ON RAINBOW 4-TERM ARITHMETIC PROGRESSIONS

M. H. SHIRDAREH HAGHIGHI* AND P. SALEHI NOWBANDEGANI

Communicated by Michel Waldschimidt

Abstract. Let \([n] = \{1, \ldots, n\}\) be colored in \(k\) colors. A rainbow AP(\(k\)) in \([n]\) is a \(k\) term arithmetic progression whose elements have different colors. Conlon, Jungić and Radoićić [3] prove that there exists an equinumerous 4-coloring of \([4n]\) which is rainbow AP(4) free, when \(n\) is even. Based on their construction, we show that such a coloring of \([4n]\) also exists for odd \(n > 1\). We conclude that for nonnegative integers \(k \geq 3\) and \(n > 1\), every equinumerous \(k\)-coloring of \([kn]\) contains a rainbow AP(\(k\)) if and only if \(k = 3\).

1. Introduction and Results

In his efforts to prove the Fermat’s last theorem, Schur [7] proved that for each nonnegative integer \(k\), every \(k\)-coloring of \([n] = \{1, \ldots, n\}\) contains a monochromatic solution of the equation \(x + y = z\), provided that \(n\) is sufficiently large. Alekseev and Savchev [1] turn this problem to rainbow solutions of the equation \(x + y = z\); i.e., solutions in which \(x, y\) and \(z\) are colored in different colors. Later on, in 2003, Jungić et al. [4] considered the rainbow arithmetic progressions arising in \(k\)-colorings of \([n]\). Jungić and Radoićić [6] proved the following theorem which is conjectured in [4].

Keywords: Rainbow arithmetic progression, 4-term arithmetic progression, AP(4), AP(\(k\)).
Received: 26 Aug 2009, Accepted: 10 January 2010.
*Corresponding author
© 2011 Iranian Mathematical Society.
Theorem 1.1. [6, Theorem 1] For every equinumerous 3-coloring of $[3n]$, there exists a rainbow $AP(3)$.

What about more than 3 colors? Axenovich and Fon-Der-Flass [2] find equinumerous $k$-coloring of $[2mk]$ which contains no rainbow $AP(k)$, for every $k \geq 5$. The most challenging case is $k = 4$ [5, Problem 1]. Sicherman [5, page 4], makes equinumerous 4-coloring of $[4n]$, for $1 < n \leq 15$, without rainbow $AP(4)$. In 2007, Conlon, et al.[3] made a rainbow free equinumerous 4-coloring of $[4n]$, whenever $n$ is even.

Theorem 1.2. ([3, Theorem 2]) For every positive integer $m$, there exists an equinumerous 4-coloring of $[8m]$ with no rainbow $AP(4)$.

Based on their construction, we prove the following theorem. The proof appears in the next section.

Theorem 1.3. For every positive integer $m$, there exists an equinumerous 4-coloring of $[8m + 4]$ with no rainbow $AP(4)$.

Hence, we have the following theorem, which in some sense finishes the story of the existence of rainbow $AP(k)$ in equinumerous random $k$-coloring of $[kn]$, $n > 1$.

Theorem 1.4. For nonnegative integers $k \geq 3$ and $n > 1$, every equinumerous $k$-coloring of $[kn]$ contains a rainbow $AP(k)$ if and only if $k = 3$.

Proof. If $k = 3$, see Theorem 1. For $k = 4$, by theorems 2 and 3, a rainbow $AP(4)$ free 4-coloring of $[4n]$ is at hand for every $n > 1$. To construct 5-coloring of $[5n]$, we use easily a equinumerous 4-coloring of $[4n]$, which has no rainbow $AP(4)$ and then color $\{4n + 1, \ldots, 5n\}$ with the fifth color. Plainly, this equinumerous 5-coloring has no rainbow $AP(5)$. One can inductively use this construction to provide equinumerous $k$-coloring of $[kn]$ for every $k > 5$, $n > 1$, with no rainbow $AP(k)$. □

Note that the construction of equinumerous $k$-coloring of $[kn]$, $k \geq 5$, by Axenovich and Fan-Der-Flaass [2], is only for $n$ even.
2. Proof of Theorem 1.3

The following equinumerous 4-coloring of \([4n]\) in the proof of Theorem 2 in [3] is rainbow AP(4) free, whenever \(n = 2m\) is even.

