An Evaluation of the Systems Reliability Using Fuzzy Lifetime Distribution

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Abstract

In this paper we consider the problem of the evaluation of system reliability, in which the lifetimes of components are described using a fuzzy exponential distribution. Formula of a fuzzy reliability function and its $\alpha$-cut set are presented. The fuzzy reliability of structures is defined on the basis of fuzzy number. Furthermore, the fuzzy reliability functions of $k$ -out-of-$m$ system, series systems and parallel systems and their FMTTF are discussed, respectively. Finally, some numerical examples are presented to illustrate how to calculate the fuzzy reliability function and its $\alpha$ – cut set.

Keywords: Fuzzy Reliability, Fuzzy Exponential Distribution, Survival Function.

1 Introduction

Reliability has always been a key role in the design of engineering systems. The most frequently used function in lifetime data analysis and reliability engineering is the reliability or survival function. This function gives the probability of an item operating for a certain amount of time without failure. Exponential distribution is the most commonly used in determining the lifetime reliability of a population of components (bearings, seals, gears etc.). A lot of methods and models in classical reliability theory assume that all parameter of lifetime density function are precise. But in the real world applications, randomness and fuzziness are often mixed up in the lifetimes of systems. However, the parameters sometimes cannot record precisely due to machine errors, experiment, personal judgment, estimation or some other unexpected situations. When parameter in the lifetime distribution is fuzzy, the conventional reliability system may have difficulty for handling reliability function. Singer [11] presented a fuzzy approach for fault tree and reliability of both serial and parallel systems analysis. Cai et al. [5, 6] gave a different insight by introducing the possibility assumption and fuzzy state assumption to replace the probability and binary state assumptions. Ramachandran et al. [14] considered the reliability network modeling. Cai et al. [7] also discussed the system reliability for coherent system based on the fuzzy state assumption and probability assumption.

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Utkin [15,16] discussed the fuzzy system reliability based on the binary state assumption and possibility assumption, and considered the fuzzy availability and unavailability and the fuzzy operational availability and unavailability [12,15-16]. Huang [10] proposed the fuzzy event “T is fuzzily bigger than t”, denoted as \( \{X > t\} \), where \( X \) is a random variable for the failure time and “\( > \)” is in a fuzzy sense. Viertl and Gurker [10] considered the fuzzy lifetime data. Chen [2,3] presented a method for fuzzy system reliability analysis using fuzzy number arithmetic operations. The collection of papers by Onisawa and Kacprzyk presents many other different approaches for the fuzzy reliability. Wu [18] proposed the fuzzy probability to discuss the system reliability in which the functioning probability of each component in the system was assumed as a nonnegative fuzzy number. Chen [3] presented a new method for analyzing the fuzzy system reliability based on vague sets. Kumar and et al. [8] and Mahapatra and Roy [9] presented a method for fuzzy system reliability analysis using idea of interval valued vague sets and intuitionist fuzzy number respectively [8]. Yao et al. (2008) applied a statistical methodology in fuzzy system reliability analysis and got a fuzzy estimation of reliability [17]. None of them found the reliability function of the system with fuzzy lifetime rate. In this paper we propose a general procedure to construct the fuzzy reliability function and its \( \alpha \)-cut set, when the lifetime rate is fuzzy. The lifetime rate of the system is represented by a trapezoidal fuzzy number. The rest of the paper is organized as follows: Section 2 reviews fuzzy theory probability. Section 3 describes the formalization of fuzzy survival functions. In section 4, we discuss the fuzzy reliability function of \( k \)-out-of-\( m \) system. Finally, Section 5 concludes the paper.

2 Fuzzy probability theory

In this section, we shall present the fuzzy distribution of both Binomial and exponential attributed to Buckley (2006).

2.1 Fuzzy Binomial distribution

In \( m \) independent Bernoulli experiment let us assume that \( p \), probability of a “success” in each experiment is not known precisely and needs to be estimated, or obtained from expert opinion. So that \( p \) value is uncertain and we substitute \( \tilde{p} \) for \( p \) and \( \tilde{q} \) for \( q \) so that there is a \( p \in [0,1] \) and a \( q \in [0,1] \) with \( p + q = 1 \). Now let \( \tilde{P}(r) \) be the fuzzy probability of \( r \) successes in \( m \) independent trials of the experiment. Under our restricted fuzzy algebra we obtain

\[
\tilde{P}(r)[\alpha] = \{C_m^r \tilde{p}^r \tilde{q}^{m-r} | s_{\alpha}\},
\]

