On the Behavior of Solution of a Certain Ordinary Differential Equation

Bahman Mehri, M. Hassan Nojumi

Department of Mathematical Sciences Sharif University of Technology, Tehran, Iran

Abstract

Results regarding bounded ness, regularity and asymptotic behavior are important both theoretically. We investigate the behavior of a certain $n$th order nonlinear ordinary differential equation in regard with bounded ness, $L^2$-regularity and asymptotic behavior of solutions. We show, using energy methods, that, under suitable conditions, every solution and its derivatives up to order $n-1$ behave asymptotically like the constant zero function.

Keywords: Bounded ness, Regularity, Asymptotic behavior, Nonlinear ODE.

1. Introduction

Here we are concerned with the $n$th order nonlinear differential equation

$$x^{(n)} + \sum_{i=0}^{n-1} p_i(t) f_i(x^{(i)}) = g(t)$$

(1)

with the $p_i$ and $g$ continuous functions on $]0, \infty[$, and the $f_i$ continuous on $IR$. Our aim is to present sufficient conditions for bounded ness and $L^2$-regularity of solutions. We also investigate the asymptotic behavior of solutions and show that under suitable conditions, every solution all its derivatives up to $(n-1)$st order tend to zero as $t$ approaches infinity.

We use energy methods [3, 7] and ideas similar to those Bihari [1], Ezeilo [4], Haas [5], Harrow [6], Levin and Nohel [9], Mehri and Zarghami [10], Nelson [11], and Swick [14] to establish our results. Besides being of high theoretical value, issues of bounded ness, regularity, and asymptotic behavior of solutions are of paramount importance in many applied fields, including dynamical system, and mathematical physics [2, 11, 12].
2. Results on Boundedness

**Theorem 1.** If the following conditions hold:

(i) $p_{n-2}$ is nonnegative and increasing on $[0,\infty[$,

(ii) $p_{n-1}$ is nonnegative on $[0,\infty[$,

(iii) $sf_{n-1}(s) \geq 0$ for all $s \in IR$,

(iv) There exists a positive constant $K > 0$ such that $|f_i(s)| \leq K|s|$ for all $i = 0,1,\ldots,n-3$ and all $s \in IR$,

(v) $\frac{g}{\sqrt{p_{n-2}}} \in L^1([0,\infty[)$,

(vi) $f_{n-2}$ is nonnegative on $IR$ and its integral

$$F_{n-2}(v) := \int_0^v f_{n-2}(s) ds$$

has the property $\lim_{v \to \pm\infty} F_{n-2}(v) = +\infty$,

then for every set of functions $b,\alpha_0,\alpha_1,\ldots,\alpha_{n-2}$ defined on $[0,\infty[$ with the following properties:

(vii) Each $\alpha_i, i = 0,1,\ldots,n-3$ is nonnegative and decreasing on $[0,\infty[$,

(viii) $b$ is nonnegative and increasing on $[0,\infty[$,

(ix) For $i = 0,1,\ldots,n-3$:

$$\sqrt[\alpha_i]{\alpha_{i+1}} , \sqrt[\alpha_{n-2}]{b \cdot p_{n-2}} , |p_i| \sqrt[b]{\frac{b}{\alpha_i} \cdot p_{n-2}} \in L^1([0,\infty[)$$

and for every $x$ a solution of (1), the following functions are bounded on $[0,\infty[$:

$$\frac{x^{(n-1)}}{\sqrt{p_{n-2}}} , x^{(n-2)} , \frac{x^{(i)}}{\sqrt{b/\alpha_i}} , i = 0,1,\ldots,n-2$$

Proof. By the standard technique, introducing $y_i := x^{(i)}$, we transform (1) into the first order system

$$\begin{cases}
    y_i = y_i + 1 & \text{for } i = 0,1,\ldots,n-2 \\
    y_{n-1} = g - \sum_{i=0}^{n-1} p_i f_i(y_i)
\end{cases} \quad (2)$$

Differentiating the energy function defined by

$$E(t) := \sum_{i=0}^{n-2} \frac{\alpha_i}{b} y_i^2 + \frac{1}{p_{n-2}} y_{n-1}^2 + 2F_{n-2}(y_{n-2})$$
we get
\[
\frac{dE}{dt} = \sum_{i=0}^{n-2} \left( \frac{\alpha_i}{b} - \frac{\alpha_i b_i}{b^2} \right)y_i^2 + \sum_{i=0}^{n-2} 2 \frac{\alpha_i}{b} y_i y_{i+1}
\]
\[
- \frac{p_{n-2}}{p_{n-2}} y_{n-1}^2 + 2 \frac{1}{p_{n-2}} y_{n-1} g - 2 \frac{p_{n-1}}{p_{n-2}} y_{n-1} f_{n-i}(y_{n-1})
\]
\[
- \sum_{i=0}^{n-2} 2 \frac{p_i}{p_{n-2}} y_{n-1} f_i(y_i)
\]