Let \(W, X, Y\) and \(Z\) be our four colors and denote by \(A\) the block \(WXYZ\) and by \(B\) the block \(ZZWX\). The coloring

\[ \underbrace{A\ldots A}_{m \text{ times}} \underbrace{B\ldots B}_{m \text{ times}} \]

is the desired coloring of \([4n] = [8m]\).

Our construction for \([4n]\), whenever \(n = 2m + 1 > 1\), is as follows:

\[ \underbrace{XWYZ\ldots A}_{m \text{ times}} \underbrace{B\ldots B}_{m \text{ times}} \underbrace{Z}_{m \text{ times}} \]

What remains is to check that this coloring of \([8m + 4]\) is rainbow AP(4) free.

To get a contradiction, let \(t_1 < t_2 < t_3 < t_4\) denote the terms of a rainbow AP(4) in (***) with common difference \(d\). Obviously, \(d > 1\).

Since (**) is rainbow AP(4) free, we must have either \(t_1 \in \{1, 2, 3\}\) or \(t_4 = 8m + 4\) or both. Since the left side (the first \(4m + 3\) numbers) of (***) is colored only by \(W, X\) and \(Y\), therefore \(t_4 > 4m + 3\). Similarly, \(t_1 \leq 4m + 3\). Now, five cases occur.

**Case 1.** \(t_1 = 1\) and \(t_4 \neq 8m + 4\).

**subcase 1a.** \(t_1 < t_2 < t_3 < t_4\). If \(d \equiv 0 \pmod{4}\), then \(t_1\) and \(t_2\) are colored \(X\). If \(d \equiv 1 \pmod{4}\), then \(t_1\) and \(t_3\) are colored \(X\). If \(d \equiv 2 \pmod{4}\), then \(t_1\) and \(t_4\) are colored \(X\). If \(d \equiv 3 \pmod{4}\), then \(t_1\) and \(t_3\) are colored \(X\).

**subcase 1b.** \(t_1 < t_2 < t_3 \leq 4m + 3 < t_4\). If \(d \equiv 0 \pmod{4}\), then \(t_1\) and \(t_2\) are colored \(X\). If \(d \equiv 1 \pmod{4}\), then \(t_2\) and \(t_3\) are colored \(Y\). If \(d \equiv 2 \pmod{4}\), then \(t_1\) and \(t_3\) are colored \(X\). If \(d \equiv 3 \pmod{4}\), then \(t_2\) and \(t_4\) are colored \(W\).

**Case 2.** \(t_1 = 2\) and \(t_4 \neq 8m + 4\).

**subcase 2a.** \(t_1 < t_2 < t_3 < t_4\). If \(d \equiv 0 \pmod{4}\), then \(t_1\) and \(t_3\) are colored \(W\). If \(d \equiv 1 \pmod{4}\), then \(t_3\) and \(t_4\) are colored \(Z\). If \(d \equiv 2 \pmod{4}\), then \(t_1\) and \(t_2\) are colored \(W\). If \(d \equiv 3 \pmod{4}\), then \(t_2\) and \(t_4\) are colored \(X\).
subcase 2b. \( t_1 < t_2 < t_3 \leq 4m + 3 < t_4 \). If \( d \equiv 0 \pmod{4} \), then \( t_2 \) and \( t_3 \) are colored \( Y \). If \( d \equiv 1 \pmod{4} \), then \( t_1 \) and \( t_3 \) are colored \( W \). If \( d \equiv 2 \pmod{4} \), then \( t_1 \) and \( t_2 \) are colored \( W \). If \( d \equiv 3 \pmod{4} \), then \( t_1 \) and \( t_3 \) are colored \( W \).