for \( 0 \leq \alpha \leq 1 \), where now \( s_{\alpha} \) is the statement, “\( s_{\alpha} = \{(p,q) | p \in \tilde{p}[\alpha], q \in \tilde{q}[\alpha], p + q = 1\} \)”. If \( \tilde{P}(r)[\alpha] = \{P^L[\alpha], P^U[\alpha]\} \) then

\[
P^L[\alpha] = \min\{C_m^r \tilde{p}^r \tilde{q}^{m-r} | s\}\]

and \( P^U[\alpha] = \max\{C_m^r \tilde{p}^r \tilde{q}^{m-r} | s\}\)

and if \( \tilde{P}[a,b] \) be the fuzzy probability of \( x \) successes so that \( a \leq x \leq b \), then

\[
\tilde{P}([a,b])[\alpha] = \sum_{s=a}^{b} C_m^s \tilde{p}^s \tilde{q}^{m-s} | s_{\alpha}\]

if \( \tilde{P}[a,b][\alpha] = [P^L([a,b])[\alpha], P^U([a,b])[\alpha]] \) then:
\[ P^L([a,b])[\alpha] = \min \left\{ \sum_{x=a}^{b} C_m^x \alpha^m (1-\alpha)^{x-m} | s_\alpha \right\} \quad \text{and} \quad P^U([a,b])[\alpha] = \max \left\{ \sum_{x=a}^{b} C_m^x \alpha^m (1-\alpha)^{x-m} | s_\alpha \right\} \]

where \( s_\alpha \) is the same with past case.

2.2 Fuzzy exponential distribution

In general, if the lifetime of a component (X) is modeled by an Exponential distribution, then
\[
f(x, \lambda) = \lambda e^{-\lambda x}, \quad x > 0,
\]
where \( \lambda \) (the failure rate of the component) is crisp. Now consider fuzzy number of \( \tilde{\lambda} \) replace \( \lambda \) in Exponential distribution. In this case we show the fuzzy probability of obtaining a value in the interval \( [c,d] \), \( c \geq 0 \) is as \( \tilde{P}(c \leq X \leq d) \) and compute its \( \alpha \)-cut as follows
\[
\tilde{P}(c \leq X \leq d)[\alpha] = \left\{ \int_{c}^{d} \lambda e^{-\lambda x} \, d\lambda, \tilde{\lambda}[\alpha] \right\} = [P^L[\alpha], P^U[\alpha]],
\]
for all \( \alpha \), where
\[
P^L[\alpha] = \min \left\{ \int_{c}^{d} \lambda e^{-\lambda x} \, d\lambda, \tilde{\lambda}[\alpha] \right\} \quad \text{and} \quad P^U[\alpha] = \max \left\{ \int_{c}^{d} \lambda e^{-\lambda x} \, d\lambda, \tilde{\lambda}[\alpha] \right\}.
\]

3 Fuzzy reliability functions

Reliability or survival function \( (S(t)) \) is the probability a unit survives beyond time \( t \). Let the random variable X denote lifetime of a system components, also let X has density function \( f(x, \theta) \) (it is known as the lifetime density function), and cumulative distribution function \( F_X(t) = P(X \leq t) \), then the reliability function at time \( t \) is defined as:
\[
S(t) = P(X > t) = 1 - F_X(t), \quad t > 0,
\]
and the unreliability function \( Q(t) \) is the probability of failure or the probability of an item failing in the time interval \( [0,t] \)
\[
Q(t) = P(X \leq t) = F_X(t), \quad t > 0.
\]

Suppose that we want to calculate reliability of component, such that the lifetime has fuzzy exponential distribution. So we represent parameter \( \tilde{\lambda} \) with a trapezoidal fuzzy number as \( \tilde{\lambda} = (a_1, a_2, a_3, a_4) \) such that we can describe a membership function \( \xi_{\tilde{\lambda}}(x) \) in the following manner:
The $\alpha$ – cut $\tilde{\lambda}$ denote as follows:

$$\tilde{\lambda}[\alpha] = [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha].$$

Trapezoidal fuzzy numbers with $a_2 = a_3$ are called triangular fuzzy numbers. Generally $\tilde{a}$ will be crisp number with $c$ value if its membership function is:

$$\tilde{\xi}_a(x) = \begin{cases} 1 & x = c \\ 0 & \text{otherwise} \end{cases}$$

So fuzzy function of component reliability is as follows:

$$\tilde{S}(t)[\alpha] = \int_t^\infty \lambda e^{-\lambda x} dx | \lambda \in \tilde{\lambda}[\alpha] = \{ e^{-\lambda} | \lambda \in \tilde{\lambda}[\alpha] \}$$

