^2}{4} \right\rfloor \\
&+ \left\lceil \frac{m}{2} \right\rceil \left(a + b + 1\right) + \frac{3}{2}m - \frac{m^2}{2} \right) \\
&+ (h+1) \left(2 + a - \left\lfloor \frac{m + 1}{2} \right\rfloor \right) + \left(2 + b - \left\lfloor \frac{m + 1}{2} \right\rceil - h\right)
\end{align*}
\]
\[-h\left(2 + a - \left\lceil \frac{m+1}{2} \right\rceil\right) - \left(b + \left\lceil \frac{m}{2} \right\rceil + h\right) - \left(a + \left\lceil \frac{m}{2} \right\rceil\right)\]

\[= -4b^2 + 4(n - m - 2h)b + 4n\left(\left\lceil \frac{m}{2} \right\rceil + h\right) - 2m^2 - 8mh - 4(m-1)\left(\left\lceil \frac{m}{2} \right\rceil - 1\right) + 4\left\lfloor \frac{m^2}{4} \right\rfloor - 4h^2 + 6h.\]

If $b = 0$, then by similar calculation, we also have the right most expression for $D'(U_{n,m-2,d+1}(a+1,b+1)) - D'(U_{n,m,d}(a,b))$ as above. Thus

\[\begin{align*}
\tau' & (U_{n,m-2,d+1}(a+1,b+1)) - \tau' (U_{n,m,d}(a,b)) \\
&= 2(n - 1)n - [D'(U_{n,m-2,d+1}(a+1,b+1)) - D'(U_{n,m,d}(a,b))] \\
&= 4b^2 - 4(n - m - 2h)b \\
&\quad + 2n^2 - 4n\left(\left\lceil \frac{m}{2} \right\rceil + h + \frac{1}{2}\right) + 2m^2 + 8mh \\
&\quad + 4(m-1)\left(\left\lceil \frac{m}{2} \right\rceil - 1\right) - 4\left\lfloor \frac{m^2}{4} \right\rfloor + 4h^2 - 6h \\
&\geq 4\left(\frac{n-m-2h}{2}\right)^2 - 4(n - m - 2h)\cdot \frac{n-m-2h}{2} \\
&\quad + 2n^2 - 4n\left(\left\lceil \frac{m}{2} \right\rceil + h + \frac{1}{2}\right) + 2m^2 + 8mh \\
&\quad + 4(m-1)\left(\left\lceil \frac{m}{2} \right\rceil - 1\right) - 4\left\lfloor \frac{m^2}{4} \right\rfloor + 4h^2 - 6h \\
&= \begin{cases} 
2m^2 + 2(2h-3)m + n^2 - 2n + 4 - 6h & \text{if } m \text{ is even} \\
2m^2 + 2(2h-3)m + n^2 - 2n + 4 - 6h \\
+ 2n - 2m + 3 & \text{if } m \text{ is odd} \\
\end{cases} \\
&\geq 2m^2 + 2(2h-3)m + n^2 - 2n + 4 - 6h \\
&\geq 2 \cdot 5^2 + 2(2h-3) \cdot 5 + n^2 - 2n + 4 - 6h \\
&= n^2 - 2n + 24 + 14h > 0. \]

Now the result follows. \qed

Let $U(n,p)$ be the set of unicyclic graphs with $n$ vertices and $p$ pendant vertices, where $0 \leq p \leq n - 3$. The case $p = 0$ is trivial.
Any graph in $U(n, n-3)$ may be obtained by attaching $n-3$ pendant vertices to vertices of a triangle, and then it is easily seen that $U_{n,3,3}$ attains maximum reverse degree distance in $U(n, n-3)$.