Case 3. \( t_1 = 3 \) and \( t_4 \neq 8m + 4 \).

subcase 3a. \( t_1 < t_2 \leq 4m + 3 < t_3 < t_4 \). If \( d \equiv 0 \pmod{4} \), then \( t_1 \) and \( t_2 \) are colored \( Y \). If \( d \equiv 1 \pmod{4} \), then \( t_2 \) and \( t_4 \) are colored \( W \). If \( d \equiv 2 \pmod{4} \), then \( t_2 \) and \( t_3 \) are colored \( X \). If \( d \equiv 3 \pmod{4} \), then \( t_1 \) and \( t_2 \) are colored \( Y \).

subcase 3b. \( t_1 < t_2 < t_3 \leq 4m + 3 < t_4 \). If \( d \equiv 0 \pmod{4} \), then \( t_1 \) and \( t_2 \) are colored \( Y \). If \( d \equiv 1 \pmod{4} \), then \( t_1 \) and \( t_3 \) are colored \( W \). If \( d \equiv 2 \pmod{4} \), then \( t_1 \) and \( t_3 \) are colored \( Y \). If \( d \equiv 3 \pmod{4} \), then \( t_1 \) and \( t_2 \) are colored \( Y \).

Case 4. \( t_1 > 3 \) and \( t_4 = 8m + 4 \).

subcase 4a. \( t_1 < t_2 \leq 4m + 3 < t_3 < t_4 \). If \( d \equiv 0 \pmod{4} \), then \( t_3 \) and \( t_4 \) are colored \( Z \). If \( d \equiv 1 \pmod{4} \), then \( t_1 \) and \( t_3 \) are colored \( X \). If \( d \equiv 2 \pmod{4} \), then \( t_2 \) and \( t_3 \) are colored \( W \). If \( d \equiv 3 \pmod{4} \), then \( t_3 \) and \( t_4 \) are colored \( Z \).

subcase 4b. \( t_1 \leq 4m + 3 < t_2 < t_3 < t_4 \). If \( d \equiv 0 \pmod{4} \), then \( t_3 \) and \( t_4 \) are colored \( Z \). If \( d \equiv 1 \pmod{4} \), then \( t_1 \) and \( t_3 \) are colored \( X \). If \( d \equiv 2 \pmod{4} \), then \( t_2 \) and \( t_4 \) are colored \( Z \). If \( d \equiv 3 \pmod{4} \), then \( t_3 \) and \( t_4 \) are colored \( Z \).

Case 5. \( t_1 \in \{1, 2, 3\} \) and \( t_4 = 8m + 4 \). In this case, since \( t_4 \equiv 0 \pmod{4} \) and \( t_4 - t_1 = 3d \), it follows that \( d \equiv t_1 \pmod{4} \). Also, \( t_2 < 4m + 3 < t_3 \) and \( t_4 \) is colored \( Z \).

subcase 5a. \( 1 = t_1 < t_2 < 4m + 3 < t_3 < t_4 = 8m + 4 \). Here, by our construction, \( t_1 \) and \( t_3 \) are colored \( X \) because \( d \equiv t_1 \equiv 1 \pmod{4} \).

subcase 5b. \( 2 = t_1 < t_2 < 4m + 3 < t_3 < t_4 = 8m + 4 \). In this subcase, we have \( d \equiv t_1 \equiv 2 \pmod{4} \). Therefore, \( t_1 \) and \( t_2 \) are colored \( W \).

subcase 5c. \( 3 = t_1 < t_2 < 4m + 3 < t_3 < t_4 = 8m + 4 \). In this subcase, since \( d \equiv t_1 \equiv 3 \pmod{4} \), \( t_1 \) and \( t_2 \) are colored \( Y \).

\[ \square \]

Acknowledgments

The authors thank the referee for constructive comments and shortening the proof.
On rainbow 4-term arithmetic progressions

References


M. H. Shirdareh Haghighi
Department of Mathematics, Shiraz University, P.O. Box 71454, Shiraz, Iran.
Email: shirdareh@susc.ac.ir

P. Salehi Nowbandegani
Department of Mathematics, Shiraz University, P.O. Box 71454, Shiraz, Iran.
Email: pouria.salehi@gmail.com