According to that the $e^{-\lambda t}$ decreasing, then:

$$\tilde{S}(t)[\alpha] = [e^{-(a_x - (a_4 - a_3)\alpha)t}, e^{-(a_x -(a_2 - a_1)\alpha)t}]$$

$\tilde{S}(t)[\alpha]$ is a two dimensional function in terms of $\alpha$ and $t$ $(0 \leq \alpha \leq 1 \text{and} t > 0)$. For $t_0$, this is a fuzzy number. In this method, for every $\alpha$ – cut, reliability curve is like a band whose width depends on the ambiguity rate lifetime. The lesser uncertainty value results in less bandwidth; and if the parameter gets a crisp value, the lower and upper bounds will become equal, which means that reliability curve is in a classic state. This reliability band has properties follows:

(i) $\tilde{\tilde{S}}(0)[\alpha] = \tilde{1}$, i.e. no one starts off dead,

(ii) $\tilde{\tilde{S}}(\infty)[\alpha] = \tilde{0}$, i.e. everyone dies eventually,

(iii) $\tilde{\tilde{S}}(t_1)[\alpha] \geq \tilde{\tilde{S}}(t_2)[\alpha] \Leftrightarrow t_1 \leq t_2$, i.e. band of $\tilde{\tilde{S}}(t)[\alpha]$ declines monotonically,

$$\tilde{\tilde{S}}(t_1)[\alpha] \geq \tilde{\tilde{S}}(t_2)[\alpha] \text{ if and only if } S^{(t_1)}(x)[\alpha] \geq S^{(t_2)}(x)[\alpha] \text{ and } S^{(t_1)}(x)[\alpha] \geq S^{(t_2)}(x)[\alpha] \text{ for all } \alpha \in [0,1], \text{ where } "\geq" \text{ means } "\text{greater than or equal to"}.\)
When the lifetimes have fuzzy exponential distributed then:

\[
MTTF[\alpha] = \int_{0}^{\infty} \lambda e^{-\lambda x} d\lambda = \lambda \tilde{\lambda}[\alpha] = \left\{ \frac{1}{\lambda} \mid \lambda \in \tilde{\lambda}[\alpha] \right\}
\]

**Example 1**

Let lifetime of electronic component is modeled by an Exponential distribution with fuzzy parameter \( \tilde{\lambda} \) that \( \tilde{\lambda} \) is "about 0.7 to 0.85" ( \( \tilde{\lambda} = (0.7,0.75,0.8,0.85) \)). Then \( \alpha \)-cut of fuzzy system reliability and FMTTF is given by

\[
\tilde{S}(t)[\alpha] = \left[ e^{-0.85 \alpha t + 0.05 \alpha}, e^{-0.7 \alpha t + 0.05 \alpha} \right],
\]

\[
MTTF[\alpha] = \left[ \frac{1}{0.85 - 0.05 \alpha}, \frac{1}{0.7 + 0.05 \alpha} \right],
\]

for all \( \alpha \).

(i) If \( t = 0.4 \) then fuzzy reliability is as:

\[
\tilde{S}(0.4)[\alpha] = \left[ e^{-0.34 + 0.02 \alpha}, e^{-0.28 - 0.02 \alpha} \right].
\]

(ii) If \( \alpha = 0 \) then reliability band is as:

\[
\tilde{S}(t)[0] = \left[ e^{-0.85 t}, e^{-0.7 t} \right].
\]

**4 k Out-of m system**

A \( k \)-out-of-\( m \) system have \( m \) component such that system works if at least \( k \) out of \( m \) components is working. Each component \( i \) can be represented by a Bernoulli random variable \( Y_i \) with fuzzy reliability (fuzzy survival probability) \( \tilde{S}(t) \) and unreliability (fuzzy failure probability) \( \tilde{Q}(t) \). That is

![Image](https://via.placeholder.com/150)

Fig 1. Fuzzy reliability with \( t = 0.4 \)

![Image](https://via.placeholder.com/150)

Fig 2. \( \alpha \)-cut of fuzzy survival function (\( \alpha = 0 \))
\[ Y_i = \begin{cases} 1, & \text{with fuzzy probability } \tilde{S}(t) \\ 0, & \text{with fuzzy probability } \tilde{Q}(t) \end{cases} \]

Then, \( Z = \sum_{i=1}^{m} Y_i \) the number of survivors at time \( t \) has the fuzzy Binomial distribution.