Assumptions of the Theorem 1 yield
\[
\frac{dE}{dt} \leq 2 \frac{1}{p_{n-2}} |g| y_{n-1} + 2 \frac{\alpha_{n-2}}{b} \frac{y_{n-1}}{y_{n-1}} |y_{n-1}|
\]
\[
+ \sum_{i=0}^{n-3} \left( 2 \frac{\alpha_i}{b} \frac{|y_i|}{|y_{i+1}|} + 2 \kappa \frac{|p_i|}{p_{n-2}} \frac{|y_{n-1}|}{|y_i|} \right)
\]

Now by the nonnegativity. Of \( F_{n-2} \) and the inequality \( 2|AB| \leq A^2 + B^2 \) we have
\[
2 \frac{1}{p_{n-2}} \frac{|g|}{y_{n-1}} \leq \left[ \frac{|g|}{\sqrt{p_{n-2}}} + \frac{|g|}{\sqrt{p_{n-2}}} \right] E
\]
\[
2 \frac{\alpha_{n-2}}{b} \frac{|y_{n-1}|}{|y_{n-1}|} \leq \frac{\alpha_{n-2}}{b} \frac{|y_{n-1}|}{p_{n-2}} E
\]
\[
2 \frac{|p_i|}{p_{n-2}} \frac{|y_{n-1}|}{|y_i|} \leq \frac{|p_i|}{\frac{b}{\alpha_i} p_{n-2}} E
\]
\[
2 \frac{\alpha_i}{b} \frac{|y_i|}{|y_{i+1}|} \leq \frac{\alpha_i}{\alpha_{i+1}} E
\]

This result in
\[
\frac{dE}{dt} \leq \frac{|g|}{\sqrt{p_{n-2}}} + \Phi E \quad (3)
\]

With
\[
\Phi := \frac{|g|}{\sqrt{p_{n-2}}} + \frac{\alpha_{n-2}}{b} \frac{p_{n-2}}{p_{n-2}} + \sum_{i=0}^{n-1} \left( \frac{\alpha_i}{\alpha_{i+1}} + K |p_i| + L |p_i| + \frac{b}{\alpha_i} \frac{p_{n-2}}{p_{n-2}} \right) \quad (4)
\]

Integrating (3), by assumption (v) and the Gronwall’s lemma, we arrive at
\[ E(t) \leq C \exp \left( \int_0^t \Phi(s) \, ds \right), \text{ for all } t \in [0, +\infty[ \]

with \( C \) some positive constant. Now \( \Phi \in L^1([0, +\infty[) \) by assumptions (v) and (ix). Therefore \( E \) is bounded and hence so are \( F_{n=2}(y_{n-2}), \sqrt{p_{n-2}}, \) and \( y_{n-2} \sqrt{b/\alpha} \) for \( i = 0,1,\ldots, n-2 \). By (vi), we conclude that \( y_{n-2} \) is also bounded. Making the proof of Theorem 1 complete.

**Remark 1.** The proof of Theorem 1 is valid for case \( g \equiv 0 \). Hence Theorem 1 is valid also for the corresponding homogeneous equation.

**Remark 2.** Theorem 1 remains valid if we relax the hypothesis \( p'_{n-2} \geq 0 \) in (i) by the weaker assumption

\[ \frac{p'_{n-2}}{p_{n-2}} \in L^1([0, +\infty[) \]

**Theorem 2.** Theorem 1 remains valid for the nonhomogeneous equation with \( g \neq 0 \) if the assumption (i), (ii), (iii), and (v) are respectively replaced by

(i) \( p_{n-2} \) is nonnegative on \([0, \infty[\)

(ii) \( p_{n-1} \) is no positive on \([0, \infty[\)

(iii) There exists a constant \( M > 0 \) such that

\[ \forall s \in \mathbb{R} \quad 0 < sf_{n-1}(s) \leq Ms^2 \]

\[ \forall t \in [0, +\infty[ \quad p'_{n-2}(t) + 2Mp_{n-2}(t)p_{n-1}(t) > 0 \]

