A CHARACTERIZATION OF CONVEX FUNCTIONS AND ITS APPLICATION TO OPERATOR MONOTONE FUNCTIONS

MASATOSHI FUJII, YOUNG OK KIM AND RITSUO NAKAMOTO

Communicated by M. S. Moslehian

Abstract. We give a characterization of convex functions in terms of difference among values of a function. As an application, we propose an estimation of operator monotone functions: If $A > B \geq 0$ and $f$ is operator monotone on $(0, \infty)$, then $f(A) - f(B) \geq f(\|B\| + \epsilon) - f(\|B\|) > 0$, where $\epsilon = \|(A-B)^{-1}\|^{-1}$. Moreover it gives a simple proof to Furuta’s theorem: If $\log A > \log B$ for $A, B > 0$ and $f$ is operator monotone on $(0, \infty)$, then there exists a $\beta > 0$ such that $f(A^\alpha) > f(B^\alpha)$ for all $0 < \alpha \leq \beta$.

1. Introduction

For a twice differentiable real-valued function $f$, its convexity is characterized by $f'' \geq 0$. Since there are many non-differentiable convex functions, we consider a characterization of general convex functions. We cannot use the differentiation, but the average rate of change is available. Roughly speaking, we claim that the convexity of a function is characterized by the non-decreasingness of average rate of change. It seems to be natural as a generalization of the condition $f'' \geq 0$. Actually it will be formulated as Lemma 1 in the next section.

To explain operator monotone functions, we introduce the operator order $A \geq B$ among selfadjoint operators $A, B$ on a Hilbert space $H$ by $(Ax, x) \geq (Bx, x)$ for all $x \in H$. In particular, $A$ is positive if $A \geq 0$, i.e., $(Ax, x) \geq 0$ for all $x \in H$. 

Date: Received: Sep. 15, 2013; Accepted: Oct. 7, 2013.

* Corresponding author.

2010 Mathematics Subject Classification. Primary 47A63; Secondary 47B10, 47BA30.

Key words and phrases. Convex function, operator monotone function, Löwner-Heinz inequality, chaotic order.
Next, a positive operator $A$ is said to be strictly positive, denoted by $A > 0$, if $A \geq c$ for some constant $c > 0$. So $A > B$ means that $A - B > 0$.

A real-valued continuous function $f$ defined on $[0, \infty)$ is called operator monotone if it preserves the operator order, i.e., $f(A) \geq f(B)$ for $A \geq B \geq 0$. One of the most important examples is the power function $t \mapsto t^p$ for $0 \leq p \leq 1$ (Löwner–Heinz inequality). In general, $f$ is called operator monotone on an interval $J$ if $f(A) \geq f(B)$ for $A \geq B$ whose spectra contained in $J$. For this, we pose $\log t$ as a fundamental example of an operator monotone function on $(0, \infty)$.

Very recently, Moslehian and Najafi [9] proposed an excellent extension of the Löwner–Heinz inequality as follows:

**Theorem MN.** If $A > B \geq 0$ and $0 < r \leq 1$, then $A^r - B^r \geq \|A\|^r - (\|A\| - \epsilon)^r > 0$, and $\log A - \log B \geq \log \|A\| - \log(\|A\| - \epsilon) > 0$, where $\epsilon = \|(A - B)^{-1}\|^{-1}$.

In this note, we apply our characterization of concave functions and give an improvement and a generalization of Theorem MN (Theorem 5). As another application, we can give a short proof to a recent result due to Furuta [6, Theorem 2.1], which is an operator inequality related to operator monotone functions and chaotic order, i.e., the order defined by $\log A \geq \log B$ among positive invertible operators.

2. A CHARACTERIZATION OF CONVEX FUNCTIONS

In this section, we propose an elementary characterization of convex functions. We essentially use average rate of change.

**Lemma 2.1.** A real valued continuous function $f$ on an interval $J = [a, b]$ with $b \in (-\infty, +\infty]$ is convex (resp. concave) if and only if, for each $0 < \epsilon < b - a$,

\[ D_\epsilon(t) = f(t + \epsilon) - f(t) \]

is non-decreasing (resp. non-increasing) on $[a, b - \epsilon]$.

**Proof.** Suppose that $f$ is convex on $J$. Take $s, t \in J$ with $s < t$ and $t + \epsilon \in J$. We may assume that $t - s < \epsilon$. Let $y = L(t)$ be the linear function through $(s, f(s))$ and $(s + \epsilon, f(s + \epsilon))$. Then we have

\[ L(t) \geq f(t) \text{ and } L(t + \epsilon) \leq f(t + \epsilon) \]

by the convexity of $f$. Hence it implies that

\[ D_\epsilon(t) = f(t + \epsilon) - f(t) \]
\[ \geq L(t + \epsilon) - L(t) \]
\[ = L(s + \epsilon) - L(s) \] by the linearity of $L$
\[ = f(s + \epsilon) - f(s) \]
\[ = D_\epsilon(s), \]

as desired.