**Theorem 4.7.** Among graphs in $U(n, p)$ with $1 \leq p \leq n-4$,

(i) if $p = 1$, then $U_{n,4,n-2}(n-4, 0)$ is the unique graph with maximum reverse degree distance;

(ii) if $p = 2$, then $U_{n,4,n-2}$ is the unique graph with maximum reverse degree distance;

(iii) if $p = 3$ and $n = 7$, then $U_{7,3,4}$ is the unique graph with maximum reverse degree distance;

(iv) if $p = 3$ and $n > 7$ is odd, then $U_{n,4,n-2}(\frac{n-3}{2}, \frac{n-5}{2})$ for $k = 0, 1$ are the unique graphs with maximum reverse degree distance;

(v) if $p = 3$ and $n \geq 6$ is even, or $4 \leq p \leq n-4$, then $U_{n,4,n-p}$ is the unique graph with maximum reverse degree distance for $\left\lfloor \frac{n-p-1}{2} \right\rfloor > \frac{n+4}{6}$, $U_{n,3,n-p}$ and $U_{n,4,n-p}$ are the unique graphs with maximum reverse degree distance for $\left\lfloor \frac{n-p-1}{2} \right\rfloor = \frac{n+4}{6}$, and $U_{n,3,n-p}$ is the unique graph with maximum reverse degree distance for $\left\lfloor \frac{n-p-1}{2} \right\rfloor < \frac{n+4}{6}$.

**Proof.** Obviously, $U(n, 1) = \left\{ U_{n,m,n-\left\lfloor \frac{m+1}{2} \right\rfloor} : 3 \leq m \leq n-1 \right\}$.

By Lemma 4.5, $\text{rD}'(U_{n,m,n-\left\lfloor \frac{m+1}{2} \right\rfloor}) < \text{rD}'(U_{n,3,n-2}(n-3, 0))$ for odd $m > 3$, and $\text{rD}'(U_{n,m,n-\left\lfloor \frac{m+1}{2} \right\rfloor}) < \text{rD}'(U_{n,4,n-2}(n-4, 0))$ for even $m > 4$. Let $G_1 = U_{n,4,n-2}(n-4, 0)$ and $G_2 = U_{n,3,n-2}(n-3, 0)$. It is easily seen that

\[
\text{rD}'(U_{n,4,n-2}(n-4, 0)) - \text{rD}'(U_{n,3,n-2}(n-3, 0)) = \text{D}'(U_{n,4,n-2}(n-4, 0)) - \text{D}'(U_{n,4,n-2}(n-4, 0)) = 4[W(G_2) - W(G_1)] + [D_{G_2}(v_0) - D_{G_1}(v_0)] - [D_{G_2}(u_0) - D_{G_1}(u_0)] = 4(n-4) + (n-5) - 1 = 5n - 22 > 0.
\]

Then (i) follows.

Suppose that $2 \leq p \leq n-4$. Let $G \in U(n, p)$, and let $d$ and $m$ be, respectively, the diameter and girth of $G$. A diametrical path of $G$ contains at most $\left\lfloor \frac{m}{2} \right\rfloor + 1$ vertices on $C_m$ and two pendant vertices, and then at most $\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) + 2 + (n-m-p) = n-p + 3 - \left\lfloor \frac{m+1}{2} \right\rfloor$ vertices in $G$, implying that $d \leq n-p + 2 - \left\lfloor \frac{m+1}{2} \right\rfloor$. 

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Note that $U_{n,n-p}$ is equal to $U_{n,n-p} \left( \frac{n-p-2}{2}, \frac{n-p-2}{2} \right)$ if $p = 2, 3$ and $n - p$ is even, $U_{n,n-p} = U_{n,n-p} \left( \left\lfloor \frac{n-p}{2} \right\rfloor, \left\lfloor \frac{n-p-3}{2} \right\rfloor \right)$ if $p = 2, 3$ and $n - p$ is odd or $4 \leq p \leq n - 4$, and $U_{n,n-p} = U_{n,n-p} \left( \frac{n-p}{2}, \frac{n-p-1}{2} \right)$.