(v) \[ \frac{g}{\sqrt{p_{n-2}}}, \quad \frac{g^2}{p_{n-2} + 2Mp_{n-2}p_{n-1}} \in L^1([0, +\infty[) \]

**Proof.** Starting in the proof of Theorem 1, this time we obtain

\[ \frac{dE}{dt} < -2M \frac{p_{n-1}}{p_{n-2}} y_{n-1}^2 - \frac{p'_{n-2}}{p_{n-2}} y_{n-1}^2 + 2 \frac{1}{p_{n-2}} |y_{n-1}| |g| \]

\[ + \sum_{i=0}^{n-3} 2 \frac{\alpha_i}{b} |y_i| |y_{i+1}| + \sum_{i=0}^{n-3} 2K \frac{|p_i|}{p_{n-2}} |y_{n-1}| |y_i| \]

Noting \( g \neq 0 \), the first three terms are equal to
The assumptions together with inequalities derived in proof of Theorem 1 result in

\[
\frac{dE}{dt} \leq \frac{g^2}{p'_{n-2} + 2Mp_{n-2}p_{n-1}} + \left( \Phi - \frac{|g|}{\sqrt{p_{n-2}}} \right) E,
\]

With \( \Phi \) defined in (4). Integrating this inequality and using assumption (v)', we again arrive at a Gronwall-type inequality; making the rest of the proof exactly as that of Theorem 1; and thus completing proof of Theorem 2.

**Remark 3.** Theorem 2 remains valid if instead of (ii)' we have (ii) in Theorem 1, and instead of (iii)', we have

(iii)’’ There exists a constant \( M > 0 \) such that

\[
\forall s \in IR \quad sf_{n-1}(s) \geq M s^2 > 0
\]
\[
\forall t \in ]0, +\infty[ \quad p'_{n-2}(t) + 2M p_{n-2}(t)p_{n-1}(t) > 0
\]

### 3 Results on \( L^2 \) Regularity

**Theorem 3.** Under the assumptions Theorem 2 every solution of (1) satisfies

\[
\sqrt{\alpha' b'} \chi^{(i)} \quad \sqrt{\frac{\alpha}{b}} \chi^{(i)} \in L^2(]0, +\infty[)
\]

for \( i = 0, 1, ..., n - 2 \).

**Proof.** Starting the same way as proof of Theorem 1, we arrive at

\[
\sum_{i=0}^{n-2} \left( \frac{\alpha b'}{b^2} \chi^{(i)} \right)^2 + \frac{p'_{n-2} + 2M p_{n-2}p_{n-1}}{p_{n-2}^2} \leq -\frac{dE}{dt} + \frac{|g|}{\sqrt{p_{n-2}}} + \Phi E
\]

Integrating both sides, using the assumptions of the theorem, and the bound ness of \( E \) and \( dE/dt \) as demonstrated in the proof of Theorem 1, we conclude that for all \( t \in ]0, +\infty[ \), the expression

\[
\sum_{i=0}^{n-2} \int_0^t \frac{\alpha b'}{b^2} \chi_i^2 ds - \sum_{i=0}^{n-2} \int_0^t \alpha''_i \chi_i^2 ds + \int_0^t \frac{p_{n-2} + 2Mp_{n-2}p_{n-1}}{p_{n-2}^2} y_{n-1}^2 ds
\]

is bounded, thereby making the proof of Theorem 3 complete.
The last statement in the proof of Theorem 3 takes us to

**Corollary.** If there exist functions \( b, \alpha_0, \ldots, \alpha_{n-2} \) satisfying the hypotheses of Theorem 2 together with

\[
(x) \quad 0 < \inf_{0 \leq t < \infty} \left\{ \frac{p'_{n-2} + 2Mp_{n-2}p_{n-1}}{p_{n-2}} \right\} < \infty
\]

then

\[
x^{(n-1)} \in L^2(0, +\infty].
\]

If, in addition,

\[(xi) \quad \text{For every } i = 0,1,\ldots,n-2,
\]

\[
0 < \inf_{0 \leq t < \infty} \left\{ \frac{\alpha_i}{b} \right\} < \infty \quad \text{or} \quad 0 < \inf_{0 \leq t < \infty} \left\{ \frac{\alpha_i b^i}{b^2} \right\} < \infty,
\]

then every solution of (1) belongs to the Sobolev space \( H^{n-1}(0, +\infty] \).

\[
\lim_{t \to +\infty} x^{(i)}(t) = 0, \quad \text{for } i = 0,\ldots,n-1.
\]

Thereby establishing Theorem 4.

**References**