So, for a \( k \)-out-of-\( m \) system consisting of \( m \) independent and identical components, the fuzzy system reliability function is given by

\[ \tilde{R}(t)[\alpha] = \tilde{P}(Z \geq k) = \left\{ \sum_{j=k}^{m} C_{m}^{j} S(t)^{j} Q(t)^{m-j} \right\} |S_\alpha|, \]

For \( 0 < \alpha < 1 \) and \( t > 0 \), where \( S \) is the statement,

\[ S_\alpha = \{(S(t),Q(t)) | S(t) \in \tilde{S}(t)[\alpha], Q(t) \in \tilde{Q}(t)[\alpha], S(t) + Q(t) = 1}\] Both parallel and series systems are special cases of the \( k \)-out-of-\( m \) system. A series system is equivalent to an \( m \)-out-of-\( m \) system while a parallel system is equivalent to a 1-out-of-\( m \) system. In the first especially case, \( k \)-out-of-\( m \) system is reducing at series systems and its fuzzy reliability function with modeled fuzzy exponential is as:

\[ \tilde{R}(t)[\alpha] = \tilde{P}(Z \geq m) = \{S(t)^{m} |S_\alpha| = [e^{-(a_{1}-(a_{1}-(a_{2}-a_{3})\alpha)^{m}}}, e^{-(b_{1}+(a_{2}-a_{3})\alpha)^{m}}] \], \]

In the second especially case, \( k \)-out-of-\( m \) system is reducing at parallel systems and its \( \alpha \) - cut fuzzy reliability function with modeled fuzzy exponential is as:

\[ \tilde{R}(t)[\alpha] = \tilde{P}(Z \geq 1) = \{1-Q^{m}(t) |S_\alpha| = [1-(1-e^{-(a_{1}-(a_{1}-(a_{2}-a_{3})\alpha)^{m}})}^{m}, 1-(1-e^{-(a_{1}+(a_{2}-a_{3})\alpha)^{m}})^{m}] \]. \]

The FMTTF of \( k \)-out-of-\( m \) system calculated as follows:

\[ MTTF[\alpha] = \left\{ \sum_{j=k}^{m} C_{m}^{j} S(t)^{j} Q(t)^{m-j} \right\} dt |S_\alpha| = \left\{ \sum_{j=k}^{m} \frac{1}{\lambda_j} \right\} |\lambda \in \tilde{\lambda}[\alpha]| \]

Example 2

Consider a 4-out-of-5 system consisting of three independent and identical components. Lifetime is modeled as Example 1. Then the reliability function is given by

\[ \tilde{R}(t)[\alpha] = \tilde{P}(Z \geq 4) = \{\sum_{j=4}^{5} C_{5}^{j} S(t)^{j}(1-S(t))^{5-j} |S(t) \in \tilde{S}(t)[\alpha] \}
\]
\[ = \{5S(t)^{4}(1-S(t)) + S(t)^{5} |S(t) \in \tilde{S}(t)[\alpha] \} \]

According to that the \( 5S(t)^{4}(1-S(t)) + S(t)^{5} \) increasing, then the \( \alpha \)-cut of fuzzy reliability function is given by

\[ \tilde{R}(t)[\alpha] = [e^{-3.4t+0.2\alpha}(5-4e^{-0.8t+0.05\alpha}), e^{-2.8t-0.2\alpha}(5-4e^{-0.7t-0.05\alpha})] \]

(i) If \( t = 0.5 \), then fuzzy reliability is as:
\[
\tilde{R}(0.5)[\alpha] = \left[ e^{-1.7+0.1\alpha} (5 - 4e^{-0.425+0.025\alpha}), e^{-1.4-0.1\alpha} (5 - 4e^{-0.35-0.025\alpha}) \right]
\]

(ii) If \( \alpha = 0 \) then reliability band is as:

\[
\tilde{R}(t)[0] = \left[ e^{-3.4t} (5 - 4e^{-0.85t}), e^{-2.8t} (5 - 4e^{-0.7t}) \right]
\]

The \( \alpha \) – cut of FMTTF of \( k \)-out-of- \( m \) is as follows:

\[
M\tilde{T}TF[\alpha] = \left[ \frac{9}{20(0.85 - 0.05\alpha)} \right], \left[ \frac{9}{20(0.7 + 0.05\alpha)} \right]
\]

5 Conclusions

The fuzzy probability theory has been successfully applied to the reliability system in this paper. Whenever, the lifetimes of components and lifetime rate contain randomness and fuzziness respectively, conventional reliability system is not feasible. Thus we invoked successfully the fuzzy distribution to overcome this difficulty. Our fuzzy theory is a generalization of conventional theory since if lifetime rate is crisp; it turns in to conventional reliability system. The proposed methodology may be used for a more general problem when lifetimes are distributed according to other fuzzy probability distributions; e.g. fuzzy normal distribution.

References