By Corollary 3.4 and Lemma 4.1,

$$D'(G) \leq D'(U_{n,n-p}) \leq D'(U_{n,n-p})$$

with equalities if and only if $G \cong U_{n,n-p}$ for $m = 3$, and

$$D'(G) \leq D'(U_{n,n-p}) \leq D'(U_{n,n-p})$$

with equalities if and only if $G \cong U_{n,n-p} \left( \frac{n-p-2}{2}, \frac{n-p-1}{2} \right)$ with $k = 0, 1$ if $p = 2, 3$ and $n - p$ is even, and $G \cong U_{n,n-p}$ if $p = 2, 3$ and $n - p$ is odd or $4 \leq p \leq n - 4$ for $m = 4$. If $m \geq 5$, then by Corollary 3.4 and Lemmas 4.1 and 4.6, we have

$$D'(G) \leq D'(U_{n,n-p}) \leq D'(U_{n,n-p})$$

where $i = \left\lfloor \frac{m-3}{2} \right\rfloor$. Thus $D'(G) < D'(U_{n,n-p})$ for odd $m > 3$, and $D'(G) < D'(U_{n,n-p})$ for even $m > 4$. We need only to compare $D'(U_{n,n-p})$ with $D'(U_{n,n-p})$. Let $G_3 = U_{n,n-p}$ and $G_4 = U_{n,n-p}$.

Suppose that $p = 2, 3$ and $n - p$ is even. By Lemma 2.3,

$$D'(U_{n,n-p}) = D'(U_{n,n-p}) = D'(U_{n,n-p})$$

It is easily seen that

$$D'(U_{n,n-p}) = D'(U_{n,n-p}) = 4[W(G_4) - W(G_3)] + (p - 1)[D_{G_4}(v_0) - D_{G_3}(v_0)]$$

$$+ [D_{G_4}(v_1) - D_{G_3}(v_2)] - (p - 2)[D_{G_4}(u) - D_{G_3}(u)]$$

$$- [D_{G_4}(u_0) - D_{G_3}(u_0)] - [D_{G_4}(u_1) - D_{G_3}(u_1)]$$
On reverse degree distance of unicyclic graphs 701

\[ rD'(U_{n,4,n-p}) - rD'(U_{n,3,n-p}) = 6 \left\lfloor \frac{n-1}{2} \right\rfloor - n - 4, \]

and thus (ii) and (v) follow easily.

Let \( U(n, \Delta) \) be the set of unicyclic graphs with \( n \) vertices and maximum degree \( \Delta \), where \( 2 \leq \Delta \leq n - 1 \). The cases \( \Delta = 2, n - 1 \) are trivial.

It is easily checked that \( U_{n,3,3} \) attains maximum reverse degree distance in \( U(n, n - 2) \).

**Theorem 4.8.** Among graphs in \( U(n, \Delta) \) with \( 3 \leq \Delta \leq n - 3 \),

(i) if \( \Delta = 3 \), then \( U_{n,4,n-2} \) is the unique graph with maximum reverse degree distance;

(ii) if \( \Delta = 4 \) and \( n = 7 \), then \( U_{7,3,4} \) is the unique graph with maximum reverse degree distance;

(iii) if \( \Delta = 4 \) and \( n > 7 \) is odd, then \( U_{n,4,n-3}(\frac{n-5}{2}, \frac{n-5}{2}) \) for \( k = 0, 1 \) are the unique graphs with maximum reverse degree distance;

(iv) if \( \Delta = 4 \) and \( n \geq 6 \) is even, or \( 5 \leq \Delta \leq n - 3 \), then \( U_{n,4,n-\Delta+1} \) is the unique graph with maximum reverse degree distance for \( \lfloor \frac{n-\Delta}{2} \rfloor > \frac{n+4}{6} \), \( U_{n,3,n-\Delta+1} \) and \( U_{n,4,n-\Delta+1} \) are the unique graphs with maximum reverse degree distance for \( \lfloor \frac{n-\Delta}{2} \rfloor = \frac{n+4}{6} \), and \( U_{n,3,n-\Delta+1} \) is the unique graph with maximum reverse degree distance for \( \lfloor \frac{n-\Delta}{2} \rfloor < \frac{n+4}{6} \).