Conversely suppose that $D_\epsilon(t)$ is non-decreasing. Take $t, s \in J$ with $s < t = s + 2\epsilon$. Since $D_\epsilon(s) \leq D_\epsilon(s + \epsilon)$, we have

\[ 2f\left(\frac{s + t}{2}\right) = 2f(s + \epsilon) \leq f(s + 2\epsilon) + f(s) = f(t) + f(s). \]

So $f$ is convex. \qed
Corollary 2.2. If \( f \) is strictly increasing and concave on an interval \([a, b + \delta]\) in \( \mathbb{R} \) for some \( \delta > 0 \), then for each \( 0 < \epsilon \leq \delta \), \( D_\epsilon(t) \geq D_\epsilon(b) > 0 \) for all \( t \in [a, b] \).

Remark 2.3. Analogous argument on convexity of functions as above has been done in [8, page 2].

3. Applications to Operator monotone functions

As an application of Corollary 2.2, we give an estimation of operator monotone functions.

Lemma 3.1. If \( f \) is non-constant and operator monotone on the interval \( \mathbb{R}_+ = [0, \infty) \), then \( f \) is strictly increasing.

Proof. First of all, we note that \( f \) is non-decreasing. Next we suppose that \( f'(c) = 0 \) for some \( c > 0 \). Noting that the Löwner matrix

\[
\begin{pmatrix}
f'(c) & f^{[1]}(c,d) \\
f^{[1]}(d,c) & f'(d)
\end{pmatrix}
\]

is positive semidefinite for any \( d > 0 \) by the operator monotonicity of \( f \), where \( f^{[1]}(c,d) = \frac{f(c)-f(d)}{c-d} \) is the divided difference.

Therefore its determinant is nonnegative, so that \( f^{[1]}(c,d) = 0 \) for any \( d > 0 \). This means that \( f \) is constant, which is a contradiction. Consequently we have \( f' > 0 \). \( \square \)

Lemma 3.2. If \( C \geq 0 \) and \( f \) is a concave and strictly increasing function on an interval \([a, d)\) containing the spectrum of \( C \), then for each \( 0 < \epsilon < d - \|C\| \),

\[ f(C + \epsilon) \geq f(C) + D_\epsilon(\|C\|). \]

Proof. We first note that for a given \( 0 < \epsilon < d - \|C\| \), we can take \( c > 0 \) satisfying \( 0 < c < d \) and \( \epsilon < c - \|C\| \). Applying Corollary 2.2 to \( b = \|C\| \) and \( \delta = c - \|C\| \), it follows that

\[ f(C + \epsilon) - f(C) \geq D_\epsilon(\|C\|). \] \( \square \)

We here give a precise estimation of [6, Theorem 2.1] and [8, Proposition 2.2], cf. [9].

Theorem 3.3. If \( A > B \geq 0 \) and \( f \) is non-constant operator monotone on \([0, \infty)\), then \( f(A) - f(B) \geq f(\|B\| + \epsilon) - f(\|B\|) > 0 \), where \( \epsilon = \|(A-B)^{-1}\|^{-1} \).

Proof. Since \( A \geq B + \epsilon \) for \( \epsilon = \|(A-B)^{-1}\|^{-1} > 0 \), we have

\[ f(A) \geq f(B + \epsilon). \]

Furthermore Lemmas 3.1 and 3.2 imply that

\[ f(B + \epsilon) \geq f(B) + D_\epsilon(\|B\|). \]

Hence we have

\[ f(A) - f(B) \geq D_\epsilon(\|B\|) = f(\|B\| + \epsilon) - f(\|B\|) > 0. \] \( \square \)
As a consequence, we have an improvement of the estimation due to Moslehian and Najafi [9]:

**Corollary 3.4.** If $A > B \geq 0$ and $0 < r \leq 1$, then $A^r - B^r \geq (\|B\| + \epsilon)^r - (\|B\|)^r > 0$, and $\log A - \log B \geq \log(\|B\| + \epsilon) - \log \|B\| > 0$, where $\epsilon = \|(A - B)^{-1}\|^{-1}$.