**Proof.** Let \( G \in U(n, \Delta) \), and let \( d \) and \( m \) be the diameter and girth of \( G \) respectively.

First we show that \( d \leq n - \Delta + 3 - \left\lfloor \frac{m+1}{2} \right\rfloor \). Let \( u \) be a vertex of degree \( \Delta \) in \( G \), and let \( N_u \) be the union of vertex \( u \) and its neighbors in \( G \). Among the vertices in \( V(C_m) \cup N_u \), any diametrical path of \( G \), say
$P(G)$ with $|P(G) \cup V(C_m)| \leq 1$, $P(G)$ contains at most $3 \leq \left\lfloor \frac{m}{2} \right\rfloor + 2$ vertices. On the other hand, any diametrical path of $G$, say $P(G)$ with $|P(G) \cup V(C_m)| \geq 2$, contains at most $\left\lfloor \frac{m}{2} \right\rfloor + 1$ vertices of the cycle $C_m$, at most three vertices in $N_u \setminus V(C_m)$ if $u \notin V(C_m)$, and at most one vertex in $N_u \setminus V(C_m)$ if $u \in V(C_m)$. Thus among the vertices in $V(C_m) \cup N_u$, $P(G)$ contains at most $\left\lfloor \frac{m}{2} \right\rfloor + 4$ vertices if $u \notin V(C_m)$, and at most $\left\lfloor \frac{m}{2} \right\rfloor + 2$ vertices if $u \in V(C_m)$. Note that $|V(C_m) \cap N_u| \leq 1$ if $u \notin V(C_m)$, and $|V(C_m) \cap N_u| = 3$ if $u \in V(C_m)$. If $u \notin V(C_m)$, then $P(G)$ contains at most $\left\lfloor \frac{m}{2} \right\rfloor + 4 + (n - |V(C_m) \cup N_u|)$ vertices, and thus

$$d \leq \left\lfloor \frac{m}{2} \right\rfloor + 4 + (n - |V(C_m) \cup N_u|) - 1$$

$$= \left\lfloor \frac{m}{2} \right\rfloor + 4 + n - |V(C_m)| - |N_u| + |V(C_m) \cap N_u| - 1$$

$$\leq \left\lfloor \frac{m}{2} \right\rfloor + 4 + n - m - (\Delta + 1) + 1 - 1$$

$$= n - \Delta + 3 - \left\lfloor \frac{m + 1}{2} \right\rfloor.$$

Similarly, if $u \in V(C_m)$, then

$$d \leq \left\lfloor \frac{m}{2} \right\rfloor + 2 + (n - |V(C_m) \cup N_u|) - 1 \leq n - \Delta + 3 - \left\lfloor \frac{m + 1}{2} \right\rfloor.$$

By Corollary 3.4 and Lemmas 4.1 and 4.6, we have

$$\gamma' \left( G \right) \leq \gamma' \left( U_{n,3,d} \right) \leq \gamma' \left( U_{n,3,n-\Delta+1} \right)$$

for $m = 3$,

$$\gamma' \left( G \right) \leq \gamma' \left( U_{n,4,d} \right) \leq \gamma' \left( U_{n,4,n-\Delta+1} \right)$$

for $m = 4$, and

$$\gamma' \left( G \right) \leq \gamma' \left( U_{n,m,d} \right) \leq \gamma' \left( U_{n,m,n-\Delta+3-[\frac{m+1}{2}]} \right)$$

$$< \gamma' \left( U_{n,m-2i,n-\Delta+3-[\frac{m+1}{2}]+i} \right)$$

$$\leq \gamma' \left( U_{n,m-2i,n-\Delta+3-[\frac{m+1}{2}]+i} \right)$$

for $m \geq 5$, where $i = \left\lceil \frac{m-3}{2} \right\rceil$. Thus $\gamma' \left( G \right) \leq \gamma' \left( U_{n,3,n-\Delta+1} \right)$ for odd $m \geq 3$, and $\gamma' \left( G \right) \leq \gamma' \left( U_{n,4,n-\Delta+1} \right)$ for even $m \geq 4$. Now the theorem follows by similar arguments appeared in the proof of Theorem 4.7. □
Finally, we give the values of the maximum reverse degree distances in Theorems 4.7 and 4.8.

(i) For $U_{n,4,n-2}(n-4,0)$ with $n \geq 6$,
\[
D'(U_{n,4,n-2}(n-4,0)) = 4W(U_{n,4,n-2}(n-4,0)) + (3 - 2)D_{U_{n,4,n-2}(n-4,0)}(v_0) + (1 - 2)D_{U_{n,4,n-2}(n-4,0)}(u_0)
\]
\[
= 4W(U_{n,4,n-2}(n-4,0)) - 3(n-4)
\]
\[
= \frac{2}{3}n^3 - \frac{35}{3}n + 36,
\]
and thus
\[
\nu D'(U_{n,4,n-2}(n-4,0)) = 2(n-1)n(n-2) - D'(U_{n,4,n-2}(n-4,0))
\]
\[
= \frac{4}{3}n^3 - 6n^2 + \frac{47}{3}n - 36.
\]

(ii) For $U_{n,4,n-2}$ with $n \geq 6$,
\[
D'(U_{n,4,n-2}) = 4W(U_{n,4,n-2}) + (3 - 2)D_{U_{n,4,n-2}}(v_0) + (1 - 2)D_{U_{n,4,n-2}}(u_0)
\]
\[
+ (3 - 2)D_{U_{n,4,n-2}}(v_1) + (1 - 2)D_{U_{n,4,n-2}}(u_1)
\]
\[
= \begin{cases} 
\frac{2}{3}n^3 - \frac{3}{2}n^2 + \frac{1}{3}n + 12 & \text{if } n \text{ is even} \\
\frac{2}{5}n^3 - \frac{3}{2}n^2 + \frac{1}{3}n + \frac{27}{2} & \text{if } n \text{ is odd}, 
\end{cases}
\]
and thus
\[
\nu D'(U_{n,4,n-2}) = 2(n-1)n(n-2) - D'(U_{n,4,n-2})
\]
\[
= \begin{cases} 
\frac{4}{3}n^3 - \frac{9}{2}n^2 + \frac{11}{3}n - 12 & \text{if } n \text{ is even} \\
\frac{4}{3}n^3 - \frac{9}{2}n^2 + \frac{11}{3}n - \frac{27}{2} & \text{if } n \text{ is odd}.
\end{cases}
\]

(iii) For $U_{n,4,n-3}$ with odd $n \geq 7$,
\[
D'(U_{n,4,n-3}) = 4W(U_{n,4,n-3}) + (4 - 2)D_{U_{n,4,n-3}}(v_0) + (1 - 2)D_{U_{n,4,n-3}}(u_0)
\]
\[
+ (1 - 2)D_{U_{n,4,n-3}}(u_1)
\]
\[
+ (1 - 2)D_{U_{n,4,n-3}}(u_2)
\]
\[
= 4W(U_{n,4,n-3}) - \frac{1}{2}n^2 + \frac{19}{2}
\]
\[
= \frac{2}{3}n^3 - \frac{5}{2}n^2 + \frac{10}{3}n + \frac{47}{2},
\]
and thus

\[
D'(U_{n,4,n-3}) = 2(n-1)n(n-3) - D'(U_{n,4,n-3}) \\
= \frac{4}{3}n^3 - \frac{11}{2}n^2 + \frac{8}{3}n - \frac{47}{2}.
\]