**Remark 3.5.** We note that Corollary 3.4 actually improves Theorem MN. Since $\|A\| - (\|A\| - \epsilon) = \epsilon = (\|B\| + \epsilon) - \|B\|$ and the function $t \mapsto t^r$ is strictly concave, it follows that

$$\|A\|^r - (\|A\| - \epsilon)^r \leq (\|B\| + \epsilon)^r - \|B\|^r.$$ 

We here pose an example:

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then $A - B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \geq 1$ and so $\epsilon = 1$. Hence we have

$$\|A\|^r - (\|A\| - \epsilon)^r = 4^r - 3^r < (\|B\| + \epsilon)^r - \|B\|^r = 3^r - 2^r.$$ 

Now Theorem 3.3 can be regarded as a difference version. So we give a ratio version of it. It is obtained by Theorem 3.3 itself:

**Corollary 3.6.** If $A > B > 0$ and $f$ is non-constant operator monotone on $(0, \infty)$, then

$$f(B)^{-\frac{1}{2}} f(A) f(B)^{-\frac{1}{2}} \geq 1 + (f(\|B\| + \epsilon) - f(\|B\|)) \|f(B)\|^{-1},$$

where $\epsilon = \|(A - B)^{-1}\|^{-1}$.

**Proof.** Put $\delta = f(\|B\| + \epsilon) - f(\|B\|)$. It follows from Theorem 3.3 that

$$f(B)^{-\frac{1}{2}} f(A) f(B)^{-\frac{1}{2}} \geq f(B)^{-\frac{1}{2}} f(B + \delta) f(B)^{-\frac{1}{2}} = 1 + \delta f(B)^{-1} \geq 1 + \delta \|f(B)\|^{-1}.$$ 

As another application of Theorem 3.3, we need the chaotic order: For $A > 0$, we can define the selfadjoint operator $\log A$. So a weaker order than the operator order appears by $\log A \geq \log B$ for $A, B > 0$. We call it the chaotic order. The chaotic order plays an substantial role in operator inequalities. Among others, it brightens the Furuta inequality [5], [2], [3], [1], [4], [7] and recent development of Karcher mean theory [11].

Now we give a simple and elementary proof to the following recent theorem [6, Theorem 2.1] due to Furuta, in which we don’t use any integral representation of operator monotone functions.

**Theorem 3.7.** If $\log A > \log B$ for $A, B > 0$ and $f$ is operator monotone on $(0, \infty)$, then there exists $\beta > 0$ such that

$$f(A^\alpha) > f(B^\alpha) \quad \text{for all} \ 0 < \alpha \leq \beta.$$
Proof. Since \( \log A > \log B \), it is known that there exists \( \beta > 0 \) such that
\[
A^\alpha > B^\alpha \quad \text{for all } 0 < \alpha \leq \beta.
\]
Therefore it follows from Theorem 3.3 that, for each fixed \( \alpha \in (0, \beta] \),
\[
f(A^\alpha) > f(B^\alpha),
\]
as desired. \( \square \)

4. A CONCLUDING REMARK.

Finally we discuss an operator extension of Lemma 2.1. Namely we may expect the following conjecture:
A real valued function \( f \) on an interval \( J = (a, b) \) with \( b \in (-\infty, +\infty) \) is operator convex if and only if, for each \( 0 < \epsilon < b - a \), \( D_\epsilon(t) \) is operator monotone on \( (a, b - \epsilon) \). Unfortunately we have a negative answer as follows: We choose the function \( f(t) = \frac{1}{t} \) on \( (0, \infty) \). It is a typical example of operator convex functions. Nevertheless, \( D_1(t) = -\frac{1}{(t+1)^2} \) is not operator monotone. As a matter of fact, we take two \( 2 \times 2 \) matrices \( A \) and \( B \):
\[
A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Note that \( D_1(A) \geq D_1(B) \) if and only if \( A(A + 1) \geq B(B + 1) \). Clearly \( A \geq B \), but
\[
A(A + 1) - B(B + 1) = \begin{pmatrix} 13 & 6 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 6 & 5 \end{pmatrix} \not\geq 0.
\]
This is a counterexample.

Incidentally, the operator convexity of the function \( \frac{1}{t} \) is easily shown as follows:
It is enough to prove the inequality
\[
\left( \frac{A + B}{2} \right)^{-1} \leq \frac{1}{2}(A^{-1} + B^{-1}).
\]
And it is simplified by putting \( C = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \) that
\[
4(1 + C^{-1})^{-1} \leq 1 + C,
\]
which follows from the numerical inequality \( 4 \leq (1 + x^{-1})(1 + x) \).

Acknowledgement. The authors would like to express their hearty thanks to the referee for his/her polite reviewing and kind suggestion.

References
5. T. Furuta, \( A \geq B \geq 0 \) assures \( (B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q} \) for \( r \geq 0, \ p \geq 0, \ q \geq 1 \) with \( (1 + 2r)q \geq p + 2r \), Proc. Amer. Math. Soc. 101 (1987), 85–88.

1 **Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan.**  
   *E-mail address: mfuji@cc.osaka-kyoiku.ac.jp*

2 **Department of Mathematics, Suwon University, Bongdamoup, Whasungsi, Kyungkido 445-743, Korea.**  
   *E-mail address: evergreen1317@gmail.com*

3 **3-4-13, Daihara-cho, Hitachi 316-0021, Japan.**  
   *E-mail address: r-naka@net1.jway.ne.jp*