(iv) For \(U_{n,4,n-p}\) with \(3 \leq p \leq n-4\),

\[
D'(U_{n,4,n-p}) = 4W(U_{n,4,n-p}) + [(p+1) - 2]Dv_{n,4,n-p}(v_0) \\
+ (1-2)Du_{n,4,n-p}(u_0) + (p-2)(1-2)Du_{n,4,n-p}(u) \\
+ (3-2)Du_{n,4,n-p}(v_2) + (1-2)Du_{n,4,n-p}(u_1) \\
\left\{ \begin{array}{ll}
\frac{2}{3}n^3 - \left(p - \frac{1}{2}\right)n^2 + \left(3p - \frac{17}{3}\right)n + \frac{p^2}{3} - \frac{p^2}{2} + \frac{14p}{3} + 10 & \text{if } n-p \text{ is even} \\
\frac{2}{3}n^3 - \left(p - \frac{1}{2}\right)n^2 + \left(3p - \frac{17}{3}\right)n + \frac{p^2}{3} - \frac{p^2}{2} + \frac{14p}{3} + \frac{7}{2} & \text{if } n-p \text{ is odd},
\end{array} \right.
\]

and thus

\[
D'(U_{n,4,n-p}) = 2(n-1)n(n-p) - D'(U_{n,4,n-p}) \\
= \left\{ \begin{array}{ll}
\frac{4}{3}n^3 - \left(p + \frac{5}{2}\right)n^2 - \left(p - \frac{17}{3}\right)n - \frac{p^3}{3} + \frac{p^2}{2} - \frac{11p}{3} - 10 & \text{if } n-p \text{ is even} \\
\frac{4}{3}n^3 - \left(p + \frac{5}{2}\right)n^2 - \left(p - \frac{17}{3}\right)n - \frac{p^3}{3} + \frac{p^2}{2} - \frac{14p}{3} - \frac{7}{2} & \text{if } n-p \text{ is odd}.
\end{array} \right.
\]

(v) For \(U_{n,3,n-p}\) with \(3 \leq p \leq n-4\),

\[
D'(U_{n,3,n-p}) = 4W(U_{n,3,n-p}) + [(p+1) - 2]Dv_{n,3,n-p}(v_0) \\
+ (1-2)Du_{n,3,n-p}(u_0) + (p-2)(1-2)Du_{n,3,n-p}(u) \\
+ (3-2)Du_{n,3,n-p}(v_1) + (1-2)Du_{n,3,n-p}(u_1) \\
\left\{ \begin{array}{ll}
\frac{2}{3}n^3 - \left(p - \frac{1}{2}\right)n^2 + \left(3p - \frac{11}{4}\right)n + \frac{p^3}{3} - \frac{p^2}{2} + \frac{2p}{3} & \text{if } n-p \text{ is even} \\
\frac{2}{3}n^3 - \left(p - \frac{1}{2}\right)n^2 + \left(3p - \frac{11}{4}\right)n + \frac{p^3}{3} - \frac{p^2}{2} + \frac{5p}{2} - \frac{7}{2} & \text{if } n-p \text{ is odd},
\end{array} \right.
\]

and thus

\[
D'(U_{n,3,n-p})
\]
\[
= 2(n - 1)n(n - p) - D'(U_{n,3,n-p}) \\
\begin{cases}
\frac{4}{3}n^3 - \left(p + \frac{5}{2}\right)n^2 - \left(p - \frac{11}{2}\right)n - \frac{p^3}{3} + \frac{p^2}{2} - \frac{2p}{3} & \text{if } n - p \text{ is even} \\
\frac{4}{3}n^3 - \left(p + \frac{5}{2}\right)n^2 - \left(p - \frac{11}{2}\right)n - \frac{p^3}{3} + \frac{p^2}{2} - \frac{5p}{2} + \frac{7}{2} & \text{if } n - p \text{ is odd.}
\end{cases}
\]

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REFERENCES


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